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SYZYGY SHEAVES ON PROJECTIVE SPACES

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Abstract: Fix a closed subscheme $Z \subset \mathbf{P}^N$ and a surjection

$$\phi : \oplus_{i=1}^{m} \mathcal{O}_{\mathbf{P}^N}(-d_i) \to \mathcal{I}_Z.$$

$\mathrm{Ker}(F_\phi)$ is called a syzygy sheaf. Here we give a condition assuring that the vector bundle $\mathrm{Ker}(F_\phi)|L$ is rigid for a general line $L \subset \mathbf{P}^N$.

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Fix an integer $N \geq 2$, a closed subscheme $Z \subset \mathbf{P}^N$ with codimension at least 2, positive integers m, d_i, $1 \leq i \leq m$ and $f_i \in H^0(\mathbf{P}^N, \mathcal{I}_Z(d_i))$ such that $\widetilde{f} := (f_1, \dots, f_m)$ defines Z scheme-theoretically. Thus $\widetilde{f}$ defines a surjection $\phi : \oplus_{i=1}^{m} \mathcal{O}_{\mathbf{P}^N}(-d_i) \to \mathcal{I}_Z$. If $Z = \emptyset$, then we will say that F_ϕ is the syzygy bundle of $\widetilde{f}$. The case $Z = \emptyset$ is the classical case studied in [3], [5], [1], and [2]. Set $F_\phi := \mathrm{Ker}(\phi)$. We will say that F_ϕ is the syzygy sheaf of $\widetilde{f}$. F_ϕ is locally free at each point of $\mathbf{P}^N \backslash Z$. Since $\oplus_{i=1}^{m} \mathcal{O}_{\mathbf{P}^N}(-d_i)$ is locally free and $\mathcal{I}_Z$ is torsion free, F_ϕ is reflexive, i.e. the natural map $F_\phi \to F_\phi^{**}$ is an isomorphism (see [4], Proposition 1.3). Hence F_ϕ is locally free if $N = 2$ (see [4], Corollary 1.4). Let $\tau(Z)$ be the mimimal integer m such that for all $d \gg 0$ Z is the scheme-theoretic intersection of m general degree d forms. Call $d_0(Z)$ the minimal positive integer d for which this is true. Notice that $\tau(\emptyset) = N + 1$. It is easy to check that $\tau(Z) \geq N + 1$ for all Z.

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Remark 1. The example in which Z is a complete intersection shows that in general $\tau(Z)$ is strictly bigger than the minimal number of homogeneous forms whose zero-locus is Z (scheme-theoretically).

Remark 2. Fix a zero-dimensional scheme $Z \subset \mathbf{P}^N$. Assume $Z \neq \emptyset$. For each $P \in Z_{red}$ let Z_P be the connected component of Z with Z as its support. For each $P \in Z_{red}$ let $\tau(Z,P)$ be the minimal number of generators of the ideal sheaf of $\mathcal{O}_{\mathbf{P}^N,P}$ induced by the germ of Z_P. It is easy to check that $\tau(Z) = \max\{N+1, \max_{P \in Z_P} \tau(Z,P)\}$.

Let A be a rank r vector bundle on $\mathbf{P}^1$. Let $a_1 \geq \cdots \geq a_r$ be the splitting type of A. A is said to be *rigid* if $a_r \geq a_1 - 1$.

Remark 3. Let A be a rank r vector bundle on $\mathbf{P}^1$. It is easy to check that A is rigid if and only if for every integer t either $h^0(\mathbf{P}^1, A(t)) = 0$ or $h^1(\mathbf{P}^1, A(t)) = 0$.

Here we will prove the following result.

Theorem 1. *Let $Z \subset \mathbf{P}^N$, $N \geq 2$, a closed subscheme with codimension at least 2 and $L \subset \mathbf{P}^N$ a line such that. Fix an integer $k > 0$ such that $h^i(\mathbf{P}^N, \mathcal{I}_Z(k-i)) = 0$ for all $1 \leq i \leq N-1$. Fix integers m, d_i, $1 \leq i \leq d_i$, such that $m \geq \tau(Z)$ and $d_i \geq \max\{k, d_0(Z)\}$ for all i. Fix general $f_i \in H^0(\mathbf{P}^N, \mathcal{I}_Z(d_i))$, $1 \leq i \leq m$. Then $(f_1, \ldots, f_m)$ defines scheme-theoretically Z. Let F_ϕ be the associated syzygy sheaf. Then $F_\phi|L$ is rigid.*

Remark 4. Assume $Z = 0$ and $N = 2$ and take any syzygy bundle F associated to Z. Let $A = \oplus A_i$ be the Artinian graded algebra associated to F (see [2], Proposition 2.1). Fix any line $L := \{\ell = 0\} \subset \mathbf{P}^2$ and assume that $F|L$ is rigid. The proof of [2], part (1) of Theorem 2.2, shows that A has the weak Lefschetz property and for each i the multiplication map $\times \ell_i : A_i \to A_{i+1}$ by ℓ has maximal rank. More precisely, if F has splitting type $a_1 \geq \cdots \geq a_r$ with either $a_1 = a_r$ or $a_1 = a_r - 1$, then $\times \ell_i : A_i \to A_{i+1}$ is injective if $i \leq -a_1$ and it is surjective if $i \geq -a_r$. Now we drop the assumption that $F|L$ is rigid and call again $a_1 \geq \cdots \geq a_r$ its splliting type. The same proof shows that $\times \ell_i : A_i \to A_{i+1}$ is injective if $i \leq -a_1$ and it is surjective if $i \geq -a_r$.

Lemma 1. *Fix integers $m \geq 2$, $d_i > 0$, $1 \leq i \leq m$, and a general m-ple $(f_1, \ldots, f_m) \in H^0(\mathbf{P}^1, \oplus_{i=1}^m \mathcal{O}_{\mathbf{P}^1}(d_i))$. The forms f_i, $1 \leq i \leq m$, have no common zero and the associated syzygy bundle is rigid.*

Proof. Since $m \geq 2$ and the m-ple is general, the forms f_i, $1 \leq i \leq m$, have no common zero. By semicontinuity it is sufficient to find an m-ple $(h_1, \ldots, h_m) \in H^0(\mathbf{P}^1, \oplus_{i=1}^m \mathcal{O}_{\mathbf{P}^1}(d_i))$ with no common zero and whose associated syzygy bundle is rigid. Let G be the only rigid bundle with rank $m-1$ and degree $-d_1 - \cdots - d_m$. It is sufficient to show the existence of an exact sequence

$$0 \to G \to \oplus_{i=1}^{m} \mathcal{O}_{\mathbf{P}^1}(-d_i) \to \mathcal{O}_{\mathbf{P}^1} \to 0\,. \tag{1}$$

The existence of the extension (1) is a very particular case of [6]. □

Proof of Theorem 1. Since $d_i \geq d_0(Z)$ for all i, $m \geq \tau(Z)$ and the forms f_i are general, $(f_1, \ldots, f_m)$ defines scheme-theoretically Z. By definition of syzygy sheaf there is an exact sequence on $\mathbf{P}^N$:

$$0 \to F_\phi \to \oplus_{i=1}^{m} \mathcal{O}_{\mathbf{P}^N}(-d_i) \xrightarrow{\phi} \mathcal{I}_Z \to 0\,. \tag{2}$$

By Castelnuovo-Mumford's Lemma we have $h^1(\mathbf{P}^N, \mathcal{I}_Z(t)) = 0$ for all $t \geq k-1$ and the homogeous ideal of Z is generated by forms of degree at most k. Take a general flag of linear subspaces $H_0 \subset \cdots \subset H_{N-2}$ such that $H_0 = L$ and $\dim(H_j) = 1 + j$ for all j. Since $L \cap Z = \emptyset$, the generality of the flag implies $\mathrm{Tor}_i(\mathcal{O}_{H_j}, \mathcal{O}_Z) = 0$ for all $j \in \{0, \ldots, N-2\}$ and all $i > 0$. Hence $N-1$ exact sequences and the regularity of $\mathcal{I}_Z(k)$ imply that the restriction map $\rho_t : H^0(\mathbf{P}^N, \mathcal{I}_Z(t)) \to H^0(L, \mathcal{O}_L(t))$ is surjective for all $t \geq k$. Hence the restriction map $\rho : H^0(\mathbf{P}^N, \oplus_{i=1}^{m} \mathcal{I}_Z(d_i)) \to H^0(\mathbf{P}^1, \oplus_{i=1}^{m} \mathcal{O}_{\mathbf{P}^1}(d_i))$ is surjective. Apply Lemma 1! □

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