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OSCILLATION CRITERIA FOR n-TH ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT DEPENDING ON THE UNKNOWN FUNCTION

N.T. Markova[1], P.S. Simeonov[2 §]

[1] Technical University
Sliven, BULGARIA

[2] Medical University of Sofia
2 Dunav Str., Sofia, 1000, BULGARIA
e-mail: simeonov@pharmfac.net

Abstract: In this paper differential equation of the type

$$L_n x(t) + f(t, x(\Delta(t, x(t)))) = 0 \tag{E}$$

is considered, where $n \geq 2$ and the deviating argument Δ depends on the independent variable t as well as on the unknown function x.

The oscillation of equation (E) is reduced to the oscillation of certain set of second order comparison differential equations.

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1. Introduction

We consider the n-th order differential equation

$$L_n x(t) + f(t, x(\Delta(t, x(t)))) = 0\,, \tag{1}$$

where the deviating argument Δ depends on the independent variable t as well as on the unknown function x.

Here $n \geq 2$ is an integer, $t \in J = [\alpha, +\infty) \subseteq \mathbb{R}_+ = [0, +\infty)$ and

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§Correspondence author

$$L_0x(t) = x(t)\,, \qquad L_kx(t) = r_k(t)(L_{k-1}x(t))'\,, \qquad k = 1,\dots,n\,.$$

The domain $\mathcal{D}(L_n)$ of L_n is defined to be the set of all functions $x : [t_0, +\infty) \to \mathbb{R}$ such that $L_kx(t)$, $k = 1,\dots,n$ exist and are continuous on the interval $[t_0, +\infty) \subseteq J$. By a *proper* solution of equation (1) is meant a function $x \in \mathcal{D}(L_n)$ which satisfies (1) for all sufficiently large t and $\sup\{|x(t)| : t \geq T\} > 0$ for $T \geq t_0$. A proper solution of equation (1) is called *oscillatory* if it has arbitrarily large zeros; otherwise it is called *nonoscillatory*. Equation (1) is said to be *oscillatory* if all its proper solutions are oscillatory. Equation (1) is said to be *almost oscillatory* if every proper solution $x(t)$ of equation (1) is either oscillatory or $|L_ix(t)| \to 0$ monotonically as $t \to +\infty$, $i = 0, 1, \dots, n-1$.

We suppose that

$$\int^{\infty} \frac{ds}{r_i(s)} = +\infty \qquad \text{and} \qquad f(t,x)\mathrm{sgn}x \geq a(t)|x|$$

for $i = 1,\dots,n$, $t \in J$ and $x \neq 0$, where $a \in C(J, \mathbb{R}_+)$.

In the present paper we are interested in conditions under which equation (1) is oscillatory (for n even) and almost oscillatory (for n odd).

Various oscillation criteria are obtained for higher order differential equations. For typical results on the subject we refer to the papers [2], [4], [5], [6], [7], [8], [9], [10], [12], [13], [14], [15], [17], [18], [19], [20], [22], [23].

In the papers [4], [5], [15] and [18] the oscillation of equation (1) is compared with the oscillation of one or several comparison equations (see [15]).

The main results (Theorems 1, 2, 3 and 4) generalize and improve the results of [15], where the oscillation of equation of the type (1) (with $f(t,x) = a(t)x$, $\Delta = t$) is reduced to the oscillation of certain set of second order linear differential equations.

2. Preliminary Notes

Introduce the following conditions:

H1. $r_i \in C(J, (0, +\infty))$, $i = 1,\dots,n-1$, $r_n \equiv 1$ and

$$\int^{\infty} \frac{ds}{r_i(s)} = +\infty\,, \qquad i = 1,\dots,n-1\,. \tag{2}$$

H2. $f \in C(J \times \mathbb{R}, \mathbb{R})$ and there exists a function $a \in C(J, \mathbb{R}_+)$ such that

$$f(t,x)\mathrm{sgn}x \geq a(t)|x| \qquad \text{for} \qquad t \in J, \quad x \neq 0\,. \tag{3}$$

H3. $\Delta \in C(J \times \mathbb{R}, \mathbb{R})$ and there exist a function $\tau \in C(J, \mathbb{R})$ and a $T_0 \geq \alpha$ such that

$$\lim_{t \to +\infty} \tau(t) = +\infty \qquad \text{and} \qquad \tau(t) \leq \Delta(t, x) \qquad \text{for} \qquad t \geq T_0\,, \quad x \in \mathbb{R}\,.$$

H4. $\tau \in C^1(J, \mathbb{R})$, $\tau'(t) \geq 0$ for $t \in J$ and $\tau(t)$ has an inverse function.

In order to formulate our results we use the following notations:

$$I_0 = 1\,,$$
$$I_j(t, s; r_j, \dots, r_1) = \int_s^t \frac{1}{r_j(u)} I_{j-1}(u, s; r_{j-1}, \dots, r_1) du\,, \qquad j = 1, \dots, n-1\,.$$

It is easy to verify that for $j = 1, \dots, n-1$

$$I_j(t, s; r_j, \dots, r_1) = (-1)^j I_j(s, t; r_1, \dots, r_j)\,, \tag{4}$$

$$I_j(t, s; r_j, \dots, r_1) = \int_s^t \frac{1}{r_1(u)} I_{j-1}(t, u; r_j, \dots, r_2) du \tag{5}$$

and the following inequality is valid (see [20], Lemma 1(v)):

$$\int_T^t \frac{1}{r_j(s)} I_{j-1}(u, s; r_1, \dots, r_{j-1}) ds \geq \frac{I_1(t, T; r_j)}{I_1(u, T; r_j)} I_j(u, T; r_1, \dots, r_j) \tag{6}$$

for $\alpha \leq T \leq t < u$.

We will need the following lemmas.

Lemma 1. (see [8]) *If $x \in \mathcal{D}(L_n)$, then for $t, s \in J$ and $0 \leq i < \nu \leq n$:*

(i) $$L_i x(t) = \sum_{j=i}^{\nu-1} I_{j-i}(t, s; r_{i+1}, \dots, r_j) L_j x(s) + \int_s^t I_{\nu-i-1}(t, u; r_{i+1}, \dots, r_{\nu-1}) \frac{L_\nu x(u)}{r_\nu(u)} du;$$

(ii) $$L_i x(t) = \sum_{j=i}^{\nu-1} (-1)^{j-i} I_{j-i}(s, t; r_j, \dots, r_{i+1}) L_j x(s) + (-1)^{\nu-i} \int_t^s I_{\nu-i-1}(u, t; r_{\nu-1}, \dots, r_{i+1}) \frac{L_\nu x(u)}{r_\nu(u)} du.$$

Lemma 2. *Suppose that condition* (2) *holds and the functions $L_n x$ and $x \in \mathcal{D}(L_n)$ are of constant sign and not identically zero for $t \geq t_* \geq \alpha$. Then:*

(i) *There exist a $t_k \geq t_*$ and an integer k, $0 \leq k < n$ with $n + k$ odd for $x(t) L_n x(t)$ nonpositive such that for every $t \geq t_k$*

$$x(t) L_i x(t) > 0\,, \qquad i = 0, 1, \dots, k\,, \tag{7}$$

$$(-1)^{k-i}x(t)L_ix(t) > 0\,, \qquad i = k+1,\dots,n-1\,; \tag{8}$$

(ii) *The following inequality is valid*

$$\frac{I_\nu(t,t_0;r_{k-\nu+1},\dots,r_k)}{|L_{k-\nu}x(t)|} \geq \frac{I_{\nu+1}(t,t_0;r_{k-\nu},\dots,r_k)}{|L_{k-\nu-1}x(t)|} \tag{9}$$

for $t \geq t_0 \geq t_k$ *and* $\nu = 0,1,\dots,k-1$.

This lemma generalizes the well-known lemma of Kiguradze [12] and can be proved similarly. The proof of part (ii) is given in [20]. From inequality (9) with $\nu = 1,\dots,k-1$ it follows

$$|x(t)| \geq \frac{I_k(t,t_0;r_1,\dots,r_k)}{I_1(t,t_0;r_k)}|L_{k-1}x(t)|\,, \qquad t > t_0 \geq t_k\,. \tag{10}$$

Consider the equations

$$(r(t)x'(t))' + p(t)x(\tau(t)) = 0\,, \tag{11}$$

$$(r(t)x'(t))' + q(t)x(\sigma(t)) = 0 \tag{12}$$

and the inequality

$$\{(r(t)x'(t))' + q(t)x(\sigma(t))\}\mathrm{sgn}x(t) \leq 0\,, \tag{13}$$

where

$$\begin{gathered} r,p,q,\sigma,\tau \in C(J,\mathbb{R}_+)\,, \qquad \int^\infty \frac{ds}{r(s)} = +\infty\,, \qquad \lim_{t\to+\infty}\tau(t) = +\infty \\ \text{and} \qquad p(t) \leq q(t)\,, \qquad \tau(t) \leq \sigma(t) \qquad \text{for} \qquad t \in J\,. \end{gathered} \tag{14}$$

Lemma 3. *Let conditions* (14) *hold. Then:*

(i) *Equation* (11) *has a nonoscillatory solution if inequality* (13) *has a nonoscillatory solution;*

(ii) *Equation* (12) *is oscillatory if equation* (11) *is oscillatory.*

We omit here the proof of Lemma 3, because it is the same as the proofs of analogous assertions (see [1], [3], [25]).

Remark 1. Various oscillation criteria for particular cases of equation (11) are given in [6], [11], [13], [16], [21], [22], [24], [26].

We note that equation (11) is oscillatory, if

$$\int^\infty p(t)dt < +\infty \tag{15}$$

and one of the following inequalities holds:

$$\lim_{t\to+\infty} \sup \int_{\alpha}^{\tau(t)} \frac{ds}{r(s)} \cdot \int_{t}^{\infty} p(s)ds > 1\,, \tag{16}$$

$$\lim_{t\to+\infty} \inf \int_{\alpha}^{\tau(t)} \frac{ds}{r(s)} \cdot \int_{t}^{\infty} p(s)ds > \frac{1}{4}\,. \tag{17}$$

These criteria generalize the criteria given in [11], [13], [21], [24] and are proved by Džurina [6] and Ohridska [22] in the case when $\tau(t)$ is differentiable and $\tau'(t) > 0$ in J. We note that the oscillation criteria (16) and (17) hold if $\tau(t)$ is continuous and nondecreasing in J (see [25], Theorems 9 and 10).

3. Main Result

Consider the following second order comparison equation

$$(r_k(t)x'(t))' + A_k(t)x(\tau(t)) = 0\,, \tag{18; k}$$

where

$$A_k(t) = \frac{1}{r_{k+1}(t)} \int_t^{\infty} I_{k-1}(\tau(u), \tau(t); r_1, \ldots, r_{k-1}) \times I_{n-k-2}(u, t; r_{n-1}, \ldots, r_{k+2})a(u)du \tag{19}$$

for $k = 1, 2, \ldots, n-3$ and

$$A_{n-1}(t) = \frac{\tau'(t)}{r_{n-2}(\tau(t))} \int_t^{\infty} I_{n-3}(\tau(u), \tau(t); r_1, \ldots, r_{n-3})a(u)du\,. \tag{20}$$

Theorem 1. *Assume that:*

1. *Conditions* H1-H4 *hold and* $n \geq 4$ *is even.*
2. *Equations* (18;k), $k \in \{1, 3, \ldots, n-1\}$ *are oscillatory.*

Then equation (1) *is oscillatory*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality we suppose that $x(t) > 0$, $t \geq t_0 \geq \alpha$. Then there exists a $t_1 \geq t_0$ such that $x(\Delta(t, x(t))) > 0$, $x(\tau(t)) > 0$ for $t \geq t_1$.

By Lemma 2(i) there exist a $t_k \geq t_1$ and an odd integer $k \in \{1, 3, \ldots, n-1\}$ such that (7) and (8) hold for $t \geq t_k$.

Suppose $1 \leq k \leq n-3$. Then from Lemma 1(ii) (with $i = k+1$ and $\nu = n$) it follows that

$$L_{k+1}x(t) = \sum_{j=k+1}^{n-1} (-1)^{j-k-1} I_{j-k-1}(s,t;r_j,\dots,r_{k+2}) L_j x(s) \\ + (-1)^{n-k-1} \int_t^s I_{n-k-2}(u,t;r_{n-1},\dots,r_{k+2}) L_n x(u) du \,.$$

Using (1), (8) and (3) and letting $s \to +\infty$ we have

$$-L_{k+1}x(t) \geq \int_t^\infty I_{n-k-2}(u,t;r_{n-1},\dots,r_{k+2}) a(u) x(\Delta(u,x(u))) du \,, \\ t \geq t_k \,. \quad (21)$$

Since $k \geq 1$, then $x(t)$ is increasing and $x(\Delta(u,x(u))) \geq x(\tau(u))$ for $t \geq t_2$, where $t_2 \geq t_k$ is such that $\tau(t) \geq t_k$, $t \geq t_2$. In this case

$$-L_{k+1}x(t) \geq \int_t^\infty I_{n-k-2}(u,t;r_{n-1},\dots,r_{k+2}) a(u) x(\tau(u)) du \,, \\ t \geq t_2 \,. \quad (22)$$

If $k \geq 3$, then using Lemma 1(i) (with $i = 0$, $\nu = k-1$, $s = t_2$ and $t \geq t_2$) we get

$$x(t) = \sum_{j=0}^{k-2} I_j(t,t_2;r_1,\dots,r_j) L_j x(t_2) \\ + \int_{t_2}^t I_{k-2}(t,u;r_1,\dots,r_{k-2}) \frac{L_{k-1}x(u)}{r_{k-1}(u)} du \,, \qquad t \geq t_2 \,.$$

Then using (7) we obtain

$$x(\tau(t)) \geq \int_{t_2}^{\tau(t)} I_{k-2}(\tau(t),u;r_1,\dots,r_{k-2}) \frac{L_{k-1}x(u)}{r_{k-1}(u)} du \,, \qquad t \geq t_2 \,. \quad (23)$$

Let $T \geq t_2$ be such that $\tau(t) \geq t_2$ for $t \geq T$. Then combining (21), (23) and taking in mind that $L_{k-1}x(u)$ is increasing we have for $t \geq T$

$$-L_{k+1}x(t) \\ \geq \int_t^\infty I_{n-k-2}(u,t;r_{n-1},\dots,r_{k+2}) a(u) \\ \times \int_{t_2}^{\tau(u)} I_{k-2}(\tau(u),s;r_1,\dots,r_{k-2}) \frac{L_{k-1}x(s)}{r_{k-1}(s)} ds du$$

$$\begin{aligned}
&\geq \int_t^\infty I_{n-k-2}(u,t;r_{n-1},\dots,r_{k+2})a(u) \\
&\quad\times \int_{\tau(t)}^{\tau(u)} I_{k-2}(\tau(u),s;r_1,\dots,r_{k-2})\frac{L_{k-1}x(s)}{r_{k-1}(s)}dsdu \\
&\geq L_{k-1}x(\tau(t))\int_t^\infty I_{n-k-2}(u,t;r_{n-1},\dots,r_{k+2})a(u) \\
&\qquad\times \int_{\tau(t)}^{\tau(u)} \frac{1}{r_{k-1}(s)} I_{k-2}(\tau(u),s;r_1,\dots,r_{k-2})dsdu\,.
\end{aligned}$$

The above inequality and equalities (5) and (19) imply

$$-\frac{L_{k+1}x(t)}{r_{k+1}(t)} \geq A_{k+1}(t)L_{k-1}x(\tau(t)) \tag{24}$$

for $3 \leq k \leq n-1$ and $t \geq T$. Inequality (24) with $k=1$ follows immediately from (22).

The function $y(t) = L_{k-1}x(t)$ is positive for $t \geq T$ and from (24) we conclude that

$$(r_k(t)y'(t))' + A_k(t)y(\tau(t)) \leq 0\,, \qquad t \geq T\,.$$

Now from Lemma 3(i) it follows that for each $k \in \{1,3,\dots,n-3\}$ equation (18;k) has a nonoscillatory solution which leads to a contradiction.

Suppose $k = n-1$. Integrating (1) we have

$$L_{n-1}x(t) \geq \int_t^\infty f(u,x(\Delta(u,x(u))))du \geq \int_t^\infty a(u)x(\tau(u))du\,, \qquad t \geq T\,. \tag{25}$$

From Lemma 1(i) (with $i=0$, $\nu = n-2$, $s = t_2$ and $t \geq t_2$) we have

$$\begin{aligned}
x(t) &\geq \sum_{j=1}^{n-3} I_j(t,t_2;r_1,\dots,r_j)L_jx(t_2) + x(t_2) \\
&\quad + \int_{t_2}^t I_{n-3}(t,u;r_1,\dots,r_{n-3})\frac{L_{n-2}x(u)}{r_{n-2}(u)}du \\
&\geq \int_{t_2}^t I_{n-3}(t,u;r_1,\dots,r_{n-3})\frac{L_{n-2}x(u)}{r_{n-2}(u)}du\,, \qquad t \geq t_2\,.
\end{aligned} \tag{26}$$

Combining (25) and (26) and changing the order of integration we get

$$L_{n-1}x(t) \geq \int_t^\infty a(u) \int_{t_2}^{\tau(u)} I_{n-3}(\tau(u), s; r_1, \dots, r_{n-3}) \frac{L_{n-2}x(s)}{r_{n-2}(s)} ds du$$
$$\geq \int_{\tau(t)}^\infty \Phi(s) L_{n-2}x(s) ds, \qquad t \geq T,$$

where

$$\Phi(s) = \frac{1}{r_{n-2}(s)} \int_{\psi(s)}^\infty I_{n-3}(\tau(u), s; r_1, \dots, r_{n-3}) a(u) du$$

and $\psi(s)$ is the inverse function of $\tau(s)$.

Hence the positive function $w(t) = L_{n-2}x(t)$ satisfies the inequality

$$w'(t) \geq \frac{1}{r_{n-1}(t)} \int_{\tau(t)}^\infty \Phi(s) w(s) ds, \qquad t \geq T. \tag{27}$$

Integrating (27) from T to t we have

$$w(t) \geq w(T) + \int_T^t \frac{1}{r_{n-1}(u)} \int_{\tau(u)}^\infty \Phi(s) w(s) ds du, \qquad t \geq T. \tag{28}$$

Denote the right-hand side of (28) by $y(t)$. By differentiation

$$(r_{n-1}(t) y'(t))' + \tau'(t) \Phi(\tau(t)) w(\tau(t)) = 0.$$

Taking into account (28), (20) and the identity $\psi(\tau(t)) = t$ we obtain

$$(r_{n-1}(t) y'(t))' + A_{n-1}(t) y(\tau(t)) \leq 0.$$

By Lemma 3(i) equation (18;$n-1$) has a nonoscillatory solution which leads to a contradiction. □

Theorem 2. *Assume that:*

1. *Conditions* H1-H4 *hold and* $n \geq 3$ *is odd.*
2. *Equations* (18;k), $k \in \{2, 4, \dots, n-1\}$ *are oscillatory and*

$$\int^\infty I_{n-1}(s, T; r_{n-1}, \dots, r_1) a(s) ds = +\infty \tag{29}$$

for each $T \geq \alpha$.

Then equation (1) *is almost oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality we suppose that $x(t) > 0$, $t \geq t_0 \geq \alpha$. Then there exists a $t_1 \geq t_0$ such that $x(\Delta(t, x(t))) > 0$, $x(\tau(t)) > 0$ for $t \geq t_1$.

By Lemma 2(i) there exist a $t_k \geq t_1$ and an even integer $k \in \{0, 2, 4, \ldots, n-1\}$ such that (7) and (8) hold for $t \geq t_k$.

For $k \geq 2$ the proof of Theorem 2 is the same as the proof of Theorem 1.

Suppose $k = 0$. Then applying Lemma 1(ii) (with $i = 0$, $\nu = n$) we get

$$x(t) \geq \int_t^\infty I_{n-1}(u, t; r_{n-1}, \ldots, r_1)a(u)x(\Delta(u, x(u)))du\,, \qquad t \geq t_k\,. \tag{30}$$

Since $x(t)$ is decreasing and positive for $t \geq t_k$ there exists $\lim_{t\to+\infty} x(t) = c \geq 0$. If $c > 0$, then there exists $T \geq t_k$ such that $2c \geq x(\Delta(u, x(u))) \geq c$, $u \geq T$. Then (30) implies the inequality

$$2c \geq \int_T^\infty I_{n-1}(u, T; r_{n-1}, \ldots, r_1)a(u)du.c\,,$$

which contradicts (29). Hence $c = 0$ and taking in mind (8) with $k = 0$ we conclude that $\lim_{t\to+\infty} |L_k x(t)| = 0$ monotonically for $k = 0, 1, \ldots, n-1$. □

Consider the n-th order differential equation

$$x^{(n)}(t) + f(t, x(\Delta(t, x(t)))) = 0 \tag{31}$$

and the second order comparison equations

$$x''(t) + a_k(t)x(\tau(t)) = 0\,, \tag{32;k}$$

where

$$a_k(t) = \int_t^\infty \frac{(u-t)^{n-k-2}}{(n-k-2)!}\frac{(\tau(u)-\tau(t))^{k-1}}{(k-1)!}a(u)du \tag{33}$$

for $k = 1, 2, \ldots, n-3$ and

$$a_{n-1}(t) = \tau'(t)\int_t^\infty \frac{(\tau(u)-\tau(t))^{n-3}}{(n-3)!}a(u)du\,. \tag{34}$$

As a consequence of Theorems 1 and 2 we obtain the following corollaries.

Crollary 1. *Let conditions* H1, H3 *and* H4 *hold and $n \geq 4$ be even. Then equation* (31) *is oscillatory, if equations* (32;k) $k \in \{1, 3, \ldots, n-1\}$ *are oscillatory.*

Corllary 2. *Let* H2, H3 *and* H4 *hold and $n \geq 3$ be odd. Then equation* (31) *is almost oscillatory, if equations* (32;k) $k \in \{2, 4, \ldots, n-1\}$ *are oscillatory and*

$$\int^\infty s^{n-1}a(s)ds = +\infty\,. \tag{35}$$

Introduce the following condition:

H5. $\Delta \in C(J \times \mathbb{R}, \mathbb{R})$ and there exists a $T_0 \geq \alpha$ such that

$$t \leq \Delta(t,x) \qquad \text{for} \qquad T \geq T_0 \qquad \text{and} \qquad x \in \mathbb{R}\,.$$

If equation (31) is of advanced type we obtain the following results.

Corollary 3. *Let conditions* H2 *and* H5 *hold and $n \geq 4$ be even. Then equation* (31) *is oscillatory if the equation*

$$x''(t) + A(t)x(t) = 0 \tag{36}$$

is oscillatory, where

$$A(t) = \int_t^\infty \frac{(u-t)^{n-3}}{(n-3)!} a(u)du\,. \tag{37}$$

Corollary 4. *Let conditions* H2 *and* H5 *hold and $n \geq 3$ be odd. Then equation* (31) *is almost oscillatory if equation* (36) *is oscillatory and* (35) *holds.*

Proof of Corollary 3 and Corollary 4. We have $\tau(t) = t$ and

$$a_k(t) = \int_t^\infty \frac{(u-t)^{n-3}a(u)}{(n-k-2)!(k-1)!}du$$
$$\geq \int_t^\infty \frac{(u-t)^{n-3}}{(n-4)!}a(u)du \geq a_{n-1}(t) = a_1(t) = A(t)$$

for $k \in \{2,3,\ldots,n-3\}$. Hence by Lemma 3(ii) equations (32;k) are oscillatory for each $k \in \{1,2,3,\ldots,n-3\}$ and $k = n-1$. □

Remark 2. Equation (36) coincides with the comparison equation used in [17] for the equation $x^{(n)}(t) + a(t)x(t) = 0$.

In the following two theorems we will use the following set of second order comparison equations

$$(r_k(t)x'(t))' + B_k(t)x(\tau(t)) = 0\,, \tag{38;k}$$

where

$$B_k(t) = \frac{I_{n-k}(t,T;r_{n-1},\ldots,r_k)}{I_1(t,T;r_k)}.\frac{I_k(\tau(t),T;r_1,\ldots,r_k)}{I_1(\tau(t),T;r_k)}a(t) \tag{39}$$

for $k \in \{1,2,3,\ldots,n-3\}$ and

$$B_{n-1}(t) = \frac{I_{n-1}(\tau(t),T;r_1,\ldots,r_{n-1})}{I_1(\tau(t),T;r_{n-1})}a(t)\,. \tag{40}$$

Theorem 3. *Assume that:*

1. *Conditions* H1-H3 *hold and* $n \geq 2$ *is even.*
2. *Equations* (38;k), $k \in \{1, 3, \dots, n-1\}$ *are oscillatory.*

Then equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality we suppose that $x(t) > 0$, $t \geq t_0 \geq \alpha$. Then there exists a $t_1 \geq t_0$ such that $x(\Delta(t, x(t))) > 0$, $x(\tau(t)) > 0$ for $t \geq t_1$.

By Lemma 2(i) there exist a $t_k \geq t_1$ and an odd integer $k \in \{1, 3, \dots, n-1\}$ such that (7) and (8) hold for $t \geq t_k$. Applying Lemma 1(ii) (with $i = k$ and $\nu = n$) we have

$$L_k x(t) = \sum_{j=k}^{n-1} (-1)^{j-k} I_{j-k}(s, t; r_j, \dots, r_{k+1}) L_j x(s) + (-1)^{n-k} \int_t^s I_{n-k-1}(u, t; r_{n-1}, \dots, r_{k+1}) L_n x(u) du\,, \qquad t \leq s\,.$$

Using (1), (8) and (3) and letting $s \to +\infty$ we have

$$L_k x(t) \geq \int_t^\infty I_{n-k-1}(u, t; r_{n-1}, \dots, r_{k+1}) a(u) x(\Delta(u, x(u))) du\,, \qquad t \geq t_k\,.$$

Since $k \geq 1$, then $x(t)$ is increasing for $t \geq t_k$ and $x(\Delta(u, x(u))) \geq x(\tau(u))$ for $t \geq T$, where $T \geq t_k$ is such that $\tau(t) \geq t_k$, $t \geq T$. In this case

$$L_k x(t) \geq \int_t^\infty I_{n-k-1}(u, t; r_{n-1}, \dots, r_{k+1}) a(u) x(\tau(u)) du\,, \qquad t \geq T\,. \tag{41}$$

Integrating (41) from T to t we get

$$\begin{aligned} L_{k-1} x(t) &= L_{k-1} x(T) + \int_T^t \frac{1}{r_k(s)} \int_s^\infty I_{n-k-1}(u, s; r_{n-1}, \dots, r_{k+1}) a(u) x(\tau(u)) du ds \\ &= L_{k-1} x(T) + \int_T^t \left[\int_T^u \frac{1}{r_k(s)} I_{n-k-1}(u, s; r_{n-1}, \dots, r_{k+1}) ds \right] a(u) x(\tau(u)) du \\ &\quad + \int_t^\infty \left[\int_T^t \frac{1}{r_k(s)} I_{n-k-1}(u, s; r_{n-1}, \dots, r_{k+1}) ds \right] a(u) x(\tau(u)) du\,. \end{aligned} \tag{42}$$

From (5) and (6) it follows

$$\int_T^u \frac{1}{r_k(s)} I_{n-k-1}(u, s; r_{n-1}, \dots, r_{k+1}) ds = I_{n-k}(u, T; r_{n-1}, \dots, r_k)$$

for $t \geq u \geq T$ and

$$\int_T^t \frac{1}{r_k(s)} I_{n-k-1}(u,s;r_{n-1},\dots,r_{k+1})ds \geq \frac{I_1(t,T;r_k)}{I_1(u,T;r_k)} I_{n-k}(u,T;r_{n-1},\dots,r_k)$$

for $T \leq t < u$. This together with (42) and (7) implies

$$L_{k-1}x(t) \geq \int_T^t I_{n-k}(u,T;r_{n-1},\dots,r_k)a(u)x(\tau(u))du + I_1(t,T;r_k)\int_t^\infty \frac{I_{n-k}(u,T;r_{n-1},\dots,r_k)}{I_1(u,T;r_k)} a(u)x(\tau(u))du\,, \qquad t \geq T\,. \tag{43}$$

Denote the right-hand side of (43) by $z(t)$. It is easy to verify that $z(t)$ is positive and satisfies the equation

$$(r_k(t)z'(t))' + \frac{I_{n-k}(t,T;r_{n-1},\dots,r_k)}{I_1(t,T;r_k)} a(t)x(\tau(t)) = 0\,. \tag{44}$$

But from (10)

$$x(\tau(t)) \geq \frac{I_k(\tau(t),T;r_1,\dots,r_k)}{I_1(\tau(t),T;r_k)} L_{k-1}x(\tau(t)) \tag{45}$$

for $t \geq t_2$, where $t_2 \geq T$ is such that $\tau(t) > T$ for $t \geq t_2$. Taking in mind (44), (45), (43) and (39), (40) we obtain

$$(r_k(t)z'(t))' + B_k(t)z(\tau(t)) \leq 0\,, \qquad t \geq t_2\,.$$

By Lemma 3(i) equation (38;k) has an eventually positive solution. This contradicts condition 2 of Theorem 3. □

Theorem 4. *Assume that:*

1. *Conditions* H1-H3 *hold and* $n \geq 3$ *is odd.*
2. *Equations* (38;k), $k \in \{2,4,\dots,n-1\}$ *are oscillatory and* (29) *holds.*

Then equation (1) *is almost oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1) which, without loss of generality, is eventually positive. Then proceeding as in the proof of Theorem 2 we conclude that there exist a $t_k \geq \alpha$ and an even integer $k \in \{0,2,\dots,n-1\}$ such that (7) and (8) hold for $t \geq t_k$ and $x(\Delta(t,x(t))) > 0$, $x(\tau(t)) > 0$, $t \geq t_k$. Further on the proof is the same as the proof of Theorem 3 if $k \in \{2,4,\dots,n-1\}$ and as the proof of Theorem 2, if $k = 0$. □

Consider equation (31) and the second order comparison equations

$$x''(t) + b_k(t)x(\tau(t)) = 0\,, \tag{46;k}$$

where

$$b_k(t) = \frac{(t-T)^{n-k-1}}{(n-k)!} \cdot \frac{(\tau(t)-T)^{k-1}}{k!} . a(t) \tag{47}$$

for $k = 1, 2, \dots, n-1$.

As a consequence of Theorems 3 and 4 we obtain the following corollaries.

Corollary 5. *Let conditions* H2 *and* H3 *hold and* $n \geq 2$ *be even. Then equation* (31) *is oscillatory if equations* (46;k), $k \in \{1, 3, \dots, n-1\}$ *are oscillatory.*

Corollary 6. *Let conditions* H2 *and* H3 *hold and* $n \geq 3$ *be odd. Then equation* (31) *is almost oscillatory if equations* (46;k), $k \in \{2, 4, \dots, n-1\}$ *are oscillatory and* (35) *holds.*

Corollary 7. *Let conditions* H2 *and* H5 *hold and* $n \geq 2$ *be even. Then equation* (31) *is oscillatory if the equation*

$$x''(t) + b(t)x(t) = 0 \tag{48}$$

is oscillatory, where

$$b(t) = \frac{(t-T)^{n-2}}{(n-1)!} a(t)\,. \tag{49}$$

Corollary 8. *Let conditions* H2 *and* H5 *hold and* $n \geq 3$ *be odd. Then equation* (31) *is almost oscillatory if equation* (48) *is oscillatory and* (35) *holds.*

Proof of Corollary 7 and Corollary 8. We have $\tau(t) = t$ and

$$b_k(t) = \frac{(t-T)^{n-2}}{(n-k)!k!} a(t) \geq \frac{(t-T)^{n-2}}{(n-1)!} a(t) = b_{n-1}(t) = b(t)$$

for $k = 1, 2, \dots, n-3$.

Hence by Lemma 3(ii) equations (46;k) are oscillatory for each $k \in \{1, 2, \dots, n-3\}$ and $k = n-1$. □

Consider the Euler equation

$$(t^{m-\beta} x^{(m)})^{(m)} + ct^{-\beta-m} x = 0\,, \qquad t \geq 1\,, \tag{50}$$

where β and $c > 0$ are real constants and $\beta \geq m-1$.

Here $n = 2m$ is even, $r_0 = r_1 = \cdots = r_{m-1} = r_{m+1} = \cdots = r_n = 1$, $r_m(t) = t^{m-\beta}$, $a(t) = ct^{-\beta-m}$.

Applying Theorem 1 and oscillation criterion (17) one can conclude (see [15]) that equation (50) is oscillatory provided c is so large that:

(i) when $m = 2$, $c > k_1 = \frac{1}{4}\beta(\beta+1)$;

(ii) when $m > 2$ is even,

$$c > k_2 = \frac{1}{4} \max \Big\{ (m-1)!\beta(\beta+1)\dots(\beta+m-1), \\ \frac{(m-1)!(m-2)!}{(2m-3)!}(\beta-m+2)(\beta-m+3)\dots(\beta+m-1) \Big\};$$

(iii) when $m > 2$ is odd,

$$c > k_3 = \frac{1}{4} \max \Big\{ (m-1)!\beta(\beta+1)\dots(\beta+m-1), \\ \frac{(m-1)!(m-2)!}{(2m-3)!}(\beta-m+1)^2(\beta-m+2)(\beta-m+3)\dots(\beta+m-1) \Big\}.$$

Applying Theorem 3 to equation (50) we obtain

Corollary 9. *Assume that:*

(i) *if $m = 2$, then the equation*

$$z''(t) + \frac{a(t)}{t-T} \int_T^t s^{\beta-2}(t-s)(s-T)ds.z(t) = 0 \tag{51}$$

is oscillatory;

(ii) *if $m > 2$ is even, then the equation*

$$z''(t) + \frac{a(t)}{t-T} \int_T^t s^{\beta-m} \frac{(t-s)^{m-1}}{(m-1)!} \frac{(s-T)^{m-1}}{(m-1)!} ds.z(t) = 0 \tag{52}$$

is oscillatory;

(iii) *if $m > 2$ is odd, then equation* (52) *and the equation*

$$(t^{m-\beta} z')' + A_m(t) z(t) = 0 \tag{53}$$

are oscillatory, where

$$A_m(t) = a(t) \left[\frac{\int_T^t s^{\beta-m} \frac{(t-s)^{m-1}}{(m-1)!} ds}{\int_T^t s^{\beta-m} ds} \right]^2. \tag{54}$$

Then equation (50) *is oscillatory.*

From Corollary 9 and oscillation criterion (17) we conclude that equation (50) is oscillatory provided c is so large that:

(j) when $m = 2$, $c > d_1 = \frac{1}{4}\beta(\beta+1)$;

(jj) when $m > 2$ is even,

$$c > d_2 = \frac{1}{4}(m-1)!\beta(\beta+1)\dots(\beta+m-1)\,;$$

(jjj) when $m > 2$ is odd,

$$c > d_3 = \frac{1}{4}\max\left\{(m-1)!\beta(\beta+1)\dots(\beta+m-1), [\beta(\beta-1)\dots(\beta-m+1)]^2\right\}.$$

Remark 3. If $n = 2$ ($m = 1$) then by Theorem 3 equation (50) is oscillatory if the comparison equation (53) is oscillatory. But equation (53) with $m = 1$ coincides with equation (50) which has the form

$$(t^{1-\beta}x')' + ct^{-\beta-1}x = 0\,. \tag{55}$$

Applying the oscillation criterion (17)we obtain that equation (55) is oscillatory if $c > \frac{\beta^2}{4}$. This condition is also necessary for equation (55) to be oscillatory.

Comparing the oscillation criteria (i)-(iii) and (j)-(jjj) we conclude that equation (50) is oscillatory provided c is so large that:

1. when $m = 2$, $c > k_1 = d_1 = \frac{1}{4}\beta(\beta+1)$;
2. when $m > 2$ is even, $c > d_2$;
3. when $m > 2$ is odd, $c > k_3$.

References

[1] F.V. Atkinson, On second-order differential inequalities, *Proc. Roy. Soc. Edinburgh*, **72** (1974), 109-127.

[2] D.D. Bainov, N.T. Markova, P.S. Simeonov, Oscillation of second order differential equations with retarded argument depending on the unknown function, *Communications in Applied Analysis*, **7**, No. 4 (2003), 593-604.

[3] J.S. Bradley, Oscillation theorems for second order delay equations, *Journal of Differential Equations*, **8** (1970), 397-403.

[4] R.S. Dahiya, O. Akinyele, Oscillation theorems of n-th order functional differential equations with forcing terms, *Jour. of Math. Anal. Appl.*, **109**, No. 2 (1985), 325-332.

[5] J. Džurina, Asymptotic analysis of n-th order differential equations, *Math. Slovaca*, **47**, No. 3 (1997), 283-289.

[6] J. Džurina, Oscillation of second order delay differential equations, *Archivum Mathematicum (Brno)*, Tomus **33**, (1997), 309-314.

[7] S.R. Grace, Oscillation theorems for n-th order differential equations with deviating arguments, *J. Math. Anal. Appl.*, **101** (1984), 268-296.

[8] S.R. Grace, Oscillatory properties of functional differential equations, *J. Math. Anal. Appl.*, **160** (1991), 60-78.

[9] S.R. Grace, B.S. Lalli, Oscillation theorems for n-th order delay differential equations, *J. Math. Anal. Appl.*, **91** (1983), 352-366.

[10] S.R. Grace, B.S. Lalli, Asymptotic and oscillatory behavior of n-th order forced functional differential equations, *J. Math. Anal. Appl.*, **140** (1989), 10-25.

[11] E. Hille, Non-oscillation theorems, *Trans. Amer. Math. Soc.*, **64** (1948), 234-252.

[12] I.T. Kiguradze, On the oscillation of solutions of the equation $\frac{d^m u}{dt^m} + a(t)|u|^n \text{sign} u = 0$, *Math. Sb.*, **65**, No. 2 (1964), 172-187, In Russian.

[13] I.T. Kiguradze, T.A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluwer Academic Publ., London (1993).

[14] Y. Kitamura, Oscillation of functional differential equations with general deviating argument, *Hiroshima Math. J.*, **15** (1985), 445-491.

[15] T. Kusano, M. Naito, Oscillation criteria for general linear ordinary differential equations, *Pacific J. Math.*, **92**, No. 2 (1981), 345-355.

[16] G.S. Ladde, V. Lakshmikantham. B.G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Pure and Applied Mathematics, **110**, Marcel Dekker (1987).

[17] D.L. Lovelady, Oscillation and even order linear differential equations, *Rocky Mountain J. Math.*, **6** (1976), 299-304.

[18] N.T. Markova, P.S. Simeonov, Oscillation theorems for n-th order nonlinear differential equations with forcing terms and deviating arguments depending on the unknown function, *Communications in Applied Analysis*, **9**, No. 3 (2005), 417-428.

[19] N.T. Markova, P.S. Simeonov, Asymptotic and oscillatory behavior of n-th order forced differential equations with deviating argument depending on the unknown function, *Pan American Mathematical Journal*, **16**, No. 1 (2006), 1-15.

[20] N.T. Markova, P.S. Simeonov, Oscillatory properties of solutions of n-th order differential equations with deviating arguments depending on the unknown function, *Communications in Applied Analisys*, **10**, No. 4 (2006), 443-460.

[21] M. Naito, On strong oscillation of retarded differential equations, *Hiroshima Math. J.*, **11**, No. 3 (1981), 553-560.

[22] J. Ohriska, Oscillation of differential equations and v-derivatives, *Czechoslovak Math. J.*, **39**, No. 114 (1989), 24-44.

[23] S. Padhi, Asymptotic behaviour of solutions to n-th order functional differential equations, *Electronic Journal of Differential Equations*, **2005**, No. 65 (2005), 1-14, ISSN: 1072-6691, URL: http://ejde.math.txstate.edu, or ftp://ejde.math.txstate.edu.

[24] Ch.G. Philos, Y.G. Sficas, Oscillatory and asymptotic behavior of second and third order retarded differential equations, *Czechoslovak Mathematical Journal*, **32** (107), No. 2 (1982), 169-182.

[25] P.S. Simeonov, Oscillation of the solutions of second order retarded impulsive differential equations, *Communications in Applied Analysis*, **9**, No. 3 (2005), 353-368.

[26] I.P. Stavroulakis, Oscillation criteria for functional differential equations, *Electronic Journal of Differential Equations*, *Conference 12* (2005), 171-185, ISSN: 1072-6691, URL: http://ejde.math.txstate.edu, or ftp://ejde.math.txstate.edu.