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LONG-TIME EXISTENCE OF FEASIBLE PATHS IN TIME EVOLUTION OF CURVES

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Abstract: We investigate the geometry of time evolution of curves. The main result is to show that every pair of points at different time in this new version of geometry is jointed by at least one shortest path.

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1. Introduction

Consider the smooth evolution of a regular space curve as time goes on. How can one say about the geometry in this phenomena? A very natural question is to ask: Does the shortest path from position p at time t_1 to position q at time t_2 exist? In this paper, we will investigate the existence of the shortest paths, which we call them *feasible paths*. Local existence of feasible paths due to the first variation formula of energy functions. Our main result is to show a long-time existence theorem for feasible paths.

Let $\gamma : [\alpha, \beta] \to \mathbf{R}^3$ be a regular space curve. Let $\sigma : [\alpha, \beta] \times [a, b] \to \mathbf{R}^3$ be a smooth map such that $\sigma|_{[\alpha,\beta]\times\{a\}} = \gamma$ and for each t, $\sigma|_{[\alpha,\beta]\times\{t\}}$ is also a regular space curve. Set

$$\Sigma = \{\sigma(u, t) | \alpha \leq u \leq \beta,\ a \leq t \leq b\}$$

and denote $\Sigma_{\bar{t}} = \sigma([\alpha, \beta] \times \{\bar{t}\})$ for some fixed $\bar{t} \in [a, b]$.

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A space curve $c : [\bar{a}, \bar{b}] \to \Sigma$ for $0 \le a \le \bar{a} < \bar{b} \le b$ is called a *time curve* if $c(t) \in \Sigma_t$ for all $t \in [\bar{a}, \bar{b}]$. That is, $c(t)$ is a time curve provided $c(t) = \sigma(u(t), t)$. A *feasible path* from $p \in \Sigma_{\bar{a}}$ to $q \in \Sigma_{\bar{b}}$ is a time curve from p to q with the shortest length. Note that the definition of the energy $\Xi(c)$ of a space curve c from a to b is

$$\Xi(c) = \int_a^b \|c'(t)\|^2 dt.$$

So a feasible path is a time curve with minimal energy. Let us first investigate the local existence of feasible paths.

Let $c(t) = \sigma(u(t), t)$ be a time curve form $p \in \Sigma_{\bar{a}}$ to $q \in \Sigma_{\bar{b}}$. Then $c'(t) = \sigma_u u'(t) + \sigma_u$ and $c''(t) = \sigma_{uu}(u'(t))^2 + 2\sigma_{ut} u'(t) + \sigma_{tt}$. We claim that $c(t)$ is a feasible path if it satisfies the following *feasible path equation*:

$$\frac{d}{dt}(Eu'(t)) - \frac{1}{2}E_u(u'(t))^2 + < \sigma_{tt}, \sigma_u >= 0\,, \tag{1}$$

where $E =< \sigma_u, \sigma_u >= \|\sigma_u\|^2$ with the standard Euclidean inner product $<,>$. Note that a curve satisfies equation (1) is in fact a stationary point of the energy function.

Indeed, let

$$c^s(t) = \sigma(u(s,t), t),\ c^s(a) = p,\ c^s(b) = q,\ \forall s \in (-\epsilon, \epsilon)\,,$$

be a proper variation of $c^0(t) = c(t)$ and denote $\dot{c}^s(t) = \frac{\partial}{\partial t} c^s(t)$. Then the first variation of the energy function is (c.f. [1] or [2])

$$\begin{aligned}
&\frac{\partial}{\partial s}\Xi(c^s(t)) = \frac{\partial}{\partial s}\int_a^b \|\dot{c}^s(t)\| dt = \frac{\partial}{\partial s}\int_a^b (E\dot{u}^2 + 2\langle \sigma_u, \sigma_t\rangle \dot{u} + \|\sigma_t\|^2) dt = \\
&\int_a^b \{(E_u \frac{\partial u}{\partial s}\dot{u}^2 + 2E\dot{u}\frac{\partial^2 u}{\partial t \partial s}) + 2(\langle \sigma_u, \sigma_t\rangle_u \dot{u}\frac{\partial u}{\partial s} + \langle \sigma_u, \sigma_t\rangle \frac{\partial^2 u}{\partial t\partial s}) + \langle \sigma_t, \sigma_t\rangle_u \frac{\partial u}{\partial s}\} dt \\
&= \int_a^b \{(E_u\dot{u}^2 + 2\langle \sigma_u, \sigma_t\rangle_u \dot{u} + \langle \sigma_t, \sigma_t\rangle_u)\frac{\partial u}{\partial s}) + 2\frac{\partial u}{\partial s}\frac{\partial}{\partial t}(E_u \dot{u} + \langle \sigma_u, \sigma_t\rangle)\} dt \\
&= \int_a^b \{E_u \dot{u}^2 + \langle \sigma_t, \sigma_t\rangle_u - 2\frac{\partial}{\partial t}(E\dot{u}) - 2\langle \sigma_{ut}, \sigma_t\rangle - 2\langle \sigma_{tt}, \sigma_u\rangle\}\frac{\partial u}{\partial s} dt \\
&= -2\int_a^b \{\frac{d}{dt}(E\dot{u}) - \frac{1}{2}E_u \dot{u}^2 + < \sigma_u, \sigma_{tt} >\}\frac{\partial u}{\partial s} dt.
\end{aligned}$$

This gives the feasible path equation (1).

Let $p = \sigma(u, t)$ and the affine tangent space $T_p\Sigma_t$ be defined by

$$T_p\Sigma_t \ = \ \{\lambda\sigma_u + \sigma_t | \lambda \in \mathbf{R}\}.$$

If $c(t) = \sigma(u(t), t)$ is a time curve, then $c'(t) = \sigma_u u'(t) + \sigma_t \in T_p\Sigma_t$ is an affine tangent vector. Now we have the following local existence theorem for feasible paths.

Local Existence Theorem. *Given $\alpha \le u_0 \le \beta$, $0 \le a \le t_0 \le b$ and $p_0 = \sigma(u_0, t_0)$. Then there exits a positive number ϵ_{t_0} depending on the time t_0 such that for $t \in [t_0, t_0 + \epsilon_{t_0})$, $u \in (u_0 - \epsilon_{t_0}, u_0 + \epsilon_{t_0})$ and $\|v\| < \epsilon_{t_0}$, where $v = \sigma_u(u_0, t_0)u'(t_0) + \sigma_t(u_0, t_0) \in T_{p_0}\Sigma_{t_0}$, there exists a unique feasible path $c = c_v(t) : [t_0, t_0 + \epsilon_{t_0}) \to \Sigma$ with the initial conditions $c(t_0) = p_0$ and $c'(t_0) = v$. Moreover, the map $c : [t_0, t_0 + \epsilon_{t_0}) \times T_{p_0}\Sigma_{t_0} \to \Sigma$ defined by $c(t, v) = c_v(t)$ is smooth.*

Here are some interesting examples.

Example 1.1. Let $\gamma(u) = (\cos u, \sin u)$ and $\sigma(u, t) = (\cos u, \sin u, \phi(t))$. Then $E = 1$, $< \sigma_{tt}, \sigma_u >= 0$ and the feasible path equation is

$$0 = \frac{d}{dt}(Eu'(t)) - \frac{1}{2}E_u(u'(t))^2 + < \sigma_{tt}, \sigma_u >= u''(t).$$

The solution is $u(t) = at + b$ for some constants a and b. In particular, a circular helix $c(t) = (\cos t, \sin t, \phi(t))$ and a time-curve with $u = constant$ are both feasible paths.

Example 1.2. Consider the unit-speed curve $\gamma(u) = (f(u), 0, g(u))$ with $(f'(u))^2 + (g'(u))^2 = 1$ rotating about the $z-$axis. Suppose γ rotates an angle $\phi(t)$ at time t. So

$$\sigma(u, t) = (f(u)\cos\phi(t), f(u)\sin\phi(t), g(u))$$

and then $E = 1$, and $\sigma_{tt} = (-\phi''(t)f(u)\sin\phi(t) - (\phi'(t))^2 f(u)\cos\phi(t),\ \phi''(t)f(u)\cos\phi(t) - (\phi'(t))^2 f(u)\sin\phi(t), 0)$.

Therefore, the feasible path equation is

$$u''(t) - f'(u(t))f(u(t))(\phi'(t))^2 = 0.$$

In particular, let $\gamma(u) = (1, 0, u)$ and compare with Example 1. Then the feasible equation is $u''(t) = 0$. Hence the parallel curve with $u = constant$ and helix are feasible paths. Moreover, let $f(u) = 2 + \cos u$ and $g(u) = \sin u$ (compare with a torus). Then the feasible path equation is

$$u''(t) + \sin u(t)(2 + \cos u(t))(\phi'(t))^2 = 0.$$

Example 1.3. Consider the circle $\gamma(u) = (a\cos u, a + a\sin u)$ of radius $a > 0$ rolling without slipping along the x-axis as time goes on. Suppose the circle rolls a distance $a\phi(t)$ at time (t). Then

$$\sigma(u(t), t) = (a\cos(u(t) - \phi(t)), a + a\sin(u(t) - \phi(t)), 0).$$

Then $E = a^2$, $< \sigma_{tt}, \sigma_u >= -a^2\phi''(t)$ and the feasible path equation is

$$0 = \frac{d}{dt}(Eu'(t)) - \frac{1}{2}E_u(u'(t))^2 + < \sigma_{tt}, \sigma_u >= a^2(u''(t) - \phi''(t)).$$

In particular, if $\phi(t) = t$ then a *cycloid*, which is a curve with $u = constant$, is a feasible path. That is, a time-curve with $u = constant$ is a feasible path if the rolling velocity is constant.

Example 1.4. Consider time evolution of the curve $\gamma(u) = (a + \cos u, \sin u)$, $a \geq 1$ with

$$\sigma(u,t) = e^{-t}(\cos u + a\cos t, \sin u + a \sin t).$$

That is, the center of γ is moving along the logarithmic spiral $(e^{-t}\cos t, e^{-t}\sin t)$ and the radius is e^{-t} at time t. Then $E = e^{-2t}$, $< \sigma_{tt}, \sigma_u >= e^{-t}(2a\sin t + \cos u, -2a\cos t + \sin u)$ and then

$$< \sigma_{tt}, \sigma_u >= -2ae^{-2t}(\sin t \sin u + \cos t \cos u) = -2ae^{-2t}\cos(u - t).$$

So the feasible path equation is

$$u''(t) - 2u'(t) - 2a\cos(u - t) = 0.$$

A very special case is when $a = 1$ and $u = \pi + t$ then $c(t) = 0$. Significantly, a time curve with $u = constant$ is never a feasible path.

Our main result is the following long-time existence theorem for feasible paths.

Main Theorem. *Let $\gamma : [\alpha, \beta] \to \boldsymbol{R}^3$ be a regular space curve. Given real numbers a, b with $0 \leq a < b$ and a smooth map $\sigma : [\alpha, \beta] \times [a, b] \to \boldsymbol{R}^3$ satisfying $\sigma|_{[\alpha,\beta]\times\{a\}} = \gamma$ and for each t, $\sigma|_{[\alpha,\beta]\times\{t\}}$ is a regular space curve. Suppose $\Sigma = \sigma([\alpha, \beta] \times [a, b])$ and $\Sigma_t = \sigma([\alpha, \beta] \times \{t\})$ for $t \in [a, b]$. Then for any $p \in \Sigma_a$ and $q \in \Sigma_b$ there exists a feasible path $c : [a, b] \to \boldsymbol{R}^3$ from $c(a) = p$ to $c(b) = q$.*

2. Long-Time Existence of Feasible Paths

Now we give a proof of the long-time existence of feasible paths. Our main tool is the Morse theory and the technique presented in Jost's book [3]. Let

$$\Lambda_{pq} = \{c(t) = \sigma(u(t), t) | c(a) = p,\ c(b) = q\}$$

be the set of all time curves with end point p, q and $c(\bar{t}) \in \Sigma_{\bar{t}}$ for $a \leq \bar{t} \leq b$. Then the energy functional $\Xi(c) = \frac{1}{2}\int_a^b |c'(t)|^2 dt$ is continuous on Λ_{pq}. For $\eta > 0$ define

$$\Lambda_{pq}^{\eta} = \{c \in \Lambda_{pq} | \Xi(c) \leq \eta\}.$$

Let $a = t_0 < t_1 < t_2 < \ldots < t_k = b$ be a partition of $[a, b]$ and $c|_{[t_{i-1},t_i]}$ denote the restriction of c to the closed interval $[t_{i-1}, t_i]$. Define

$$\Lambda_{pq}(t_1, \ldots, t_{k-1}) = \{c \in \Lambda_{pq} | c|_{[t_{i-1},t_i]} \text{ is feasible}\}.$$

Now we denote

$$\Lambda_{pq}^{\eta}(t_1, \ldots, t_{k-1}) = \Lambda_{pq}(t_1, \ldots, t_{k-1}) \cap \Lambda_{pq}^{\eta}.$$

By the local existence of feasible paths, there exists $\rho_0 > 0$ such that for $a \leq t_j$, $t_m \leq b$, $|t_j - t_m| < \rho_0$ and $x \in \Sigma_{t_j}$, $y \in \Sigma_{t_m}$ there is a unique feasible path from x to y. Now we choose a partition $a = t_0 < t_1 < \ldots t_k = b$ of $[a, b]$ with

$$t_i - t_{i-1} < \frac{\rho_0^2}{2\eta}$$

for $i = 1, 2, \ldots, k$. Then, for each $c \in \Lambda_{pq}^{\eta}(t_1, \ldots, t_{k-1})$,

$$\begin{aligned} d(c(t_{i-1}), t_i)^2 \leq Length(c|_{[t_{i-1},t_i]})^2 &= 2(t_i - t_{i-1})\Xi(c|_{[t_{i-1},t_i]}) \\ &\leq 2(t_i - t_{i-1})\Xi(c) \leq 2(t_i - t_{i-1})\eta < \rho_0^2. \end{aligned}$$

Therefore, the feasible path from $c(t_{i-1})$ to $c(t_i)$ is unique and hence coincides with $c|_{[t_{i-1},t_i]}$.

Moreover, the piecewise feasible path c is uniquely determined by

$$(c(t_1), \ldots, c(t_k)) \in \Sigma \times \ldots \times \Sigma = \Sigma^{k-1}.$$

Thus,

$$c \to (c(t_1), \ldots, c(t_k))$$

defines a homeomorphism of the interior of $\Lambda_{pq}^{\eta}(t_1, \ldots, t_{k-1})$ onto an open subset of Σ^{k-1} and hence the interior of $\Lambda_{pq}^{\eta}(t_1, \ldots, t_{k-1})$ may be equipped with the structure of a differentiable manifold. Then for $c \in \Lambda_{pq}^{\eta}(t_1, \ldots, t_{k-1})$, we have the formula

$$\Xi(c) = \sum_{i=1}^{k} \Xi(c|_{[t_{i-1},t_i]}) = \sum_{i=1}^{k} \frac{d(c(t_{i-1}), c(t_i))^2}{2(t_i - t_{i-1})}.$$

In particular, the restriction of Ξ to $\Lambda_{pq}^{\eta}(t_1, \ldots, t_{k-1})$ is differentiable. Moreover, it can be shown that (c.f. [3]) the energy function Ξ is differentiable on Λ_{pq}.

Lemma 2.1. *All critical points of Ξ on Λ_{pq}^{η} are contained in $\Lambda_{pq}^{\eta}(t_1, \ldots, t_{k-1})$.*

Proof. Let $c \in \Lambda_{pq}^{\eta}$. Then

$$d(c(t_{i-1}), c(t_i))^2 \leq 2(t_i - t_{i-1})\Xi(c) < \rho_0^2.$$

This implies that the map

$$r : \Lambda_{pq}^{\eta} \to \Lambda_{pq}^{\eta}(t_1, \ldots, t_{k-1})$$

is well-defined. Moreover, r is continuous and

$$\Xi(r(c)) \leq \Xi(c)$$

Now we define a family $(r_t)_{0 \leq t \leq 1}$ of maps $r_t : \Lambda_{pq}^{\eta} \to \Lambda_{pq}^{\eta}$ by the following. For $i = 1, \ldots, k-1$, let

$$r_t(c)|_{[t_{i-1}, t_{i-1}+t(t_i - t_{i-1})]}$$

be the feasible path from $c(t_{i-1})$ to $c(t_{i-1} + t(t_i - t_{i-1}))$ and let

$$r_t(c)|_{[t_{i-1}+t(t_i-t_{i-1})]} = c|_{[t_{i-1}+t(t_i-t_{i-1})]}.$$

Then we have $r_0(c) = c$, $r_1(c) = r(c)$ and $r_t(c)$ is continuous in t and c. This proves that $\Lambda_{pq}^{\eta}(t_1, \ldots, t_{k-1})$ is a deformation retract of Λ_{pq}^{η}. Since the critical points of Ξ are feasible paths and so are piecewise feasible paths, they lie in $\Lambda_{pq}^{\eta}(t_1, \ldots, t_{k-1})$ if their energy is $\leq \eta$. $\square$

Let another partition $(\tau_1, \ldots, \tau_k)$ be given by

$$t_0 < \tau_1 < t_1 < \tau_2 < \ldots < \tau_k < t_k$$

and we also assume that

$$\tau_i - \tau_{i-1} < \frac{\rho_0^2}{2\eta}$$

for $i = 1, \ldots, k$ with $\tau_0 = \tau_k - b$.

Let γ be a time-curve from $p \in \Sigma_a$ to $q \in \Sigma_b$. Note that this curve always exists due to the local existence theorem. Assume

$$\Xi(\gamma) \leq \eta.$$

Now we use the curve shortening process to prove the long time existence of the feasible paths. Let $r_1(c)$ be the piecewise feasible path for which $r_1(c)|_{[t_{i-1}, t_i]}$ is the feasible path from $c(t_{i-1})$ to $c(t_i)$. By the local existence theorem and the choice of ρ_0, this determines $r_1(c)$ uniquely. Then define $r_2(c)$ as the piecewise feasible path for which $r_2(c)|_{[\tau_{i-1},\tau_i]}$ is the feasible path from $c(\tau_{i-1})$ to $c(\tau_i)$. Note that $r_2(c)$ is likewise uniquely determined. Define

$$P(c) \equiv r_2 \circ r_1(c).$$

Lemma 2.2. *We have the inequality*

$$\Xi(P(c)) \leq \Xi(c),$$

with equality holds if and only if c is a feasible path.

Proof. Note that

$$\Xi(r_1(c)) \leq \Xi(c),$$

which the equality holds if and only if c is a piecewise feasible path from p to q with nodes $c(t_1), \ldots, c(t_{k-1})$. Likewise, for each time-curve $\tilde{c}$

$$\Xi(r_2(\tilde{c})) \leq \Xi(\tilde{c}),$$

with equality holds if and only if $\tilde{c}$ is a piecewise feasible path from p to q with nodes $\tilde{c}(\tau_1), \ldots, \tilde{c}(\tau_{k-1})$. Thus, if $\Xi(P(\gamma)) = \Xi(\gamma)$, then all segments $c|_{[t_i,t_{i-1}]}$ as well as all segments $c|_{[\tau_i,\tau_{i-1}]}$ are feasible paths. Hence c is a feasible path from p to q. $\square$

Lemma 2.3. *Let c be a time curve with energy $\Xi(c) \leq \eta$. Then a subsequence of $P^n(c) \equiv P \circ \ldots \circ P(c)$ converges uniformly to a feasible path.*

Proof. First note that each curve $P^n(c)$ for $n = 1, 2, \ldots$ is a piecewise feasible path with nodes $P^n(c(\tau_1)), \ldots, P^n(c(\tau_k))$, where the individual segments are feasible paths between those nodes. Hence, each such curve may be identified with a $k-$tuple

$$P^n(c(\tau_1)), \ldots, P^n(c(\tau_k)) \in \Sigma^k.$$

Since Σ^k is compact, $P^n(c(\tau_1)), \ldots, P^n(c(\tau_k))$ converges to some $(p_1, \ldots, p_k)$ $\in \Sigma^k$ and hence $P^n(c)$ converges uniformly to the piecewise feasible path c_0 with nodes $c_0(\tau_i) = p_i$ for $i = 1, \ldots, k$, whose individual segments $c_0|_{[\tau_{i-1},\tau_i]}$ again are the feasible paths from $c_0(\tau_{i-1})$ to $c_0(\tau_i)$ since the limit of feasible paths is a feasible path. Denote the convergent subsequence of $(P^n(c))_{n\in N}$ by $c_m := (P^{n_m}(c))_{m\in N}$. Then

$$\Xi(c_0) = \lim_{m\to\infty} \Xi(c_m),$$

as follows. Moreover,

$$\Xi(c_0) = \lim \Xi(c_{m+1}) = \lim \Xi(P^{n_m} c_m) \leq \lim \Xi(P(c_m)) \leq \lim \Xi(c_m) = \Xi(c_0).$$

Therefore, the equality must hold throughout. Hence $P(c_m)$ converges to $P(c_0)$ and we have

$$\Xi(P(c_0)) = \lim_{m\to\infty} \Xi(P(c_m)) = \Xi(c_0).$$

Finally, by Lemma 2.2, c_0 is a feasible path from p to q. □

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References

[1] M.P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall (1976).

[2] J. Jost, *Riemannian Geometry and Geometric Analysis*, Prentice-Hall (1995).

[3] A. Pressley, *Elementary Differential Geometry*, Springer (2003).