International Journal of Pure and Applied Mathematics

Volume 36 No. 4 2007, 345-352

DYNAMICS OF A LASER RESONATOR

Vicente Aboites

Centro de Investigaciones en Optica A.C.
Loma del Bosque 115
Col. Lomas del Campestre, 37150, Leon, Guanajuato, MEXICO
e-mail: aboites@cio.mx

Abstract: The dynamics of a laser resonator is presented. It is shown that a non-linear dynamic behaviour takes place when a chaos generating element is introduced within the resonator. The analysis of a laser resonator using ABCD matrix formalism is showed for the case where these elements are present in the resonator. Assuming a ray inside the resonator with parameters $y(z)$ and $\theta(z)$ for the effective distance to the optical axis z and angle to the same axis confined in the resonator we obtain expressions for the n-th trip $y(z)_n$ and $\theta(z)_n$. In particular an expression for $y(z)_{n+1}$ of the form: $y_{n+1} = ay_n(l - y_n)$ is obtained. Chaotic regions are shown in a simple bifurcation diagram. The dynamics of the resonator can be modified by the change of the resonator parameters. Finally the characteristics of the chaos generating elements is discussed and the matrix elements of a chaos generating matrix $[a, b, c, e]$ are presented.

AMS Subject Classification: 78A60
Key Words: laser dynamics, laser, resonator

1. Introduction

Chaos has been found and studied in many systems [4], [5], [1], [2], ; optical, mechanical, electrical, chemical and many others. The study of a chaotic physical system allows for the search of appropriate theoretical or computational models to describe it. Also, the study of chaotic theoretical and computational results often stimulates the search for physical systems that behave according to the model predictions. In this work it is shown that a beam within a laser resonator may be described by a logistic map and the conditions for this to

Received: August 16, 2006

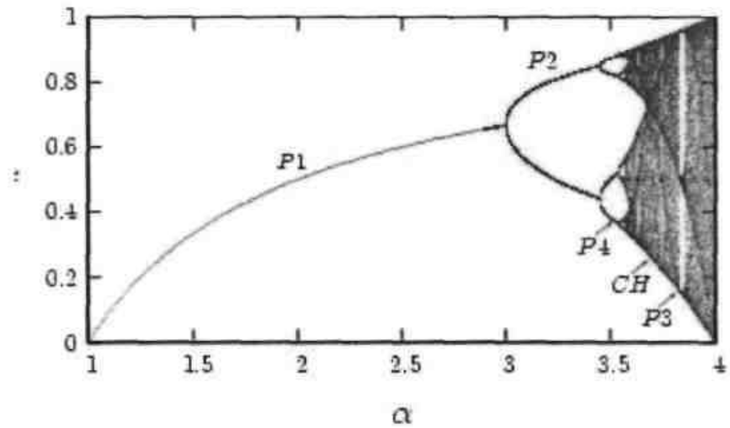

Figure 1: Bifurcation diagram of the logistic map

occur are discussed. Section 2 presents the basic features of logistic maps, Section 3 discusses the matrix optics elements on which this work is based, Section 4 shows the main characteristics of the chaos generation matrix and finally Section 5 presents the conclusion.

2. Logistic Map

The logistic map [3] is given by the difference equation

$$N_{n+1} = N_n a(1 - N_n)\,, \tag{1}$$

where $N_n \in [0, 1]$. This map is one-dimensional and non-linear. The time sequence produced by the mapping is obtained by choosing a value of α, plotting the corresponding quadratic curve and repetitively generating subsequent points starting with some initial value. The process is repeated. Where $N_{n+1} = N_n$ we call it is a fixed point. Changing for an appropriate value of α it is possible to see that after an initial transient, N_n, oscillates between two values so that $N_{n+2} = N_n$. Higher values of a may lead to further bifurcations and chaos. Figure 1 shows the bifurcation diagram obtained for the logistic map and Table 1 summarizes the most relevant features of it.

Figure 2 shows the logistic map for three different cases: (a) shows the case of a stable fixed point, (b) shows a stable period two points and, (c) shows a chaotic sequence.

Parameter a	**Main characteristic of the logistic map**
1	Starts stable fixed point
2	Stable fixed point
3	Starts stable period 2
3.445	Starts stable period 4
3.54409	Starts stable period 8
3.5644	Starts stable period 16
3.568759	Starts stable period 32
3.569945672 0	Stats of chaos
3.828427125	Starts stable period 3
3.9	Chaos
4	End of chaos

Table 1: Characteristics of the logistic map as a function of the parameter α

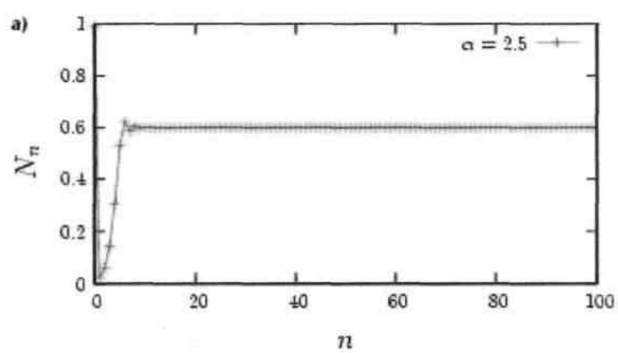

Figure 2: (a) Stable fixed point

3. Matrix Optics

Any optical element may be described by a matrix $[A, B, C, D]$. Assuming radial symmetry around the optical axis and defining the perpendicular distance of any ray to the optical axis and the angle to the same axis at a given position z as; $y(z)$ and $\theta(z)$. Then within the paraxial approximation $(\sin\theta \approx \tan\theta \approx \theta)$ the relation of the parameters y and θ at two different points along the optical

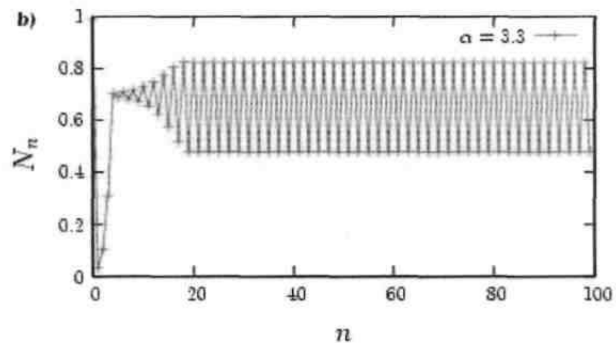

Figure 2: (b) Period 2

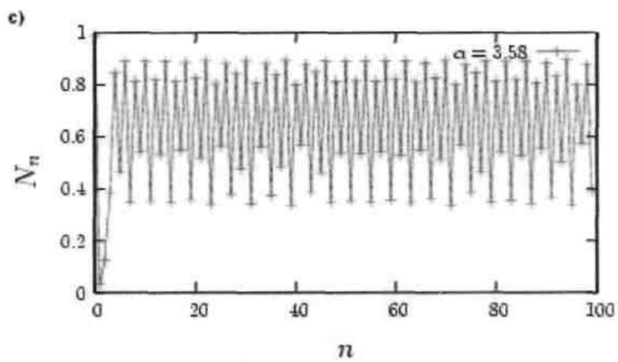

Figure 2: (c) Chaos

axis is given by:

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} . \tag{2}$$

For any optical system it is always possible to obtain the total $[A, B, C, D]$ matrix taking the matrix product of each one of the optical elements in the system.

A resonator formed by two mirrors separated by a distance d, with radii of curvature R_1, and R_2 is shown in Figure 3. In addition, an unknown chaos generating element represented by the matrix $[a, b, c, e]$ is located in the middle of the resonator at a distance $d/2$, as shown in Figure 3.

For this system, the total transformation matrix $[A, B, C, D]$ for a complete round trip is given by

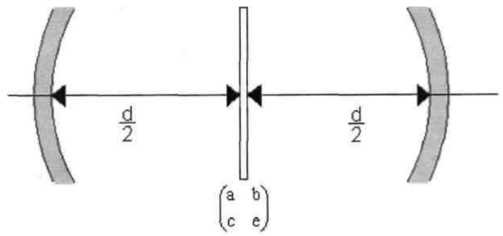

Figure 3: Two mirror laser resonator with a chaos generating element

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{R_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{R_2} & 1 \end{bmatrix} \times \begin{bmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{bmatrix} . \quad (3)$$

For the sake of simplicity we will consider a confocal laser resonator with the same radius of curvature for both mirrors and equal to the separation distance d. Then the transformation matrix $[A, B, C, D]$ for a round trip resonator acquires the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{d} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{d} & 1 \end{bmatrix} \times \begin{bmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{bmatrix} . \quad (4)$$

The elements of the above total transformation matrix of the resonator are

$$A = \frac{1}{2}adc + \frac{1}{4}d^2c^2 - ae\frac{2ab}{d} ,$$

$$B = \frac{1}{4}cd^2e - be - ab + \frac{1}{8}c^2d^3 + \frac{1}{4}acd^2 - \frac{2b^2}{d} , \quad (6)$$

$$C = \frac{4ab}{d^2} + ac , \quad (7)$$

$$D = \frac{2ab}{d} - \frac{1}{2}acd + \frac{4b^2}{d^2} - ae . \quad (8)$$

Assuming a symmetric ring ray beam inside the resonator with parameters $y(z)$ and $\theta(z)$ confined in the resonator, and taking y_0 as the initial radius of the beam at $z = 0$, we may obtain expressions for the n-th trip $y(z)_n$ and $\theta(z)_n$.

4. Chaos Generating Matrix

Using the round trip transformation matrix $[A, B, C, D]$ we can find an expression to obtain y_{n+1} and θ_{n+1} from y_n and θ_n in the following form

$$\begin{pmatrix} y_{n+1} \\ \theta_{n+1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_n \\ \theta_n \end{pmatrix} . \tag{9}$$

Since we are looking for the simplest chaos-generating mapping of the general form

$$N_{n+1} = N_n \alpha (1 - N_n) . \tag{10}$$

The matrix $[a, b; c, e]$ of the chaos generating element needs to be found, so that equation (9) is able to generate an equation for y_{n+1} in the form (10). From (9) we obtain

$$y_{n+1} = A y_n + B \theta_n , \tag{11.a}$$

$$\theta_{n+1} = C y_n + D \theta_n . \tag{11.b}$$

In order to obtain a map equation as (10) we must have

$$A = \alpha , \tag{12.a}$$

$$B = -\frac{\alpha^2 y_n}{\theta_n} . \tag{12.b}$$

If we now consider a chaos generating element sufficiently thin, we may then assume $b = 0$, and since the determinant of the matrix $[a, b; c, e]$ must be equal to 1, then, necessarily the product $ae = 1$, then elements A, B, C, D of the transformation matrix (5), (6), (7) and (8) acquire the form:

$$A = \frac{1}{2}\frac{dc}{e} + \frac{1}{4} d^2 c^2 - 1 , \tag{13}$$

$$B = \frac{1}{4} c d^2 e + \frac{1}{8} c^2 d^3 + \frac{1}{4}\frac{c d^2}{e} , \tag{14}$$

$$C = \frac{c}{e} , \tag{15}$$

$$D = -\frac{1}{2}\frac{cd}{e} - 1 . \tag{16}$$

Replacing (12.a) and (12.b) in equations (13) and (14) we obtain a system of equations for e and c of the following form

$$\alpha = \frac{1}{2}\frac{dc}{e} + \frac{1}{4} d^2 c^2 - 1 , \tag{17}$$

$$-\frac{\alpha y_n^2}{\theta_n} = \frac{1}{4}cd^2e + \frac{1}{8}c^2d^3 + \frac{1}{4}\frac{cd^2}{e}\,. \tag{18}$$

When we solve the system of equations for elements c and e, using the fact that $\alpha = 1/e$, the results are

$$a = \frac{(1+\alpha)\theta_n d}{\alpha\left[(1+\alpha)(d\theta_n + 2y_n^2)(d\theta_n\alpha + 2y_n^2\alpha + d\theta_n)\right]^{\frac{1}{2}}}\,, \tag{19}$$

$$b = 0\,, \tag{20}$$

$$c = \frac{\left[\alpha(1+\alpha)(d\theta_n + 2y_n^2)(d\theta_n\alpha + d\theta_n)\right]^{\frac{1}{2}}}{(d\theta_n + 2y_n^2)\alpha d}\,, \tag{21}$$

$$e = -\frac{\left[\alpha(1+\alpha)(d\theta_n + 2y_n^2)(d\theta_n\alpha + 2y_n^2\alpha + d\theta_n)\right]^{\frac{1}{2}}}{(1+\alpha)\theta_n d}\,. \tag{22}$$

Taking the solutions (19), (20), (21) and (22) for the elements of the matrix $[a, b; c, e]$ into equation (4), the total transformation matrix $[A, B; C, D]$ for a round trip is now written:

$$\begin{pmatrix} \alpha & -\dfrac{\alpha y_n^2}{\theta_n} \\ -2\dfrac{\theta_n(1+\alpha)}{\alpha(d\theta_n + 2y_n^2)} & \dfrac{dR_n(1+\alpha)}{\alpha(d\theta_n + 2y_n^2)} - 1 \end{pmatrix}\,. \tag{23}$$

Substituting in equation (9), we may obtain an equation for y_n as:

$$y_{n+1} = \alpha y_n(1 - y_n)\,. \tag{24}$$

This equation is isomorphic to equation (10). When the value of the parameter α is between 0 and 4, the corresponding chaos and bifurcation diagram shown in Figure 1, is found.

5. Conclusions

When a sufficiently thin element is introduced within a two mirror concentric laser resonator with curvature radii equal to the separation among them, chaos such as the article describes can appear so that an iteration for y_{n+1} of the form $y_{n+1} = \alpha y_n(1 - y_n)$ is obtained. As it is well known, this represents a simple expression for chaos generation. It is known that depending on the value of α chaos or stable values for y are obtained as shown in the bifurcation diagram in Figure 1.

The matrix elements of α chaos generating element $[a, b, c, e]$ are found for the first time to our knowledge, and used to obtain a laser resonator able to produce chaos described by a well known iterative mapping with clear chaos regions as is shown in a bifurcation diagram.

References

[1] G.L. Baker, J.P. Gollub, *Chaotic Dynamics*, Cambridge University Press, New York (2002).

[2] P. Berge, Y. Pomeau, C. Vidal, *Order within Chaos*, John Wiley and Sons, New York (1994).

[3] M.J. Feigenbaum, *J. Stat. Phys.*, **19**, No. 25 (1978).

[4] H.O. Peitgen, H. Jurgens, D. Saupe, *Chaos and Fractals*, Springer Verlag, New York (1992).

[5] S.H. Strogatz, *Nonliniear Dynamics and Chaos*, Perseus Publishing, Cambridge (2000).