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ON A GENERALIZATION OF THE HAHN-BANACH THEOREM

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Abstract: A vectorial norm is a mapping from a linear space into a real ordered vector space with the properties of a usual norm. Here we consider the ordered vector space to be a unitary Archimedean-Riesz space (Yosida space), Dedekind complete and such that the intersection of all its hypermaximal bands is the zeroelement of the space ($\mathcal{B}$-regular Yosida space). Let E be a linear space and X, Y $\mathcal{B}$-regular Yosida spaces. In Theorem 2.2.1 we define a vectorial norm G on the linear space $\mathcal{L}(E, Y)$ of all bounded linear operators from E into Y and with range in the partially ordered linear space $\mathcal{L}(X, Y)$ of all continuous linear operators from X into Y. Next, in Theorem 3.1 we establish the following result: If t is a bounded linear operator on a linear subspace F of E into Y, then there exists a bounded linear operator T defined on E that is an extension of t and with the same vectorial norm, i.e. $G(T) = G(t)$. We finish with some consequences of this result.

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1. Introduction

1.1. Yosida Spaces. Vectorially Normed Spaces

Let E be a linear space and L a Riesz space (vector lattice). A mapping p of

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E into L is said to be a vectorial seminorm if it satisfies: $p(\lambda u) = |\lambda| p(u)$ and $p(u+v) \preceq p(u) + p(v)$, if p has the further property that $p(u) \neq 0$ if $u \neq 0$ then p is called a vectorial norm. The space E with a vectorial norm is named a vectorially normed space.

A unitary Archimedean-Riesz space is called a Yosida space. We will say that a Yosida space is $\mathcal{B}$-regular if it is Dedekind complete and the intersection of all its hypermaximal bands is the zeroelement.

In Marques [6] we have asserted that any $\mathcal{B}$-regular Yosida space is Riesz isomorphic to the space of all bounded real-valued mappings on a certain set. Let X and Y be two $\mathcal{B}$-regular Yosida spaces, that we assume to be, without loss of generality, the spaces $B(D)$ and $B(A)$ of all bounded real-valued mappings defined on the sets D and A respectively. In the $\mathcal{B}$-regular Yosida space $B(D)$ we denote by $\mathcal{B}_{B(D)}$ the set of all its hypermaximal bands and we have that $\mathcal{B}_{B(D)} = \{J_\alpha : \alpha \in D\}$, where $J_\alpha = \{x \in B(D) : x(\alpha) = 0\}$. We consider in $B(D)$ the topology induced by the family of seminorms $(\tau_\alpha)_{\alpha \in D}$ defined by

$$\tau_\alpha(x) = |x(\alpha)|, \quad \text{for all} \quad \alpha \in D.$$

What we have done in the space $B(D)$ can be done analogously in $B(A)$. We will consider in $B(A)$ the topology induced by the family of seminorms $(\varphi_\beta)_{\beta \in A}$ defined by

$$\varphi_\beta(y) = |y(\beta)|, \quad \text{for all} \quad \beta \in A.$$

Let E be a linear space, p a vectorial norm on E and with range into $B(D)$.

We also consider in E the topology induced by the family of seminorms $(\theta_\alpha)_{\alpha \in D}$ defined by

$$\theta_\alpha(u) = (p(u))(\alpha), \quad \text{for all} \quad \alpha \in D.$$

We will denote this topological space by (E, Θ).

The kernel of θ_α will be denoted by V_α, i.e. $V_\alpha = \{u \in E : (p(u)(\alpha)) = 0\}$ and we also consider for each $\alpha \in D$ the linear subspace $W_\alpha := \{u \in E : (p(u)(\beta)) = 0 \,, \beta \neq \alpha\}$.

For each finite subset S of D, we define $W(S)$ as the direct sum

$$W(S) = \bigoplus_{\alpha \in S} W_\alpha.$$

Denoting by $\mathcal{PF}(D)$ the set of all finite subsets of D, let

$$W := \bigcup_{S \in \mathcal{PF}(D)} W(S).$$

For each $\alpha \in D$, let

$$Z_\alpha := W_\alpha \oplus V_\alpha$$

we have that

$$W \subset Z_\alpha, \quad \text{for all} \quad \alpha \in D.$$

Definition 1.1.1. Let p be a vectorial norm defined on the topological linear space (E, Θ) and with range in $B(D)$. The vectorial norm p is said to be regular if $\overline{W} = E$.

Let us define the following elements of $B(D)$. For each $\alpha \in D$ the element $e_\alpha \in B(D)$ is such that

$$e_\alpha(\gamma) = \begin{cases} 1\,, & \text{if } \gamma = \alpha\,, \\ 0\,, & \text{if } \gamma \neq \alpha\,. \end{cases}$$

Now we present some concepts and results for purposes of later references.

A binary relation $\geq$ directs a set I if I is non-void and: i) if i, j and k are members of I such that $i \geq j$ and $j \geq k$, then $i \geq k$; ii) if $i \in I$, then $i \geq i$; iii) if i and j are members of I, then there is $k \in I$ such that $k \geq i$ and $k \geq j$.

A directed set is a pair $(I, \geq)$ such that $\geq$ directs I. A net is a pair $(S, \geq)$ such that S is a function and $\geq$ directs the domain of S. We will say that a net $\{S_i, i \in I, \geq\}$ is eventually in a set A if there is an element i of I such that, if $j \in I$ and $j \geq i$ then $S_j \in A$. A net $\{S_i, i \in I, \geq\}$, in a topological space, converges to an element s if and only if it is eventually in each neighborhood of s. A net $\{S_i, i \in I, \geq\}$, in a topological linear space is called a Cauchy net if, for each neighborhood $\mathcal{U}$ of zero there is some $i_0 \in I$ such that $S_i - S_j \in \mathcal{U}$ if $i \geq i_0$ and $j \geq i_0$. If a net converges to an element of the space, then it is a Cauchy net. If every Cauchy net converges to some element of the space, then we will say that the space is complete.

Let f be a function whose domain includes a set J and whose values line in a complete Hausdorff topological linear space E. Let $\mathcal{J}$ be the family of all finite subsets of J, and for $F \in \mathcal{J}$ let S_F be the sum of $f(a)$ for a in F. The family $\mathcal{J}$ is directed by $\supset$, and $\{S_F, F \in \mathcal{J}, \supset\}$ is a net in E. If this net converges to a member $u \in E$, then f is said to be summable over J, u is defined to be the sum of f over J, and we write $u = \sum\{f(a) : a \in J\} = \sum_{a \in J} f(a)$.

Lemma 1.1.1. *Let T be a linear operator on $B(D)$ into $B(A)$. Then T is continuous on $B(D)$ if and only if for each $\beta \in A$ there corresponds some positive number k and some $\alpha \in D$ such that*

$$\varphi_\beta(Tx) \leq k\, \tau_\alpha(x), \quad \textit{for all} \;\; x \in B(D).$$

Lemma 1.1.2. *Let $B(D)$ be topologized by the family of seminorms $(\tau_\alpha)_{\alpha\in D}$ and $\mathcal{D} = \{e_\alpha : \alpha \in D\}$. Then:*

1. The linear subspace spanned by $\mathcal{D}$ is dense in $B(D)$.

2. Given $x \in B(D)$ the family $\{x(\alpha)e_\alpha\}_{\alpha\in D}$ is summable with the sum x, i.e. $x = \sum_{\alpha\in D} x(\alpha)e_\alpha$.

Proposition 1.1.1. *Let P_o be a linear, positive and continuous operator on $B(D)$ into $B(A)$. Given $x \in B(D)$, the family $\{x(\alpha)P_o e_\alpha\}_{\alpha\in D}$ is summable in $B(A)$ (with respect to the topology induced by the family of seminorms $(\varphi_\beta)_{\beta\in A}$) and with the sum $P_o x$.*

Proof. A neighborhood $\mathcal{V}$ of $P_o x$ is characterized by a finite subset $\{\beta_1, \beta_2, ..., \beta_n\}$ of A and a finite number of positive real constants $\epsilon_1, \epsilon_2, ..., \epsilon_n$ such that $\mathcal{V} = \{y \in B(A) : \varphi_{\beta_i}(y - P_o x) \leq \epsilon_i, \quad i = 1, 2, ..., n\}$.

Since P_o is continuous, for each β_i we know that there exists a real number $k_i > 0$ and $\alpha_i \in D, \quad i \in \{1, 2, ..., n\}$ such that

$$\varphi_{\beta_i}(P_o x) \leq k_i\, \tau_{\alpha_i}(x).$$

Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $T \in \mathcal{PF}(D)$ such that $S \subset T$.

Let us show that $x^* := \sum_{\gamma\in T} x(\gamma)P_o e_\gamma \in \mathcal{V}$, what will prove the desired result.

Given β_i , $i \in \{1, 2, ..., n\}$ we have that

$$\varphi_{\beta_i}(x^* - P_o x) = \varphi_{\beta_i}(P_o(\textstyle\sum_{\gamma\in T} x(\gamma)e_\gamma - x)) \leq k_i\, \tau_{\alpha_i}(\textstyle\sum_{\gamma\in T} x(\gamma)e_\gamma - x)$$

$$= k_i\, |(\textstyle\sum_{\gamma\in T} x(\gamma)e_\gamma - x)(\alpha_i)| = k_i\, |x(\alpha_i) - x(\alpha_i)| = 0$$

what shows that $x^* \in \mathcal{V}$ and completes our proof. □

A proof of the following lemma can be found in Zaanen [7].

Lemma 1.1.3. *Let q be a sublinear mapping from the arbitrary real linear space E into the Dedekind complete Riesz space X. This means that $q(x + y) \preceq q(x) + q(y)$ for all $x, y \in E$ and $q(ax) = aq(x)$ for all real numbers $a \geq 0$. Furthermore, let t be an operator from the linear subspace F of E into X such that $t(x) \preceq q(x)$ for all $x \in F$.*

Then there exists a linear operator T from E into X such that $Tx = tx$ for all $x \in F$ and $Tx \preceq q(x)$ for all $x \in E$.

2. Bounded Operators

2.1. Bounded Linear Operators and Vectorial Seminorms

Let us consider the real topological linear space (E, Θ), where the vectorial norm p defined on E has range into the $\mathcal{B}$-regular Yosida space $B(D)$. Let T be a linear operator from E into the $\mathcal{B}$-regular Yosida space $B(A)$. The linear operator T is said to be bounded if there exists a linear, positive and continuous operator P from $B(D)$ into $B(A)$ such that

$$|Tu| \preceq P(p(u)), \quad \text{for all} \quad u \in E.$$

Analogously a vectorial seminorm Φ from E into $B(A)$ is said to be a bounded vectorial seminorm if there exists a linear, positive and continuous operator L from $B(D)$ into $B(A)$ such that

$$\Phi(u) \preceq L(p(u)), \quad \text{for all} \quad u \in E.$$

Let Φ be a bounded vectorial seminorm from E into $B(A)$ and let S_Φ be the set of all $\alpha \in D$ such that, there exists $u \in W_\alpha$ with $\Phi(u) \neq 0$.

Given $\alpha \in S_\Phi$, let u_α be a nonzero element of W_α. As

$$\Phi(u_\alpha) \preceq L(p(u_\alpha))$$

(where L is an operator in the refered conditions) and since

$$L(p(u_\alpha)) = p(u_\alpha)(\alpha) L e_\alpha \,,$$

it follows that

$$\frac{\Phi(u_\alpha)}{p(u_\alpha)(\alpha)} \preceq L e_\alpha \,, \quad \text{for all} \quad u_\alpha \neq 0, u_\alpha \in W_\alpha \,,$$

and the element

$$\sup_{u_\alpha \in W_\alpha \,,\; u_\alpha \neq 0} \left\{ \frac{\Phi(u_\alpha)}{p(u_\alpha)(\alpha)} \right\}$$

exists, since $B(A)$ is Dedekind complete.

Let P_Φ be the linear and positive operator on the subspace of $B(D)$ spanned by the set $\mathcal{D} = \{e_\alpha \,,\; \alpha \in D\}$ into $B(A)$, defined by

$$P_\Phi e_\alpha := \begin{cases} \sup_{u_\alpha \in W_\alpha \,,\; u_\alpha \neq 0} \left\{ \dfrac{\Phi(u_\alpha)}{p(u_\alpha)(\alpha)} \right\} \,, & \text{if } \alpha \in S_\Phi \,, \\ 0 \,, & \text{if } \alpha \notin S_\Phi \,. \end{cases}$$

Given $u \in W$ there exists a set $S \in \mathcal{PF}(D)$ such that $u = \sum_{\alpha \in S} u_\alpha \,,\; u_\alpha \in W_\alpha \,,\; \alpha \in S$ and we have

$$\Phi(u) \preceq \sum_{\alpha\in S} \Phi(u_\alpha) \preceq \sum_{\alpha\in S} P_\Phi(p(u_\alpha)) = P_\Phi(\sum_{\alpha\in S} p(u_\alpha)) = P_\Phi(p(u)),$$

i.e.

$$\Phi(u) \preceq P_\Phi(p(u)), \quad \text{for all} \quad u \in W.$$

Next we want to show that the linear operator P_Φ is continuous in $\mathcal{D} = \{e_\alpha\ ,\ \alpha \in D\}$.

By hypothesis the linear operator L is not only positive but also continuous, hence by the last Lemma 1.1.1, given $\beta \in A$ we have

$$\varphi_\beta(Lx) \leq k\,\tau_\alpha(x), \quad \text{for all} \quad x \in B(D).$$

for some positive number k and some $\alpha \in D$.

Given $x \in \mathcal{D}$ and $\beta \in A$ we have

$$\varphi_\beta(P_\Phi x) = |(P_\Phi x)(\beta)| = |P_\Phi x|(\beta) \leq (P_\Phi|x|)(\beta) \leq (L|x|)(\beta) = |(L|x|(\beta))|$$

$$= \varphi_\beta(L|x|) \leq k\,\tau_\alpha(x)$$

and it follows that P_Φ is continuous in $\mathcal{D}$.

Our objective now is to prove that P_Φ admits one and only one linear continuous extension on $B(D)$, for this we will prove the following steps:

1. P_Φ admits a linear extension on $B(D)$, that we will denote by Q_Φ,
2. Q_Φ is continuous,
3. $Q_\Phi x = \sum_{\alpha\in D} x(\alpha) P_\Phi e_\alpha$.

The operator defined on $B(D)$ into $B(A)$ by

$$q(x) := L(x^+)$$

is clearly sublinear. We have that

$$P_\Phi x \preceq q(x), \quad \text{for all} \quad x \in \mathcal{D},$$

since

$$P_\Phi x = P_\Phi(x^+ - x^-) = P_\Phi x^+ - P_\Phi x^- \preceq P_\Phi x^+ \preceq L(x^+) = q(x).$$

Applying Lemma 1.1.3, there exists a linear operator Q on $B(D)$ into $B(A)$ such that

$$P_\Phi x = Qx, \quad \forall x \in \mathcal{D}, \quad \text{and} \quad Qx \preceq q(x), \quad \text{for all} \quad x \in B(D).$$

Before we show that Q is continuous let us show that Q is a positive operator. In fact, given $x \succeq 0$ we have that

$$Q(-x) \preceq q(-x) = L((-x)^+) = L(0) = 0,$$

or equivalently

$$Q(x) \succeq 0.$$

Let us show now that Q is continuous. Given $\beta \in A$ we have that

$$\begin{aligned}\varphi_\beta(Qx) = |(Qx)(\beta)| = |(Qx)|(\beta) \leq (Q|x|)(\beta) \leq q(|x|)(\beta) = (L(|x|^+))(\beta) \\ = (L(|x|))(\beta) = |(L(|x|))(\beta)| = \varphi_\beta((L|x|)).\end{aligned}$$

Using the continuity of L we know that there exists a real positive number k and a set $\alpha \in D$ such that

$$\varphi_\beta((L|x|)) \leq k\,\tau_\alpha(|x|) = k\,\tau_\alpha(x).$$

At all we have shown that

$$\varphi_\beta(Qx) \preceq k\,\tau_\alpha(x),$$

i.e. Q is continuous.

Using Proposition 1.1.1 we are now in conditions to say that, given $x \in B(D)$ it is summable the family

$$\{x(\alpha)Qe_\alpha\}_{\alpha\in D} = \{x(\alpha)P_\Phi e_\alpha\}_{\alpha\in D}$$

with the sum Qx, i.e. $Qx = \sum_{\alpha\in D} x(\alpha)P_\Phi e_\alpha$. We will denote this operator Q by Q_Φ.

2.2. A Vectorial Norm on $\mathcal{L}(\mathcal{E}, \mathcal{B}(\mathcal{A}))$

In this subsection we will be assuming that the vectorial norm p is regular. Let T be a bounded linear operator on E into $B(A)$, i.e. there exists a linear, positive and continuous operator P on $B(D)$ into $B(A)$ such that

$$|Tu| \preceq P(p(u)), \quad \text{for all} \quad u \in E.$$

Obviously the mapping $\Phi(u) := |Tu|$ is a bounded vectorial seminorm and the results of the last subsection allows us to define a linear, positive and continuous operator P_Φ on the linear subspace of $B(D)$ spanned by $\mathcal{D}$ such that

$$\Phi(u) \preceq P_\Phi(p(u)), \quad \text{for all} \quad u \in W.$$

We also show that P_Φ admits a well defined linear, positive and continuous extension Q_Φ at the entire space $B(D)$.

The operator $Hu := \Phi(u) - Q_\Phi(p(u))$ is continuous on E and we have that

$$Hu \preceq 0, \quad \text{for all} \quad u \in W.$$

Next we want to show that

$$Hu \preceq 0, \quad \text{for all} \quad u \in E.$$

Let $u \in E$ and $\epsilon > 0$. Given $\beta \in A$ let us consider the following neighborhood of Hu

$$\mathcal{V}_{Hu} = \{a \in B(A) : \varphi_\beta(a - Hu) < \epsilon\}.$$

The continuity of H allows us to choose a neighborhood $\mathcal{V}_u$ of u such that, if $v \in \mathcal{V}_u$ then $Hv \in \mathcal{V}_{Hu}$. Since p is a regular vectorial norm we have that $\mathcal{V}_u \cap W \neq \emptyset$. Choosing $w \in \mathcal{V}_u \cap W$ we have $Hw \preceq 0$ and $\varphi_\beta(Hw - Hu) < \epsilon$. As

$$(Hu - Hw)(\beta) \leq \varphi_\beta(Hw - Hu) < \epsilon$$

it follows that

$$(Hu)(\beta) - \epsilon < (Hw)(\beta) \leq 0.$$

From this result, it follows readily that

$$Hu \preceq 0, \quad \text{for all} \quad u \in E,$$

or equivalently

$$\Phi(u) \preceq Q_\Phi(p(u)), \quad \text{for all} \quad u \in E.$$

Let us suppose now that S is a linear, positive and continuous operator on $B(D)$ into $B(A)$ such that

$$\Phi(u) \preceq S(p(u)), \quad \text{for all} \quad u \in E.$$

It is obvious that

$$Q_\Phi(p(u)) \preceq S(p(u)), \quad \text{for all} \quad u \in W$$

As the operator $V := Q_\Phi - S$ is continuous on $B(D)$, $V(x) \preceq 0$, for all $x \in \mathcal{D}^+$ and $\mathcal{D}^+$ is dense in $(B(D))^+$, by an argument similar to the one used with the operator H it follows that $V(x) \preceq 0, \quad \text{for all} \quad x \in (B(D))^+$, i.e.

$$Q_\Phi(x) \preceq S(x), \quad \text{for all} \quad x \in (B(D))^+,$$

hence

$$Q_\Phi(p(u)) \preceq S(p(u)), \quad \text{for all} \quad u \in E.$$

Theorem 2.2.1. *The mapping*

$$\begin{array}{rcl} G : \mathcal{L}(E, B(A)) & \longrightarrow & \mathcal{L}(B(D), B(A)) \\ T & \longrightarrow & G(T) = Q_\Phi \end{array}$$

defines a vectorial norm on $\mathcal{L}(E, B(A))$ of all bounded linear operators on E into $B(A)$, with range in $\mathcal{L}(B(D), B(A))$ of all continuous linear operators on $B(D)$ into $B(A)$ partially ordered by the relation:

$$M_1 \preceq M_2 \text{ if and only if } M_1(x) \preceq M_2(x), \quad \text{for all} \quad x \in (B(D))^+.$$

Proof. Let $T \in \mathcal{L}(E, B(A))$ and suppose that $G(T) = \emptyset_{\mathcal{L}(B(D),B(A))}$. From inequality

$$|Tu| \preceq (G(T))(p(u)), \quad \text{for all} \quad u \in E,$$

it follows that

$$Tu = 0, \quad \text{for all} \quad u \in E,$$

i.e.

$$T = \emptyset_{\mathcal{L}(E,B(A))}.$$

Let $T_1, T_2 \in \mathcal{L}(E, B(A))$. On the one hand we have that

$$|(T_1 + T_2)(u)| \preceq G(T_1 + T_2)(p(u)), \quad \text{for all} \quad u \in E,$$

on the other, since

$$|(T_1)(u)| \preceq G(T_1)(p(u)), \quad \text{for all} \quad u \in E,$$

$$|(T_2)(u)| \preceq G(T_2)(p(u)), \quad \text{for all} \quad u \in E,$$

we also have that

$$\begin{aligned} |(T_1 + T_2)(u)| &\preceq |T_1 u| + |T_2 u| \preceq G(T_1)(p(u)) + G(T_2)(p(u)) \\ &= (G(T_1) + G(T_2))(p(u)), \quad \text{for all} \quad u \in E, \end{aligned}$$

and so

$$G(T_1 + T_2) \preceq G(T_1) + G(T_2).$$

Let us show that

$$G(\lambda T) = |\lambda| G(T), \qquad \text{for all} \quad \lambda \in \mathbb{R}.$$

We can assume $\lambda \neq 0$, and we have that

$$|\lambda||Tu| \preceq |\lambda|(G(T))(p(u)),$$

or equivalently

$$|(\lambda T)u| \preceq (|\lambda|(G(T)))(p(u)),$$

which implies

$$G(\lambda T) \preceq |\lambda| G(T).$$

The inequality

$$|(\lambda T)u| \preceq (G(\lambda T))(p(u))$$

is equivalent to

$$|Tu| \preceq (|\lambda|^{-1}(G(\lambda T))(p(u)),$$

which implies

$$G(T) \preceq (|\lambda|^{-1}(G(\lambda T))),$$

i.e.

$$|\lambda| G(T) \preceq G(\lambda T).$$

Our proof is now complete. □

3. The Hahn-Banach Theorem

Let F be a proper linear subspace of (E, Θ). The linear subspace F is also a vectorially normed space with the restriction of p to F. The restriction of p to F will be a regular vectorial norm if $\overline{W_F} = F$, where

$$W_F := \bigcup_{S \in \mathcal{PF}(D)} W_F(S), \quad W_F(S) = \bigoplus_{\alpha \in S} W_\alpha^F,$$

and

$$W_\alpha^F = W_\alpha \cap F.$$

If t is a bounded linear operator on F into $B(A)$, defining $\Phi(u) = |tu|$ and being S_Φ^F the set of all $\alpha \in D$ such that, there exists $u \in W_\alpha^F$ with $\Phi(u) \neq 0$, by what was seen in the previous section, we have that

$$(G(t))x = \sum_{\alpha \in D} x(\alpha) P_\Phi e_\alpha$$

with

$$P_\Phi e_\alpha := \begin{cases} \sup\limits_{u_\alpha \in W_\alpha^F,\ u_\alpha \neq 0} \left\{ \dfrac{\Phi(u_\alpha)}{p(u_\alpha)(\alpha)} \right\}, & \text{if } \alpha \in S_\Phi^F, \\ 0, & \text{if } \alpha \notin S_\Phi^F. \end{cases}$$

Theorem 3.1. (Hahn-Banach Theorem) *Let F be a proper linear subspace of (E, Θ), p a regular vectorial norm on E into the $\mathcal{B}$-regular Yosida space $B(D)$ and suppose that the restriction of p to F is still regular.*

Let t be a bounded linear operator from F into the $\mathcal{B}$-regular Yosida space $B(A)$, i.e.

$$|tu| \preceq P(p(u)), \quad \textit{for all} \quad u \in F$$

(with P a positive continuous linear operator on $B(D)$ into $B(A)$).

Then there exists a bounded linear operator T on E into $B(A)$, i.e.

$$|Tu| \preceq Q(p(u)), \quad \textit{for all} \quad u \in E$$

(with Q a positive continuous linear operator on $B(D)$ into $B(A)$) such that

$$Tu = tu, \quad \textit{for all} \quad u \in F \;\; \textit{and} \quad G(T) = G(t).$$

Proof. Under the considered hypotheses it is defined a vectorial norm G on $\mathcal{L}(F, B(A))$ into $\mathcal{L}(B(D), B(A))$ and we have

$$|tu| \preceq (G(t)(p(u)), \quad \text{for all} \quad u \in F.$$

Since the mapping $r(u) := (G(t)(p(u)))$ on E into $B(A)$ is sublinear by Lemma 1.1.3 there exists an operator T on E into $B(A)$ such that

$$Tu = tu, \quad \text{for all} \quad u \in F \text{ , and} \quad Tu \preceq r(u), \quad \text{for all} \quad u \in E.$$

We also have

$$-Tu = T(-u) \preceq r(-u) = r(u),$$

hence

$$|Tu| \preceq G(t)(p(u), \quad \text{for all} \quad u \in E.$$

From this last inequality follows that T is not only a bounded operator but also that

$$G(T) \preceq G(t).$$

But since T is a bounded linear operator that is an extension of t, it follows readily that

$$G(t) \preceq G(T)\,,$$

what completes our proof. □

In what follows we will consider in this context some of the most known consequences of the Classical Hahn-Banach Theorem. We start presenting the following proposition.

Proposition 3.1. *Let us suppose that the real topological linear space (E,Θ) is complete and the vectorial norm p is defined on E and has range into the $\mathcal{B}$-regular Yosida space $B(D)$. Then:*

1. $E = V_\alpha \oplus W_\alpha$ for all $\alpha \in D$.

Given $\alpha \in D$, let P_α be the projection of $E = V_\alpha \oplus W_\alpha$ onto W_α.

2. The family $\{w_\alpha\}_{\alpha\in D}$, where for each $\alpha \in D$ the element w_α is an arbitrary fixed element in W_α, is summable.

3. Given $u \in E$ let $P_\alpha u := u_\alpha$. The family $\{u_\alpha\}_{\alpha\in D}$ is summable with the sum u, and we will write $u = \sum_{\alpha\in D} u_\alpha$.

Proof. 1. Let us show that for each $\alpha \in D$ the linear subspace $Z_\alpha = W_\alpha \oplus V_\alpha$ is closed. Given $z \in \overline{Z_\alpha}$, there exists a net $\{z_i\,,\, i \in I\}$ in Z_α such that $z_i \longrightarrow z$. For each $i \in I$ we have that $z_i = x_i + y_i$, $x_i \in W_\alpha$, $y_i \in V_\alpha$. Since $\{z_i\,,\, i \in I\}$ is a Cauchy net and for each $\beta \in D$, $\theta_\beta(x_i - x_j) \le \theta_\beta(z_i - z_j)$ and $\theta_\beta(y_i - y_j) \le \theta_\beta(z_i - z_j)$ it follows that $\{x_i\,,\, i \in I\}$ and $\{y_i\,,\, i \in I\}$ are also Cauchy nets. As W_α and V_α are closed linear subspaces of a complete linear space the nets $\{x_i\,,\, i \in I\}$ and $\{y_i\,,\, i \in I\}$ converges to some $x \in W_\alpha$ and to some $y \in V_\alpha$, respectively. The linear complete space E is also a Hausdorf space hence $z = x + y$ and Z_α is closed. Since $W \subset Z_\alpha$ and p is regular it follows that $E = Z_\alpha = V_\alpha \oplus W_\alpha$.

2. It is obvious that to say that the family $\{w_\alpha\}_{\alpha\in D}$ is summable is the same to say that the net $\{w_S = \sum_{\alpha\in S} u_\alpha\,,\, S \in \mathcal{PF}(D)\}$, with $\mathcal{PF}(D)$ directed by $\supset$, is convergent. Let us prove that $\{w_S\,,\, S \in \mathcal{PF}(D)\}$ is a Cauchy net. A neighborhood $\mathcal{V}$ of zero is of the form $\mathcal{V} = \{w \in E \,:\, \theta_\beta(w) \le \epsilon_\beta\,, \beta \in S_0\}$, where $S_0 \in \mathcal{PF}(D)$ and ϵ_β are positive real constants.

Let us show that given $S_1, S_2 \in \mathcal{PF}(D)$, if $S_0 \subset S_1$ and $S_0 \subset S_2$, then $w_{S_1} - w_{S_2} \in \mathcal{V}$.

We have that

$$w_{S_1} = \sum_{\alpha\in S_0} w_\alpha + \sum_{\alpha\in S_1\setminus S_0} w_\alpha,$$

$$w_{S_2} = \sum_{\alpha\in S_0} w_\alpha + \sum_{\alpha\in S_2\setminus S_0} w_\alpha,$$

and for each $\beta \in S_0$

$$\begin{aligned}\theta_\beta(w_{S_1} - w_{S_2}) &= \theta_\beta(\textstyle\sum_{\alpha\in S_1\setminus S_0} w_\alpha - \sum_{\alpha\in S_2\setminus S_0} w_\alpha)\\ &= p(\textstyle\sum_{\alpha\in S_1\setminus S_0} w_\alpha - \sum_{\alpha\in S_2\setminus S_0} w_\alpha)(\beta) = 0,\end{aligned}$$

what shows that $w_{S_1} - w_{S_2} \in \mathcal{V}$ and $\{w_S\ ,\ S \in \mathcal{PF}(D)\}$ is a Cauchy net, hence convergent.

3. Given $u \in E$ and $\alpha \in D$, let $P_\alpha u := u_\alpha$ be the projection of u onto W_α. According to the previous point the family $\{u_\alpha\}_{\alpha\in A}$ is summable, hence it remains to show that its sum is u. A neighborhood $\mathcal{V}$ of u is characterized by a finite subset $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ of D and positive real constants $\epsilon_1, \epsilon_2, ..., \epsilon_n$ such that $\mathcal{V} = \{v \in E\ :\ \theta_{\alpha_i}(u - v) < \epsilon_i\ ,\ i = 1, 2, ..., n\}$.

Being T a finite subset of D such that $S \subset T$, for each $\alpha_i \in S$ we have that

$$\theta_{\alpha_i}(u - \sum_{\alpha\in T} u_\alpha) = p(u - \sum_{\alpha\in T} u_\alpha)(\alpha_i) = p(u_{\alpha_i} - u_{\alpha_i})(\alpha_i) = 0 < \epsilon_i\,,$$

what shows that $\sum_{\alpha\in T} u_\alpha \in \mathcal{V}$ and completes our proof. □

Let (E, Θ) be as before, p a regular vectorial norm on E into the $\mathcal{B}$-regular Yosida space $B(D)$ let us also suppose $B(A) = B(D)$.

Let u be an arbitrary fixed element in E and P_α the projection of E onto W_α. Let A be the linear space generated by the set $\{u_\alpha : \alpha \in D\}$, where $u_\alpha = P_\alpha u$ if $P_\alpha u \neq 0$ and u_α is an arbitrary element in $W_\alpha \setminus \{0\}$ if $P_\alpha u = 0$. Being M the closure of A let us show that the restriction of p to M is regular.

Since

$$W_\alpha^M = W_\alpha \cap M = W_\alpha \cap A = W_\alpha^A\,,$$

it follows that

$$W_M = \bigcup_{S\in\mathcal{PF}(D)} W_M(S) = \bigcup_{S\in\mathcal{PF}(D)} W_A(S) = W_A = A\,.$$

Thus

$$\overline{W}_M = \overline{A} = M$$

and the restriction of p to M is regular as wished.

We point out that the elements of M are of the form $\sum_{\alpha\in D} a_\alpha u_\alpha$ where a_α are real constants. We define on M the following linear mapping

$$\begin{array}{cccc} f: & M & \longrightarrow & B(D)\\ & w = \sum_{\alpha\in D} a_\alpha u_\alpha & \longrightarrow & f(w) = \sum_{\alpha\in D} a_\alpha p(u_\alpha)\end{array}$$

We remark that the mapping f is well defined since the families $\{a_\alpha u_\alpha\}_{\alpha\in D}$ and $\{a_\alpha p(u_\alpha)\}_{\alpha\in D}$ are summable in E and $B(D)$ respectively. As

$$|f(w)| = |\sum_{\alpha\in D} a_\alpha p(u_\alpha)| = \sum_{\alpha\in D} p(a_\alpha u_\alpha) = p(w)$$

it follows that f is a bounded linear operator, more precisely we have that, given $x \in B(D)$

$$(G(f))x = \sum_{\alpha\in D} x(\alpha) P_\Phi e_\alpha = \sum_{\alpha\in D} x(\alpha) \sup_{\lambda\in \mathbb{R},\, \lambda\neq 0} \left\{ \frac{|f(\lambda u_\alpha)|}{p(\lambda u_\alpha)(\alpha)} \right\}$$

$$= \sum_{\alpha\in D} x(\alpha) \sup_{\lambda\in \mathbb{R},\, \lambda\neq 0} \left\{ \frac{p(u_\alpha)}{p(u_\alpha)(\alpha)} \right\} = \sum_{\alpha\in D} x(\alpha) e_\alpha = x\,.$$

The Hahn-Banach Theorem allows us to say that there exists a bounded linear mapping F of E into $B(D)$ such that

$$F(w) = f(w), \quad for\ all\ w \in M, \quad \text{and} \quad G(f) = G(F) = I_{B(D)},$$

with $I_{B(D)}$ the identity operator of $B(D)$.

In particular the element u arbitrary fixed in E is an element of M and so

$$F(u) = p(u).$$

Let us show now that

$$p(u) = \sup_{T\in\mathcal{L}(E,B(D))\,,\; G(T)=I_{B(D)}} |Tu|$$

On the one hand it is obvious that if $T \in \mathcal{L}(E, B(D))$ and $G(T) = I_{B(D)}$, then

$$|Tu| \preceq p(u),$$

hence

$$\sup_{T\in\mathcal{L}(E,B(D))\,,\; G(T)=I_{B(D)}} |Tu| \preceq p(u)\,.$$

On the other hand, it was already proved that there exists $F \in \mathcal{L}(E, B(D))$ such that

$$G(T) = I_{B(D)} \text{ and } F(u) = p(u),$$

so

$$\sup_{T\in\mathcal{L}(E,B(D))\,,\; G(T)=I_{B(D)}} |Tu| \succeq F(u) = p(u)\,,$$

and we have the desired result.

Now it follows easily that, if

$$Tu = 0 \text{ for all } T \in \mathcal{L}(E, B(D))\,,$$

then

$$u = 0\,.$$

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