International Journal of Pure and Applied Mathematics

Volume 36 No. 4 2007, 407-417

THE TEST FOR ADEQUACY OF SHELL MODELS FOR A TURBULENCE PROBLEM

Igor M. Gaissinski[1 §], Vladimir Y. Rovenski[2]

[1] Aerospace Engineering Department
Technion – Israel Institute of Technology
Haifa, 32000, ISRAEL
e-mail: ricka1993@012.net.il

[2]Department of Mathematics
University of Haifa
Haifa, 31905, ISRAEL
e-mail: rovenski@math.haifa.ac.il

Abstract: The nonlinear interactions between the base functions used in a shell model are described by the tensor T_{NML}, obtained using Galerkin projection of Navier–Stokes equations on the wavelets base. The coefficients T_{NML} must satisfy the conservation laws for the energy and the enstrophy or the helicity (depending on the dimension 2 or 3). The paper deduces these relations and presents their solution as the product of the partial solution and $G(\eta_{LNM}, \eta_{NML}, \eta_{MLN})$, where G is any odd function invariant under transpositions of arguments, and a tensor η has zero diagonal terms and satisfies the conditions $\eta_{NML} = -\eta_{LMN}$, $\eta_{N+k,M+k,L+k} = \eta_{NML}$. One may use these relations and their solution to verify the adequacy of wavelet functions base to the turbulence problem.

AMS Subject Classification: 76E20, 76J20, 76N99

Key Words: Navier-Stokes equations, enstrophy, helicity, shell model, turbulence

1. Introduction

The hierarchical models of a turbulence are based on the assumption that the

Received: January 29, 2007

§Correspondence author

turbulence is an ensemble of vortices of progressively diminishing scales. In this way there was obtained, by Kolmogorov [8], [3], the spectral law (K41) for homogeneous and isotropic turbulence. The models, see [1], [5], [9], etc. differing by some details, preserve the fundamental properties of Navier–Stokes equations (NSE) – satisfy in inviscid limit the conservation laws. Since the end of 1990s the interest to shell models in turbulence problems is decreased, that apparently is connected with the fact that later advanced models did not satisfy conditions of the main invariants conservations laws. In particular, the reason of infringement of these conditions in models developed in [2], [10] is covered in updating original basis by artificial entering multiplicative members picked up thus that K41 to be carried out. However, the same factors 'have spoiled' this perfectly found basis. The coupling terms (among the various shells) of obtained system of hierarchical of quadratic ordinary differential equations (ODE) have to be chosen according to the main symmetries and conservation laws of NSE. The choice of shell model for the energy cascade in a fully developed turbulence is a key-point in finding consistent solutions to NSE

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu_0 \nabla^2 \mathbf{u}, \tag{1}$$

where ν_0 is the fluid viscosity. The *kinetic energy* $E = \frac{1}{2}\int |\mathbf{u}|^2 \, d\mathbf{r}$ is an invariant of NSE: $\frac{dE}{dt} = 0$. Taking the curl of (1) and using identity $\nabla u^2 = 2\mathbf{u}\cdot\nabla\mathbf{u} + 2\mathbf{u}\times(\nabla \times \mathbf{u})$, we obtain the *vorticity equation* ($\omega = \nabla \times \mathbf{u}$)

$$\partial_t \omega = \nabla \times (\mathbf{u} \times \omega) + \nu_0 \nabla^2 \omega. \tag{2}$$

Enstrophy $\Omega = \frac{1}{2}\int |\omega|^2 \, d\mathbf{r} = \frac{1}{2}\int |\nabla \times \mathbf{u}|^2 \, d\mathbf{r}$ is invariant of (2) for $2D$ case, i.e., $\frac{d\Omega}{dt} = 0$. The inviscid fluid dynamics is governed by the *helicity evolution equation* [7] (for viscid case see [3])

$$\partial_t (\mathbf{u} \cdot \omega) + (\mathbf{u} \cdot \nabla)(\mathbf{u} \cdot \omega) = \nabla \cdot [\omega(u^2/2 - p)]. \tag{3}$$

The *helicity* $H = \frac{1}{2}\int \mathbf{u} \cdot \omega \, d\mathbf{r}$ is an invariant of (3) for 3D-case: $\frac{dH}{dt} = 0$. Using Galerkin projection for NSE on the wave-vectors base $\mathbf{u}_{N\nu}(\mathbf{r})$, one may determine the tensor T_{NML}, satisfying the conservation laws for the energy and the enstrophy or the helicity (depending on the dimension two or three). This leads to the *main relation* (Section 3) for the tensor $Q_{NML} = T_{NML} + T_{NLM}$:

$$\frac{Q_{NML}}{Q_{LNM}} = -\frac{q^{L-M} - 1}{q^{N-M} - 1}, \qquad q = 2^{4-D}. \tag{4}$$

Section 4 presents its solutions as the product of the partial solution and $G(\eta_{LNM}, \eta_{NML}, \eta_{MLN})$, where G is any odd function invariant under transpositions of arguments, and a tensor η has zero diagonal terms and satisfies the conditions $\eta_{NML} = -\eta_{LMN}$, $\eta_{N+k,M+k,L+k} = \eta_{NML}$. One may use (4) and its solution to verify the adequacy of wavelet functions base $\mathbf{u}_{N\nu}(\mathbf{r})$ to the turbulence problem. The present criterion, naturally, cannot be panacea from all troubles while its applicability consists in that to warn the researchers at an early stage of study. The next step is to solve the system of hierarchical ODE (see, for example, [4] for the triangular system).

2. Hierarchical Bases for the Shell Models

The functions of the hierarchical base are constructed in such a way that Fourier-images of vortices of a single level occupy only a single octave in the wave-number space and regions of localization of different levels in the Fourier space do not overlap (the ensemble of vortices of the same size forms a *level*). The wave-number space may be divided by ring zones $D_N = \{\mathbf{k} \in \mathbb{R}^3 : \pi 2^N < |\mathbf{k}| < \pi 2^{N+1}\}$, $N \in \mathbb{Z}$. The vector potential of the velocity is represented as a Fourier integral $\mathbf{a}(\mathbf{r}, t) = \int \mathbf{A}(\mathbf{k}, t) \exp(2\pi i \mathbf{k} \cdot \mathbf{r})\, d\mathbf{k}$. The function $\mathbf{A}(\mathbf{k}, t)$ must be expanded inside each layer D_N in terms of the discrete totality of the base functions $(2\pi|\mathbf{k}|)^{-1}\mathbf{e}_\nu \exp(2\pi i\, \mathbf{r}_N \cdot \mathbf{k})$, where $\mathbf{e}_\nu$ ($\nu = 1, 2, 3$) form the orthonormal coordinate basis. The random points $\mathbf{r}_N$ are uniformly distributed in the region filled with a fluid with an average density ρ_N. The value ρ_N should be chosen such that the product of the volume of D_N by the volume ρ_N^{-1} of the base function in $\mathbf{r}$-space is equal to unity. As a result we obtain $\rho_N = (7/9)\,\pi 2^{3N}$. The derivation of the Fourier transform $\mathbf{u}^F(\mathbf{k}) = 2\pi i(\mathbf{k} \times \mathbf{A})$ of the velocity field $\mathbf{u}(\mathbf{r}, t)$ gives the normalized base functions, [2],

$$\mathbf{u}^F_{N\nu}(\mathbf{k}) = \begin{cases} \frac{3i}{\sqrt{7\pi}}\, 2^{-3N/2}\, \frac{\mathbf{k} \times \mathbf{e}_\nu}{|\mathbf{k}|}\, e^{-2\pi i \mathbf{k} \cdot \mathbf{r}_N}, & \mathbf{k} \in D_N, \\ 0, \quad \mathbf{k} \notin D_N. \end{cases}$$

Using inverse Fourier transform [10], they get the divergence-free vector wavelets

$$\mathbf{u}_{N\nu}(\mathbf{r}) = -(9/14)\rho_N^{1/2}\mathbf{e}_\nu \times \overrightarrow{\nabla}_{\mathbf{s}_N} F_N,$$

where $F_N = \frac{\cos|\mathbf{s}_N| - \cos 2|\mathbf{s}_N|}{s_N^2}$, $\mathbf{s}_N = \pi 2^N (\mathbf{r} - \mathbf{r}_N) = s_{Nx}\mathbf{e}_1 + s_{Ny}\mathbf{e}_2 + s_{Nz}\mathbf{e}_3$, and $\overrightarrow{\nabla}_{\mathbf{s}_N} F_N = \frac{\partial F_N}{\partial s_{Nx}}\mathbf{e}_1 + \frac{\partial F_N}{\partial s_{Ny}}\mathbf{e}_2 + \frac{\partial F_N}{\partial s_{Nz}}\mathbf{e}_3$. Here, each wavelet is considered as an axisymmetric vortex with its axis along a unit vector $\mathbf{e}_\nu$. The kinetic energy and the helicity can be represented as sums $E = \sum_N E_N$, $H = \sum_N H_N$, where

$$E_N = \frac{1}{2}\sum\nolimits_\nu \int |\mathbf{u}_{N\nu}|^2\, d\mathbf{r}, \qquad H_N = \frac{1}{2}\sum\nolimits_\nu \int \mathbf{u}^*_{N\nu} \cdot (\nabla \times \mathbf{u}_{N\nu})\, d\mathbf{r},$$

where * is a complex conjugation. The vectors $\mathbf{e}_\nu$ are orthonormal, and hence $\int \mathbf{u}^*_{N\nu}(\mathbf{r}) \cdot \mathbf{u}_{M\mu}(\mathbf{r})\, d\mathbf{r} = \delta_{NM}\delta_{\nu\mu}$.

Remark 1. ($2D$ Case)The $2D$-velocity field $\mathbf{u}\,(\mathbf{r}, t)$ can be filtrated as

$$\mathbf{u}\,(\mathbf{r}, t) = \int \mathbf{u}\,(\mathbf{r}', t) \cdot g_N\,(\mathbf{r} - \mathbf{r}')\, d\mathbf{r},$$

where the Fourier transform of g_N is equal to unity in D_N and equal to zero outside it. The velocity and vorticity fields can be written as $\mathbf{u}\,(\mathbf{r}, t) = \sum_N \mathbf{u}_N(\mathbf{r}, t)$ and $\omega(\mathbf{r}, t) = \sum_N \omega_N(\mathbf{r}, t)$. The kinetic energy and enstrophy are represented as sums $E = \sum_N E_N$, $\Omega = \sum_N \Omega_N$, where $E_N = \frac{1}{2}\int |\mathbf{u}_N|^2\, d\mathbf{r}$, $\Omega_N = \frac{1}{2}\int |\omega_N|^2\, d\mathbf{r}$, because $\int \mathbf{u}_N\mathbf{u}_M d\mathbf{r} = \delta_{NM}$, $\int \omega_N\omega_M d\mathbf{r} = \delta_{NM}$. One may decompose $\mathbf{u}_N$ into sums of functions each of them describes the velocity or vorticity oscillations of concrete scale in a concrete space region

$$\mathbf{u}\,(\mathbf{r}, t) = \sum\nolimits_N A_N(t)\mathbf{u}_N\,(\mathbf{r} - \mathbf{r}_N),$$

where $\mathbf{r}_N$ is the radius-vector of the vortex center. The coefficients $A_N(t)$ will be found from infinite system of hierarchical ODEs to be obtained using Galerkin projection method [10], [7], [4], [6] from (1) and (2), accordingly. The Fourier form of N-th level function for the vorticity is

$$\Psi_N = \begin{cases} \frac{2^{1-N}}{\sqrt{3}} \exp\left(-i\,\mathbf{k} \cdot \mathbf{r}_N\right), & \mathbf{k} \in D_N, \\ 0, \quad \mathbf{k} \notin D_N. \end{cases}$$

The orthonormal base functions for the velocity are

$$\mathbf{u}_N(\mathbf{r}) = \frac{1}{\sqrt{3\pi}} \frac{\mathbf{r} \times \mathbf{e}}{\mathbf{r} - \mathbf{r}_N} \cdot \frac{2J_0(2s_N) - J_0(s_N)}{s_N}, \qquad s_N = 2^N\pi,$$

where $\mathbf{e}$ is a right unit normal to the considered plane, J_0 are Bessel functions.

3. The Conservation Laws for the Hierarchical System of Differential Equations

The functions of a given base correspond to a hierarchy of eddies of different scale. In fact, after substitution $\mathbf{u}_{N\nu}$ into (1), multiplying by $\mathbf{u}^*_{N\nu}$ and integration the result over the entire fluid-filled volume, we obtain

$$\dot{A}_{N\nu} + \sum\nolimits_{M\mu,\, L\lambda} \Phi_{N\nu,M\mu,L\lambda} A_{M\mu} A_{L\lambda} = \nu_0 \sum\nolimits_{M\mu} \Lambda_{N\nu,M\mu} A_{M\mu}, \tag{5}$$

where

$$\Phi_{N\nu,M\mu,L\lambda}=\int \mathbf{u}^*_{N\nu}(\mathbf{r})\,(\mathbf{u}_{M\mu}(\mathbf{r})\cdot\nabla)\,\mathbf{u}_{L\lambda}(\mathbf{r})\,d\mathbf{r},\ \ \Lambda_{N\nu,M\mu} =\int \mathbf{u}^*_{N\nu}(\mathbf{r})\cdot\nabla^2\mathbf{u}_{M\mu}(\mathbf{r})\,d\mathbf{r}.$$

Note that the term of the pressure gradient vanishes due to the base functions to be satisfied the equation of continuity, i.e. $\mathbf{u}_{N\nu}$ tends to zero as $N \to \infty$. By Plancherel Theorem the base functions are orthonormal with respect to the shell index N. The kinetic energy may be expressed as

$$E = \sum\nolimits_N E_N = \frac{1}{2}\sum\nolimits_N A_N^2(t)\sum\nolimits_{\nu\mu} a_{N\mu}(t)a_{N\nu}(t) = \frac{1}{2}\sum\nolimits_N \tilde{\beta}_N(t)A_N^2(t),$$

where $\tilde{\beta}_N(t) = \sum_{\nu\mu} a_{N\nu}(t)a_{N\mu}(t)$. Define the representation

$$A_{N\nu}(t) = a_{N\nu}(t)A_N(t), \tag{6}$$

where $a_{N\nu} \geq 0$ and $\sum_\nu |a_{N\nu}|^2 = 1$ are some functions. Substituting (6) in (5) leads to the system of hierarchical quadratic ODEs

$$\dot{A_N} = -\sum\nolimits_{M,L} T_{NML}A_M(t)A_L(t) + \nu_0 K_N A_N(t) + f_N, \tag{7}$$

where $T_{NML} = \sum_{\nu\mu\lambda} a_{N\nu}a_{M\mu}a_{L\lambda}\Phi_{N\nu,M\mu,L\lambda}$, $K_N = \sum_{\nu\mu} a_{N\nu}a_{N\mu}\Lambda_{N\nu,N\mu}$. For $D = 2$ the formulas above are without indices ν, μ, λ.

The solutions of (7) must satisfy the main conservation laws: for the energy and the enstrophy (2D-case), or the helicity (3D-case).

Theorem 1. (Conservation Laws for Shell Models) *The components of the symmetric (by the last two indices) tensor* $Q_{NML} := T_{NML}+T_{NLM}$ *satisfy*

$$\begin{cases} Q_{NML} + Q_{MLN} + Q_{LNM} = 0, \\ q^N Q_{NML}+q^M Q_{MLN}+q^L Q_{LNM}=0, \end{cases} \quad q = 2^{4-D}=\begin{cases} 2, & \text{for } 3D \text{ case}, \\ 4, & \text{for } 2D \text{ case}, \end{cases} \tag{8}$$

and symmetries

$$Q_{NML} = Q_{NLM}, \tag{9}$$

$$Q_{N+k,M+k,L+k} = Q_{NML}, \quad \forall k. \tag{10}$$

The 1-st relation of (8) means the conservation kinetic energy law, the 2-nd one of (8) means the conservation laws for enstrophy ($q = 4$) or helicity ($q = 2$).

Proof. a) From definition of Q it follows (9). The tensor Q_{NML} (and T_{NML}) satisfies Navier-Stokes equations, and follow self-similarity of turbulence it should be invariant under parallel shifts (see [3]), that means (10).

b) The system (7) may be rewritten as

$$\begin{aligned}\dot{A_N} = &-\sum_{p,q>0}[(T_{N,N+p,N+p+q}+T_{N,N+p+q,N+p})\,A_{N+p}A_{N+p+q}\\ &+(T_{N,N+p,N-q}+T_{N,N-q,N+p})\,A_{N+p}A_{N-q}\\ &+(T_{N,N-p,N-p-q}+T_{N,N-p-q,N-p})\,A_{N-p}A_{N-p-q}]\\ &=-\sum_{p,q>0}(Q_{N,N+p,N+p+q}A_{N+p}A_{N+p+q}+Q_{N,N+p,N-q}A_{N+p}A_{N-q}\\ &+Q_{N,N-p,N-p-q}A_{N-p}A_{N-p-q})\,.\end{aligned}$$

Accordingly, the energy conservation law may be written as

$$\begin{aligned}-\dot{E} = -\sum_N A_N\dot{A_N} = &\sum_{N;\,p,q>0}(Q_{N,N+p,N+p+q}A_NA_{N+p}A_{N+p+q}\\ &+Q_{N,N+p,N-q}A_NA_{N+p}A_{N-q}+\;Q_{N,N-p,N-p-q}A_NA_{N-p}A_{N-p-q})=0.\end{aligned}$$

Using (10) we obtain

$$\begin{aligned}\sum_{N;\,p,q>0}Q_{N,N+p,N-q}A_NA_{N-p}A_{N-q} &= \sum_{N;\,p,q>0}Q_{N+q,N+p+q,q}A_{N+q}A_{N+p+q}A_N\,,\\ \sum_{N;\,p,q>0}Q_{N,N-p,N-p-q}A_NA_{N-p}A_{N-p-q} &= \sum_{N;\,p,q>0}Q_{N+p+q,N+q,N}A_{N+p+q}A_{N+q}A_N.\end{aligned}$$

Thus, the energy conservation law is 1-st of (8): $Q_{NML}+Q_{MLN}+Q_{LNM}=0.$

c) Let $D=2,3$ be the dimension of the turbulence problem. By Plancherel Theorem the functions $\omega_N(\mathbf{r})=\nabla\times\mathbf{u}_N(\mathbf{r})$ are orthonormal, since their Fourier images are orthogonal due to shells construction, see Remark 1. As a result, we obtain the conservation laws for *enstrophy* ($D=2$) and for *helicity* ($D=3$),

$$\begin{aligned}\frac{d\Omega}{dt} &= (\nabla\times\mathbf{u})\cdot\partial_t(\nabla\times\mathbf{u})\\ &=\sum_N(a_N^2\int\mathbf{u}_N^*(\mathbf{r})\cdot\mathbf{u}_N(\mathbf{r})\,d\mathbf{r})2^{2N}A_N\dot{A_N}=\sum_N 2^{2N}A_N\dot{A_N}=0,\\ \frac{dH}{dt} &= \frac{1}{2}\,[\,\partial_t\mathbf{u}\cdot(\nabla\times\mathbf{u})+\mathbf{u}\cdot\partial_t\,(\nabla\times\mathbf{u})]\\ &=\sum_{N,\nu\mu}(a_{N\nu}a_{N\mu}\int\mathbf{u}_{N\nu}^*(\mathbf{r})\cdot\mathbf{u}_{N\mu}(\mathbf{r})\,d\mathbf{r})2^NA_N\dot{A_N}=\sum_N 2^NA_N\dot{A_N}=0.\end{aligned}$$

Similar manipulations as in the part b) of the proof, lead to (8) for $q=2^{4-D}$, where $D=2,3$. □

Theorem 2. *a) The components of a tensor Q satisfy the relation*

$$\frac{Q_{NML}}{Q_{LNM}} = -\frac{q^{L-M}-1}{q^{N-M}-1}, \quad \text{where} \quad N \neq M \neq L. \tag{11}$$

The diagonal elements of a tensor Q are zero

$$Q_{NNL} = Q_{NMM} = Q_{NMN} = 0. \tag{12}$$

Moreover, the system (8)-(10) is equivalent to the system (9)-(12).

b) A series of solutions to (9)-(12) may be presented by

$$Q_{NML} = q^{\frac{2M-N-L}{3}} \left(1-q^{L-M}\right) G\left(\eta_{LNM}, \eta_{NML}, \eta_{MLN}\right), \tag{13}$$

where a third rank tensor η with zero diagonal terms satisfies the symmetries

$$\eta_{NML} = -\eta_{LMN}, \qquad \eta_{N+k,M+k,L+k} = \eta_{NML}, \quad \forall k, \tag{14}$$

and $G : \mathbb{R}^3 \to \mathbb{R}$ is an arbitrary function with the symmetries

$$G(\sigma(x,y,z)) = G(x,y,z), \quad G(-x,y,z) = -G(x,y,z), \ \forall\, x,y,z\in\mathbb{R}, \ \sigma\in S_3, \tag{15}$$

where S_3 is the set of transpositions of three indices.

Proof. From (8:a) the equality $Q_{MMM} = 0$ for all $M \in \mathbb{Z}$ follows. If $N = M$, $L \neq M$ then from (8) follows

$$Q_{MML} + Q_{MLM} + Q_{LMM} = 0, \quad q^M (Q_{MML} + Q_{MLM}) + q^L Q_{LMM} = 0.$$

The difference of equations above multiplying by q^N reads $(q^M - q^L)\, Q_{LMM}$ =0. Hence $Q_{LMM} = 0$ for $L \neq M$, and $Q_{MML} + Q_{MLM} = 0$. From this and symmetry property (9) it follows $Q_{MML} = Q_{MLM} = 0$. Hence (12) is proven.

The relation (11) is obtained excluding Q_{MLN} from equations (8) for $D = 2,3$ or $D = 2,4$. Then substituting in (8) with $q = 1$ we obtain

$$\frac{Q_{MLN}}{Q_{LNM}} = -\frac{q^{N-M} - q^{L-M}}{q^{N-M}-1}. \tag{16}$$

This relation is inverse to (11) and hence gives nothing new. Assume $F(i,j) = q^{-(i+j)/3}\left(1-q^j\right) f_0(i,j)$, where f_0 is given by (19) in what follows. Using Theorem 3 and inverse transformations to $i = N-M, \ j = N-L$ we get (13). ☐

So, if we neglect non-diagonal elements of Q_{NML} that have been done by [10], we receive only the trivial solution $A_N = \text{const}$.

Example 1. For $\eta_{NML} = a^{N-L} - a^{L-N}$ $(a > 0)$ relations (14) are satisfied. Function $G(x, y, z)$ may be presented as odd function or polynomial of elementary symmetric functions $S_1 = x + y + z$ and $S_3 = xyz$. Namely, $G = F(S_1, S_3)$, where F has the property $F(-a, b) = F(a, -b) = -F(a, b)$. In class of analytical functions or polynomials we obtain a unique presentation

$$G = \sum_{\beta\geq 0}[\sum_{0\leq\alpha\leq 2\beta+1} c_{\beta\alpha}(x + y + z)^{\alpha}(xyz)^{2\beta+1-\alpha}].$$

Example 2. One may verify that the tensor elements, see [10]

$$Q_{NML} = c \cdot \begin{cases} 2^{5M/2}, & M = L,\ N < L, \\ 2^{5L/2}, & N = M,\ L < N, \\ 2^{N+3M/2}, & N = L,\ M < N, \end{cases} \qquad \text{where} \quad c = \text{const.} \neq 0,$$

do not satisfy (12). Hence, solution (see [10]) does not satisfy the conservation laws.

Example 3. (Gledzer Shell Model, see [6])The governing equations are

$$(d/dt + \nu_0\, k_n^2)u_n = a_n u_{n+1}u_{n+2} + b_n u_{n-1}u_{n+1} + c_n u_{n-1}u_{n-2} + f_n.$$

Assume $k_n = q^n$ (where $q = 2^{4-D}$) and $a_n{=}k_n a_0,\ b_n{=}-k_{n-2}b_0,\ c_n{=}-k_{n-3}c_0$. For the values $a_0 = -c_0,\ b_0 = -(q + 1)c_0,\ c_0 \in \mathbb{R}$ conditions of Theorem 1 are satisfied. Namely, formula (11) is satisfied for $D = 2, 3, 4$.

4. Solution of the Functional Relation

This section proves a series of solutions to the relation (11). By (10) the components of a tensor Q do not depend on three indices but only from their differences. Thus we define the function $F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ by the formula $F(i, j) := Q_{i\,0\,j}$. Note that $F(i, 0) = F(0, j) = F(i, i) = 0$, see (12).

Theorem 3. *a) The function F satisfies the following relations for $i \neq j \neq 0$:*

$$\frac{F(i, i-j)}{F(-j, -i)} = -\frac{q^{i-j} - 1}{q^i - 1}, \qquad \frac{F(i, i-j)}{F(j, j-i)} = 1. \tag{17}$$

b) A solution of (17) may be presented as $F = q^{-(i+j)/3}\left(1 - q^j\right) f(i, j)$, where $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ satisfies the "homogeneous" system of relations

$$f(i, i-j) = f(-j, -i), \quad f(i, j) = -f(j, i), \quad f(i, 0) = f(0, j) = 0. \tag{18}$$

c) A series of solutions of (18) may be found in the form

$$f_0 : (i,j) \to G\left(g(i,j), g(-j, i-j), g(j-i, -i)\right), \tag{19}$$

where $g : \mathbb{Z}^2 \to \mathbb{R}$ is an arbitrary function with $g(i,0) = g(0,j) = 0$ and the skew symmetry: $g(i,j) = -g(j,i)$, and $G : \mathbb{R}^3 \to \mathbb{R}$ satisfies (15).

Example 4. In Theorem 3 one may take $g(i,j) := h(j-i)$, where $h : \mathbb{R} \to \mathbb{R}$ any odd function of one variable: $h(-t) = -h(t)$. For $h(t) = 2\sinh(t)$ we get $g = a^{j-i} - a^{i-j}$. For example, if $G(x,y,z) := xyz$, then (19) has a view

$$f_0 : (i,j) \to \left(1 - a^{i-j}\right)\left(1 - a^{-i}\right)\left(1 - a^{j}\right) \quad (a = \text{const.} > 0). \tag{20}$$

Proof of Theorem 3. a) For $i = N-M$, $j = N-L$ we get $i-j = L-M$. Using (10) we get

$$\begin{aligned} Q_{LNM} &= Q_{L-N,0,M-N} = F(L-N, M-N) = F(-j,-i), \\ Q_{NML} &= Q_{N-M,0,L-M} = F(N-M, L-M) = F(i, i-j), \\ Q_{MLN} &= Q_{M-L,0,N-L} = F(M-L, N-L) = F(j-i, j). \end{aligned}$$

From this it follows that (11) is transformed to (17:a). The relation (9) transforms to (17:b). Note that (16) is transformed to the relation inverse for (17:a),

$$\frac{F(j-i,j)}{F(-j,-i)} = -\frac{q^i - q^{i-j}}{q^i - 1}. \tag{21}$$

b) Following Remark 2 in Appendix one may check that the function $F_0 : (i,j) \to q^{-(i+j)/3}\left(1 - q^j\right)$ satisfies (17:a) and we get (18:a). Also F_0 satisfies satisfies (21), but (17:b) with opposite sign of RHS,

$$F_0(i, i-j) = -F_0(j, j-i). \tag{22}$$

Hence (21) and (17:b) are transformed to

$$f(j-i, j) = f(-j,-i), \qquad f(i, i-j) = -f(j, j-i). \tag{23}$$

From (23) and (18:a) we get the "skew symmetry" property (18:b),

$$f(j-i,j) \overset{(23:a)}{=} f(-j,-i) \overset{(18:a)}{=} f(i, i-j) \overset{(23:b)}{=} -f(j, j-i).$$

To show that (23) is equivalent of (17:a) one may use transformation $(i', i'-j') \to (-j,-i)$ and get $i' = -j$, $j' = i-j$, and then $(-j', -i') \to (j-i, j)$.

c) One may check (18:a) directly. Using the skew symmetry of g, we prove that f_0 satisfies (18:b).

References

[1] V.N. Desnyansky, E.A. Novikov, The evolution of turbulence spectra to the similarity regime, *Izv. Akad. Nauk SSSR Fiz. Atm. Okeana*, **10** (1974), 127-136.

[2] P.G. Frik, Hierarchical model of two-dimensional turbulence, *J. Magnetic Hydrodynamic*, **1** (1983), 60-66.

[3] U. Frisch, *Turbulence. The Legacy of A.N. Kolmogorov*, Cambridge (1995).

[4] I.M. Gaissinski, Recurrence algorithm for computation of infinitive systems of the linear differential equations with triangular matrix, *Sov. J. Comp. Math. and Math. Phys.*, **4** (1983), 1003-1007.

[5] E.B. Gledzer, System of hydrodynamic type admitting two quadratic integral of motion, *Sov. Phys. Dokl.*, **18** (1973), 216-217.

[6] Ye.B. Gledzer, A.L. Makarov, On the construction of a cascade model of two-dimensional turbulence, *Akademiia Nauk SSSR, Izvestiia, Fizika Atmosfery i Okeana*, **15**, No. 9 (1979), 899-906, In Russian.

[7] R.M. Kerr, Helicity generation in 3D Euler and turbulence, In: *Elementary Vortices and Coherent Structures, Proceedings of the IUTAM Symposium Kyoto*, October 2004, 1-8.

[8] A.N. Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large Reynolds number, *Dokl. Akad. Nauk USSR*, **30** (1941), 9-13.

[9] K. Ohkitani, M. Yamada, Temporal intermittency in the energy cascade process and local Lyapunov analysis in fully-developed model turbulence *Progr. Theor. Phys.*, **89** (1989), 329-341.

[10] V. Zimin, F. Hussain, Wavelet based model for small-scale turbulence, *Phys. Fluids*, **7** (1995), 2925-2927.

5. Appendix

Lemma 1. *Let $q > 1$ and $\epsilon \in \{1, -1\}$. Then the functional equation*

$$\frac{F(\alpha_1 i + \beta_1 j,\, \alpha_2 i + \beta_2 j)}{F(\alpha_3 i + \beta_3 j,\, \alpha_4 i + \beta_4 j)} = \frac{q^{\xi_1 i + \eta_1 j} + \epsilon\, q^{\xi_2 i + \eta_2 j}}{q^{\xi_3 i + \eta_3 j} + \epsilon\, q^{\xi_4 i + \eta_4 j}} \tag{24}$$

(with real parameters), where

$$\alpha_1\beta_2 \neq \alpha_2\beta_1, \quad \alpha_3\beta_4 \neq \alpha_4\beta_3, \quad (\alpha_1-\alpha_3)(\beta_2-\beta_4) \neq (\beta_1-\beta_3)(\alpha_2-\alpha_4), \tag{25a}$$

$$\frac{\beta_s(\xi_1-\xi_2) - \alpha_s(\eta_1-\eta_2)}{\alpha_1\beta_2 - \alpha_2\beta_1} = \frac{\beta_{2+s}(\xi_3-\xi_4) - \alpha_{2+s}(\eta_3-\eta_4)}{\alpha_3\beta_4 - \alpha_4\beta_3}, \quad s = 1, 2, \tag{25b}$$

has a solution of the view

$$F(i,j) = \left(q^{h_1 j + h_2 i} + \epsilon\right) q^{g_1 j - g_2 i} \widetilde{F}(i,j), \tag{26}$$

where $\widetilde{F}$ is a solution of "homogeneous" (24) with $q = 1$, and for $s = 1, 2$

$$h_s = \frac{\alpha_s\,(\eta_1 - \eta_2) - \beta_s\,(\xi_1 - \xi_2)}{\alpha_1\beta_2 - \alpha_2\beta_1}, \tag{27a}$$

$$g_s = \frac{(\alpha_s - \alpha_{s+2})\,(\eta_2 - \eta_4) - (\beta_s - \beta_{s+2})\,(\xi_2 - \xi_4)}{(\alpha_1 - \alpha_3)\,(\beta_2 - \beta_4) - (\beta_1 - \beta_3)\,(\alpha_2 - \alpha_4)}. \tag{27b}$$

Remark 2. Conditions (25a,b) are obviously satisfied in the case of

$$\alpha_3 = \beta_1 = \beta_4 = \xi_1 = \xi_4 = \eta_1 = \eta_3 = \eta_4 = 0,$$
$$\alpha_1 = \alpha_2 = \xi_2 = \xi_3 = 1, \quad \epsilon = \alpha_4 = \beta_2 = \beta_3 = \eta_2 = -1,$$

and we get $h_1 = -1$, $h_2 = 0$, $g_1 = 2/3$, $g_2 = 1/3$. Equation (24) has a view $\frac{F(i,i-j)}{F(-j,-i)} = \frac{1-q^{i-j}}{q^i-1}$, where $i \neq j \neq 0$, see (17:a), and its particular solution (26) has a form $F(i,j) = q^{-\frac{i+j}{3}}\left(1 - q^j\right)$ and also satisfies (17:b) and (22).

Proof of Lemma 1. We will find the solution in the form

$$F(i,j) = \left(q^{h_1 j + h_2 i} + \epsilon\right) C\,(i,j)\,. \tag{28}$$

After substituting (28) into (24) and simple manipulations we obtain

$$\begin{cases} h_1\alpha_2 - h_2\alpha_1 = \xi_1 - \xi_2\,, \\ h_1\beta_2 - h_2\beta_1 = \eta_1 - \eta_2\,, \end{cases} \qquad \begin{cases} h_1\alpha_4 - h_2\alpha_3 = \xi_3 - \xi_4\,, \\ h_1\beta_4 - h_2\beta_3 = \eta_3 - \eta_4\,, \end{cases} \tag{29}$$

and

$$\frac{C\,(\alpha_1 i + \beta_1 j,\, \alpha_2 i + \beta_2 j)}{C\,(\alpha_3 i + \beta_3 j,\, \alpha_4 i + \beta_4 j)} = q^{(\xi_2-\xi_4)i + (\eta_2-\eta_4)j}. \tag{30}$$

The first of the systems (29) leads to (27a), while the second one gives

$$h_s = \frac{\alpha_{s+2}\,(\eta_3 - \eta_4) - \beta_{s+2}\,(\xi_3 - \xi_4)}{\alpha_3\beta_4 - \alpha_4\beta_3}, \quad s = 1, 2. \tag{31}$$

In view of (25b) we may represent h_1, h_2 by any of (31) or (27a). Similarly we suppose $C(i,j) = q^{g_1 j - g_2 i}\widetilde{F}(i,j)$. Its substitution into (30) gives (27b). □