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DISTRIBUTIONAL DUNKL TRANSFORM AND APPLICATIONS

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Abstract: The purpose of this paper is to extend the Dunkl transform on new spaces of distributions. We study the fundamental properties of this extension and we give some applications.

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1. Introduction

In this paper, we consider the Dunkl transform $\mathcal{F}_\alpha$ given for all $f \in L^1(A_\alpha(x)\, dx)$ by

$$\forall \ \lambda \in \mathbb{R}, \mathcal{F}_\alpha(f)(\lambda) = \int_{\mathbb{R}} \psi_\lambda^\alpha(x) f(x) A_\alpha(x) dx \,,$$

where $A_\alpha(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}$ and for every $\lambda \in \mathbb{C}$, ψ_λ^α represent the unique solution of the system

$$\begin{cases} \Lambda_\alpha \psi_\lambda^\alpha(x) &= -i\lambda\psi_\lambda^\alpha(x), \ \lambda \in \mathbb{C}, \\ \psi_\lambda^\alpha(0) &= 1. \end{cases}$$

Λ_α denotes the differential-difference operator on $\mathbb{R}$, given by

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$$\Lambda_\alpha f(x) = \frac{d}{dx}f(x) + \frac{(2\alpha+1)}{x}(\frac{f(x)-f(-x)}{2}), \tag{1}$$

with $\alpha \geq \frac{-1}{2}$. This operator is called Dunkl operator on $\mathbb{R}$ of index $(\alpha + \frac{1}{2})$ associated with the reflection group $\mathbb{Z}_2$ (see [1], [3], [4] and [5]).

It can also be written in the form

$$\Lambda_\alpha f(x) = \frac{d}{dx}f(x) + \frac{A'_\alpha(x)}{A_\alpha(x)}(\frac{f(x)-f(-x)}{2}).$$

Motivated by the studies of J.J. Betancor, J.D. Betancor and J.M.R. Mendez [2] and H. Ben Mohamed and K. Trimèche [1], we extend the Dunkl transform on new spaces of distributions.

As application, we solve some distributional differential-difference equations.

The organization of this paper is as follows. After a preliminary section, we introduce in Section 3 the new Fréchet functions spaces H_χ(where χ is a continuous function on $\mathbb{R}$ which is zero free on $\mathbb{R}$ such that the function $x \mapsto x\chi(x)$ is bounded), consisting of all C^∞-functions ϕ on $\mathbb{R}$ such that

$$\forall k \in \mathbb{N},\ p_k(\phi) = \sup_{x\in\mathbb{R}} \left| \chi(x)\frac{d^k}{dx^k}\phi(x)\right| < +\infty. \tag{2}$$

We consider on H_χ, the topology generated by the family of seminorms $\{p_k\}_{k\in\mathbb{N}}$. The space $S(\mathbb{R})$ of $C^\infty-$ functions on $\mathbb{R}$ rapidly decreasing together with their derivatives, equipped with the topology of seminorms $P_{k,l}$, $k,\ l \in \mathbb{N}$, given by

$$P_{k,l}(\phi) = \sup_{x\in\mathbb{R}} \left[(1+x^2)^k \left|\frac{d^l}{dx^l}\phi(x)\right|\right],\ k,\ l \in \mathbb{N}, \tag{3}$$

is continuously contained in H_χ. We denote by $\mathcal{H}_\chi$ the closure of $\mathcal{D}(\mathbb{R})$ in H_χ. Next, we investigate the Dunkl transform $\mathcal{F}_\alpha$ on the topological dual $\mathcal{H}'_\chi$ of the space $\mathcal{H}_\chi$, and we give their properties.

Section 4 is devoted to study the existence of solutions of a differential-difference equation.

2. Preliminaries

In this section we collect some notations and results about the Dunkl operator, the Dunkl kernel and the Dunkl transform (see [1], [3], [6], [7] and [8]).

Notations. We denote by:

• $\mathcal{C}^\infty(\mathbb{R})$, the space of $C^\infty-$ functions on $\mathbb{R}$.

• $\mathcal{S}(\mathbb{R})$, the space of $C^\infty-$ functions on $\mathbb{R}$ rapidly decreasing together with their derivatives, equipped with the topology of seminorms $P_{k,l}$, $k,\ l \in \mathbb{N}$, given by

$$P_{k,l}(\phi) = \sup_{x\in\mathbb{R}} \left[(1+x^2)^k \left|\frac{d^l}{dx^l}\phi(x)\right|\right], \ k,\ l \in \mathbb{N},$$

• $\mathcal{D}_a(\mathbb{R})$, $a > 0$, the space of $C^\infty-$ functions on $\mathbb{R}$ supported in $[-a, a]$, equipped with the topology of uniform convergence of the function and all its derivatives.

• $\mathcal{D}(\mathbb{R}) = \bigcup_{a>0} \mathcal{D}_a(\mathbb{R})$ endowed with the inductive limit topology.

• $L^p(d\mu_\alpha)(d\mu_\alpha(x) = A_\alpha(x)dx)$, $1 \le p \le +\infty$, the space of measurable functions on $\mathbb{R}$, such that

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p \, d\mu_\alpha(x)\right)^{\frac{1}{p}} < +\infty, \ 1 \le p < +\infty,$$

$$\|f\|_{\infty,\alpha} = \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)| < +\infty.$$

• $\mathbb{H}_a(\mathbb{C})$, $a > 0$, the space of entire functions on $\mathbb{C}$, rapidly decreasing and of exponential type. More precisely a function f is in $\mathbb{H}_a(\mathbb{C})$ if and only if f is entire on $\mathbb{C}$ and for all $m \in \mathbb{N}$

$$\rho_m(f) = \sup_{\lambda\in\mathbb{C}} (1+|\lambda|)^m \, |f(\lambda)| \, e^{-a|\mathrm{Im}(\lambda)|} < +\infty.$$

Note that the topology of $\mathbb{H}_a(\mathbb{C})$ is given by the seminorms ρ_m, $m \in \mathbb{N}$.

• $\mathbb{H}(\mathbb{C}) = \bigcup_{a>0} \mathbb{H}_a(\mathbb{C})$, is equipped with the inductive limit topology.

2.1. The Dunkl Kernel

Proposition 2.1. *For $\lambda \in \mathbb{C}$, the system*

$$\begin{cases} \Lambda_\alpha u(x) &= -i\lambda u(x), \ \lambda \in \mathbb{C}, \\ u(0) &= 1, \end{cases} \tag{4}$$

has a unique solution Ψ_λ^α (called the Dunkl kernl) given by

$$\forall x \in \mathbb{R}, \ \Psi_\lambda^\alpha(x) = j_\alpha(\lambda x) - \frac{i\lambda x}{2(\alpha+1)} j_{\alpha+1}(\lambda x), \tag{5}$$

where j_α is the normalized Bessel function of index α, defined by

$$\forall x \in \mathbb{R}, \ j_\alpha(x) = \Gamma(\alpha+1) \sum_{n\ge 0} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} (\frac{x}{2})^{2n}. \tag{6}$$

The following gives some properties of the Dunkl operator and the dunkl kernel, proof of which can be found in [1], [3] and [7].

Basic Properties. 1. In the case $\alpha=-\frac{1}{2}$, we have

$$\forall\lambda\in\mathbb{C},\ \forall x\in\mathbb{R},\ \Psi_\lambda^\alpha(x)=e^{-i\lambda x}.$$

2. For $\alpha>-\frac{1}{2}$, the function Ψ_λ^α has the following integral representation

$$\forall\lambda\in\mathbb{C},\ \forall x\in\mathbb{R},\ \Psi_\lambda^\alpha(x)=\ a_\alpha\int_{-1}^{1}e^{-i\lambda xy}(1-y)^{\alpha-\frac{1}{2}}(1+y)^{\alpha+\frac{1}{2}}dy, \qquad (7)$$

where

$$a_\alpha=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}.$$

3. For all $\lambda,\ x\in\mathbb{R}$, we have

$$\overline{\Psi_\lambda^\alpha}(x)=\Psi_{-\lambda}^\alpha(x),\quad \Psi_\lambda^\alpha(x)=\Psi_x^\alpha(\lambda).$$

4. Let $k\in\mathbb{N}$, we have

$$\forall\ \lambda\in\mathbb{R},\ \forall\ x\in\mathbb{R},\ \left|\frac{d^k}{dx^k}\Psi_\lambda^\alpha\left(x\right)\right|\le\left|\lambda\right|^k, \qquad (8)$$

in particular we have

$$\forall\ \lambda\in\mathbb{R},\ \forall\ x,\in\mathbb{R},\ \left|\Psi_\lambda^\alpha\left(x\right)\right|\le 1. \qquad (9)$$

5. Suppose that f is in $\mathcal{C}^\infty\left(\mathbb{R}\right)$ and $\alpha>-\dfrac{1}{2}$. Then:

(a) For all $n\in\mathbb{N}$, the function $\wedge_\alpha^n f$ defined by $\wedge_\alpha^0 f=f$ and $\wedge_\alpha^n f=\wedge_\alpha\left(\wedge_\alpha^{n-1}f\right)$, $n\ge 1$, is a C^∞-function on $\mathbb{R}$.

(b) For every $R>0$ and $k\in\mathbb{N}^*$, there exists a positive constant C_k satisfying: For all $x\in[-R,R]$, there exist $\zeta_j=\zeta_j(x,k)$, $j=1,\dots,k$, such that

$$\left|\wedge_\alpha^k f(x)\right|\le\left|f^{(k)}(x)\right|+C_k\sum_{j=1}^{k}\left|f^{(k)}(\zeta_j)\right|. \qquad (10)$$

(c) For all $k\in\mathbb{N}^*$, and $R>0$, there exists a positive constant $C_k^{'}$, such that for all $x\in\mathbb{R}$, satisfying $|x|>R:$

$$\left|\wedge_\alpha^k f(x)\right|\le\left|f^{(k)}(x)\right|+C_k^{'}\sum_{i=0}^{k-1}\left(\left|f^{(i)}(x)\right|+\left|f^{(i)}(-x)\right|\right). \qquad (11)$$

6. The operator $\wedge_\alpha$ maps continuously $\mathcal{S}\left(\mathbb{R}\right)$ (resp. $\mathcal{D}(\mathbb{R})$) into itself.

7. Let $n\in\mathbb{N}$, for all f in $\mathcal{S}(\mathbb{R})$ and g in $\mathcal{C}^\infty\left(\mathbb{R}\right)$, we have

$$\int_{\mathbb{R}}\wedge_\alpha^n(f)(x)g(x)d\mu_\alpha(x)=(-1)^n\int_{\mathbb{R}}f(x)\wedge_\alpha^n(g)(x)d\mu_\alpha(x). \qquad (12)$$

2.2. The Dunkl Transform

Definition 2.2. The Dunkl transform $\mathcal{F}_\alpha$ of f in $L^1(d\mu_\alpha)$ is defined by

$$\forall \lambda \in \mathbb{R}, \quad \mathcal{F}_\alpha(f)(\lambda) := \int_{\mathbb{R}} f(x)\Psi_\lambda^\alpha(x)d\mu_\alpha(x).$$

The proof of the following theorem is given in [6], [8] (see also [5]).

Theorem 2.3. *i) The Dunkl transform $\mathcal{F}_\alpha$ is a topological isomorphism*
- *from $\mathcal{S}(\mathbb{R})$ onto itself.*
- *from $\mathcal{D}(\mathbb{R})$ onto $\mathcal{H}$ $(\mathbb{C})$.*

Moreover f belongs to $\mathcal{D}_a(\mathbb{R})$ if and only if $\mathcal{F}_\alpha(f)$ is in $\mathcal{H}_a(\mathbb{C})$.
ii) The inverse transform $\mathcal{F}_\alpha^{-1}$ is given by

$$\forall x \in \mathbb{R}, \quad \mathcal{F}_\alpha^{-1}(f)(x) = \int_{\mathbb{R}} f(\lambda)\Psi_{-\lambda}^\alpha(x)d\mu_\alpha(\lambda), \quad f \in \mathcal{S}(\mathbb{R}). \tag{13}$$

3. The Spaces H_χ of Functions and Their Dual

Definition 3.1. Let χ be a continuous function on $\mathbb{R}$ which is zero free on $\mathbb{R}$ such that the function $x \mapsto x\chi(x)$ is bounded. We denote by H_χ the space of functions ϕ in $C^\infty(\mathbb{R})$, such that

$$\forall\, m \in \mathbb{N}, \quad p_m(\phi) = \sup_{x\in\mathbb{R}} |\chi(x)\frac{d^m}{dx^m}\phi(x)| < \infty\,.$$

The topology of this space is given by the seminorms $\{p_m\}_{m\in\mathbb{N}}$. Standard arguments allow us to see that H_χ as a Fréchet space.

We now give an alternative description of the space H_χ that will be useful in the sequel.

Proposition 3.2. *Let ϕ be in $C^\infty(\mathbb{R})$. Then $\phi \in H_\chi$ if and only if, for every $m \in \mathbb{N}$,*

$$q_m(\phi) = \sup_{x\in\mathbb{R}} |\chi(x)\Lambda_\alpha^m\phi(x)| < \infty\,.$$

Moreover, the system of seminorms $\{q_m\}_{m\in\mathbb{N}}$ generates the topology of H_χ.

Proof. Suppose first that $\phi \in H_\chi$. Then, according to (10) and (11), it is easy to see that for all $m \in \mathbb{N}$, there exists a positive constant C_m such that

$$q_m(\phi) \leq C_m \sum_{i=0}^{m} p_i(\phi). \tag{14}$$

Then

$$\forall m \in \mathbb{N}, \quad q_m(\phi) < +\infty. \tag{15}$$

This implies that $\{q_m\}_{m\in\mathbb{N}}$ defines on H_χ a topology weaker than the one associated with $\{p_m\}_{m\in\mathbb{N}}$.

Reciprocally, we consider ϕ in $C^\infty(\mathbb{R})$ satisfying the condition (15). From (1) we obtain

$$\forall x \in \mathbb{R}, \quad |x| \geq 1,\ |\chi(x)||\Lambda_\alpha\phi(x) - \phi'(x)| \leq Cq_0(\phi).$$

On the other hand since $q_1(\phi) < +\infty$ and as the function $x \longmapsto \chi(x)\frac{d}{dx}\phi(x)$ is bounded on $[-1,\ 1]$, we deduce that the function $x \longmapsto \chi(x)\frac{d}{dx}\phi(x)$ is bounded on $\mathbb{R}$ and then $p_1(\phi) < +\infty$.

The relation (1) and a proof by induction show that for all $n \in \mathbb{N}^*$ and $x \in \mathbb{R}^*$, we have

$$\Lambda_\alpha^n\phi(x) = \phi^n(x) + \sum_{i=0}^{n-1} P_{n-i}(\frac{1}{x})\phi^{(i)}(x) + Q_{n-i}(\frac{1}{x})\phi^{(i)}(-x)), \tag{16}$$

where $P_{n-i}(.,.)$, and $Q_{n-i}(.,.)$ are polynomials of degree $n-i$ in the variable $\frac{1}{x}$.

We suppose now that for all $j \in \{0,1,\dots,k-1\}$, $k \geq 2$, $p_j(\phi) < +\infty$. Then according to (16), we show that for all $k \in \mathbb{N}$, $k \geq 2$, there exists a positive constant C_k such that

$$\forall\, x \in \mathbb{R}, \quad |x| \geq 1,\ |\chi(x)||\Lambda_\alpha^k\phi(x) - \frac{d^k}{dx^k}\phi(x)| \leq C_k \sum_{j=1}^{k-1} p_j(\phi).$$

On the other hand for all $k \in \mathbb{N}$, the function $x \longmapsto \chi(x)\frac{d^k}{dx^k}\phi(x)$ is bounded on $[-1,\ 1]$. Then for all $k \in \mathbb{N}$, the function $x \longmapsto \chi(x)\frac{d^k}{dx^k}\phi(x)$ is bounded on $\mathbb{R}$. Thus

$$\forall k \in \mathbb{N}, \quad p_k(\phi) < +\infty.$$

Moreover, by arguing in a standard way (see [2]) we can see that the families $\{q_k\}_{k\in\mathbb{N}}$ define Fréchet topology on H_χ, then from (14) and the Open Mapping Theorem, $\{p_k\}_{k\in\mathbb{N}}$ and $\{q_k\}_{k\in\mathbb{N}}$ generate the same topology on H_χ.

Notation. We denote by $\mathcal{H}_\chi$, the closure of $D(\mathbb{R})$ in H_χ.

In the following, we give a characterization of $\mathcal{H}_\chi$.

Proposition 3.3. *Let $\phi \in C^\infty(\mathbb{R})$. Then the following assertions are equivalent:*

i) $\phi \in \mathcal{H}_\chi$.

ii) For every $m \in \mathbb{N}$:

$$\lim_{|x|\to+\infty} \chi(x)\frac{d^m}{dx^m}\phi(x) = 0. \tag{17}$$

Proof. i) $\Longrightarrow$ ii). Let ϕ be in $\mathcal{H}_\chi$, then there exists a sequence $\{\phi_j\}_{j\in\mathbb{N}}$ in $D(\mathbb{R})$ such that $\phi_j \to \phi$, as $j \to +\infty$, in $\mathcal{H}_\chi$. Let $\varepsilon > 0$ and $m \in \mathbb{N}$. There exists $j_0 \in \mathbb{N}$, such that

$$\forall\, x \in \mathbb{R},\ |\chi(x)\frac{d^m}{dx^m}\phi(x)| \leq \sup_{t\in\mathbb{R}}[|\chi(t)||\frac{d^m}{dx^m}(\phi(t)-\phi_{j_0}(t))|] + |\chi(x)\frac{d^m}{dx^m}\phi_{j_0}(x)| \leq \varepsilon + |\chi(x)\frac{d^m}{dx^m}\phi_{j_0}(x)|.$$

Since ϕ_{j_0} is in $D(\mathbb{R})$, then there exist $R>0$ such that

$$\forall\, |x| > R,\quad \forall\, m \in \mathbb{N},\quad \chi(x)\frac{d^m}{dx^m}\phi_{j_0}(x) = 0\,.$$

Hence, we obtain

$$\forall\, m \in \mathbb{N},\quad \lim_{|x|\to\infty}\chi(x)\frac{d^m}{dx^m}\phi(x) = 0\,.$$

Now we prove that ii) $\Longrightarrow$ i). Let $\zeta \in D(\mathbb{R})$, such that $\zeta(x) = 1$, $|x| \leq 1$ and $\zeta(x) = 0$, $|x| \geq 2$.

For every $n \in \mathbb{N}^*$, we put

$$\forall\, x \in \mathbb{R},\quad \zeta_n(x) = \zeta(\frac{x}{n})e^{-\frac{x^2}{n}}\,.$$

It is clear that

$$\forall\, n \in \mathbb{N}^*,\ \phi\zeta_n \in D(\mathbb{R})\,.$$

On the other hand, for all $j \in \mathbb{N}$, there exists a positive constant M_j, such that

$$\forall\, x \in \mathbb{R},\ |\frac{d^j\zeta_n(x)}{dx^j}| \leq M_j. \tag{18}$$

Let $m \in \mathbb{N}^*$. Leibniz's rule leads to

$$\forall\, x \in \mathbb{R},\ \frac{d^m}{dx^m}[(\phi\zeta_n - \phi)(x)] = \sum_{j=1}^{m}\binom{j}{m}\frac{d^{m-j}\phi(x)}{dx^{m-j}}\frac{d^j\zeta_n(x)}{dx^j} + \frac{d^m\phi(x)}{dx^m}(\zeta_n(x)-1) \tag{19}$$

Then using (18), we deduce that there exists a positive constant C_m, such that

$$\forall\, x \in \mathbb{R},\ |\frac{d^m}{dx^m}(\phi\zeta_n - \phi)(x)| \leq C_m\sum_{j=0}^{m}|\frac{d^j\phi(x)}{dx^j}|.$$

Thus, from this relation and (19), we deduce that for all $\varepsilon > 0$ there exits $R > 0$, such that

$$\forall x \in [-R,R]\ ,\ |\chi(x)\frac{d^m}{dx^m}(\phi\zeta_n - \phi)(x)| \leq \varepsilon.$$

Moreover, if $n > R$, we have

$$\forall x \in [-R,R],\ \zeta_n(x) = e^{-\frac{x^2}{n}}\,.$$

Then from (19), there exist two positive constants C'_m and C''_m, such that

$$\forall\, x \in \mathbb{R},\ |\chi(x)\frac{d^m}{dx^m}(\phi\zeta_n - \phi)(x)| \leq \frac{C'_m}{n} + C''_m(1 - e^{-\frac{x^2}{n}}).$$

Hence, there exists $N \in \mathbb{N}$, $N > R$, such that for all $x \in \mathbb{R}$, we have

$$\forall\, n \geq N,\ |\chi(x)\frac{d^m}{dx^m}(\phi\zeta_n - \phi)(x)| \leq \varepsilon\,.$$

Thus, we conclude that $\phi\zeta_n \to \phi$, as $n \to +\infty$ in H_χ. Then the proof is finished. □

Remarks. 1) The space $\mathcal{H}_\chi$, does not coincide with H_χ. Indeed for $\chi(x) = \frac{1}{1+|x|}$ and $\phi(x) = x$ we have $\phi \in H_\chi \backslash \mathcal{H}_\chi$.

2) The space $S(\mathbb{R})$ is continuously contained in $\mathcal{H}_\chi$.

Proposition 3.4. *For every $\lambda \in \mathbb{R}$, the function Ψ^α_λ belongs to $\mathcal{H}_\chi$.*

Proof. Let $k \in \mathbb{N}$. From (9), we have

$$\forall x \in \mathbb{R},\quad \forall \lambda \in \mathbb{R},\qquad \left|\frac{d^k}{dx^k}\Psi^\alpha_\lambda(x)\right| \leq |\lambda|^k\,.$$

Then $\forall k \in \mathbb{N}$ we have

$$\lim_{|x|\to+\infty} \chi(x)\frac{d^k}{dx^k}\Psi^\alpha_\lambda(x) = 0\,.$$

Using the last proposition we deduce that Ψ^α_λ belongs to $\mathcal{H}_\chi$. □

Notation. The topological dual of $\mathcal{H}_\chi$, will be denoted by $\mathcal{H}'_\chi$.

Definition 3.5. The Dunkl transform $\mathcal{F}_\alpha(T)$ of T in $\mathcal{H}'_\chi$, is defined by

$$\forall\, \lambda \in \mathbb{R},\quad \mathcal{F}_\alpha(T)(\lambda) = \langle T,\ \Psi^\alpha_\lambda\, \rangle. \tag{20}$$

Proposition 3.6. *Let T be in $\mathcal{H}'_\chi$. There exists a polynomial Q such that*

$$\forall \lambda \in \mathbb{R},\quad |\mathcal{F}_\alpha(T)(\lambda)| \leq Q(|\lambda|). \tag{21}$$

Proof. Let T be in $\mathcal{H}'_\chi$, there exist $r \in \mathbb{N}$ such that

$$\begin{aligned}\forall\, \lambda \in \mathbb{R},\quad |\mathcal{F}_\alpha(T)(\lambda)| = |\langle T,\ \Psi^\alpha_\lambda\,\rangle| &\leq\ C \max_{0\leq k\leq r}\left|\frac{d^k}{dx^k}\Psi^\alpha_\lambda(x)\right| \\ &\leq\ C \max_{0\leq k\leq r} |\lambda|^k \leq C(1+|\lambda|)^r.\end{aligned}$$

Lemma 3.7. *Let T be in $\mathcal{H}'_\chi$, $\phi \in \mathcal{S}(\mathbb{R})$ and φ the function given by*

$$\forall\ x,\ y \in \mathbb{R},\quad \varphi(x,y) = \int_0^y \Psi^\alpha_t(x)\phi(t)\,|t|^{2\alpha+1}\,dt,$$

then the function G given by $\forall y \in \mathbb{R},\ G(y) = \langle T_x,\ \varphi(x,y)\rangle$, is differentiable on $\mathbb{R}$ and we have

$$\forall\ y \in \mathbb{R},\quad G'(y) = \left\langle T_x,\ \frac{\partial\varphi}{\partial y}(x,y)\right\rangle.$$

Proof. Let $y_0 \in \mathbb{R}$ and $h \in \mathbb{R}^*$, such that $|h| < 1$. Using Taylor formula, we obtain

$$\varphi(x,y_0+h)=\varphi(x,y_0)+h\int_0^1\frac{\partial\varphi}{\partial y}(x,y_0+th)dt.$$

Then

$$\frac{G(y_0+h)-G(y_0)}{h}-\left\langle T_x,\ \frac{\partial\varphi}{\partial y}(x,y_0)\right\rangle=\langle T_x,\ R(x,y_0,h)\rangle,$$

where

$$R(x,y_0,h)=\int_0^1\left[\frac{\partial\varphi}{\partial y}(x,y_0+th)-\frac{\partial\varphi}{\partial y}(x,y_0)\right]dt.$$

From (7), we have

$$\forall\ x,\ y\in\mathbb{R},\quad \frac{d^k\Psi_y^\alpha(x)}{dx^k}=(-iy)^kQ_k(x,y),$$

where

$$\forall\ x,\ y\in\mathbb{R},\quad Q_k(x,y)=\ a_\alpha\int_{-1}^1 e^{-ixyt}(1-t)^{\alpha-\frac{1}{2}}(1+t)^{\alpha+\frac{1}{2}}t^kdt,$$

Then by derivation under the integral sign we obtain

$$\left|\frac{d^k}{dx^k}R(x,y_0,h)\right|\leq\int_0^1\left|P_k(x,y_0+th)-P_k(x,y_0)\right|dt,$$

where

$$P_k(x,y)=Q_k(x,y)\phi(y)\left|y\right|^{2\alpha+1}y^k.$$

But from Taylor formula, we have

$$\begin{aligned}P_k(x,y_0+th)&-P_k(x,y_0)\\&=Q_k(x,y_0)\phi(y_0)\left[\left|y_0+th\right|^{2\alpha+1}(y_0+th)^k-\left|y_0\right|^{2\alpha+1}y_0^k\right]\\&\qquad\qquad+h\ g_k(x,y_0,t,h)\left|y_0+th\right|^{2\alpha+1}(y_0+th)^k,\end{aligned}$$

where g_k is a function satisfying for all $x\in\mathbb{R}$, $h\in\mathbb{R}^*$, $|h|<1$, $t\in[0,1]$:

$$|g_k(x,y_0,t,h)|\leq C(1+|x|),$$

where C is a positive constant. Then there exists a positive constant $C^{'}$ such that

$$\begin{aligned}&\left|\chi(x)\frac{d^k}{dx^k}\left[R(x,y_0,h)\right]\right|\\&\qquad\leq C^{'}\left[|h|+\int_0^1\left|\left|y_0+th\right|^{2\alpha+1}(y_0+th)^k-\left|y_0\right|^{2\alpha+1}y_0^k\right|dt\right].\end{aligned}$$

Hence we deduce that

$$p_k\left[R(x,y_0,h)\right]\longrightarrow 0,\ \text{ as } h\longrightarrow 0,$$

which prove that G is differentiable at y_0 and we have

$$G^{'}(y_0)=\left\langle T_x,\ \frac{\partial\varphi}{\partial y}(x,y_0)\right\rangle.$$

Then the lemma is proven. □

Remarks. i) The Dunkl transform can be defined on the topological dual space $\mathcal{S}'(\mathbb{R})$ of $\mathcal{S}(\mathbb{R})$ by transposition. That is if T is in $\mathcal{S}'(\mathbb{R})$, the Dunkl transform $\mathcal{F}_\alpha(T)$ of T is defined by

$$\langle \mathcal{F}_\alpha(T), \phi \rangle = \langle T, \mathcal{F}_\alpha(\phi) \rangle , \quad \phi \in \mathcal{S}(\mathbb{R}). \tag{22}$$

ii) Since $\mathcal{S}(\mathbb{R})$ is continuously contained in $\mathcal{H}_\chi$, then $\mathcal{H}_\chi^{'}$ is contained in $\mathcal{S}'(\mathbb{R})$. Hence for all T in $\mathcal{H}_\chi^{'}$, we can define the Dunkl transform $\mathcal{F}_\alpha(T)$ of T in two apparently different ways namely, by (20) and by (22).

In the following proposition we establish that both definitions coincide.

Proposition 3.8. *Let T be in $\mathcal{H}_\chi^{'}$, We put*

$$\forall\, y \in \mathbb{R}, \quad F(y) = \left\langle T\ , \Psi_y^\alpha \ \right\rangle .$$

Then

$$\forall \phi \in \mathcal{S}(\mathbb{R}), \quad \langle S_F, \phi \rangle = \langle T,\ \mathcal{F}_\alpha(\phi) \rangle , \tag{23}$$

where S_F is the distribution in $\mathcal{S}'(\mathbb{R})$ given by the function F.

Proof. Let ϕ be in $\mathcal{S}(\mathbb{R})$ and $n \in \mathbb{N}, n > \alpha + \frac{1}{2} d^0 Q + 1$, where Q is the polynomial given by (22). From Proposition 3.6, we have

$$|\langle S_F, \phi \rangle| \leq \left\{ \int_\mathbb{R} \frac{|Q(|\lambda|)|}{(1+\lambda^2)^n} d\mu_\alpha(\lambda) \right\} P_{0,n}(\phi),$$

where $P_{0,n}$ is the seminorm given by (3). This proves that S_F defines an element of $\mathcal{S}'(\mathbb{R})$.

Now we prove that for every ϕ in $\mathcal{S}(\mathbb{R})$, we have

$$\int_\mathbb{R} \langle T,\ \Psi_y^\alpha \rangle \phi(y) d\mu_\alpha(y) = \left\langle T,\ \int_\mathbb{R} \Psi_y^\alpha(\cdot) \phi(y) d\mu_\alpha(y) \right\rangle . \tag{24}$$

Note that by invoking Proposition 3.3, it is not hard to see that for every $\eta,\ \delta \in \mathbb{R}$ and ϕ in $\mathcal{S}(\mathbb{R})$, the functions $x \mapsto \int_{-\infty}^{\eta} \Psi_y^\alpha(x)\phi(y)d\mu_\alpha(y)$ and $x \mapsto \int_{\delta}^{+\infty} \Psi_y^\alpha(x)\phi(y)d\mu_\alpha(y)$ belong to $\mathcal{H}_\chi$.

We will prove in the following that

$$\lim_{\eta \to -\infty} \int_{-\infty}^{\eta} \Psi_y^\alpha(x)\phi(y)d\mu_\alpha(y) = \lim_{\delta \to +\infty} \int_{\delta}^{+\infty} \Psi_y^\alpha(x)\phi(y)d\mu_\alpha(y) = 0, \text{ in } H_\chi.$$

By (4) and (8), for every $k \in \mathbb{N}$, we can write

$$\left| \chi(x) \wedge_\alpha^k \left\{ \int_{-\infty}^{\eta} \Psi_y^\alpha(x)\phi(y)d\mu_\alpha(y) \right\} \right| \leq |\chi(x)| \int_{-\infty}^{\eta} |y|^k \, |\phi(y)| \, d\mu_\alpha(y).$$

Using the fact that the function $y \mapsto |y|^k \, |\phi(y)|$ belongs to $L^1(d\mu_\alpha)$, we deduce

that
$$\lim_{\eta\to-\infty}\int_{-\infty}^{\eta}\Psi_y^\alpha(x)\phi(y)d\mu_\alpha(y)=0,\ \text{in } H_\chi.$$
By the same argument we prove that
$$\lim_{\delta\to+\infty}\int_{\delta}^{+\infty}\Psi_y^\alpha(x)\phi(y)d\mu_\alpha(y)=0,\ \textit{in } H_\chi.$$
It remains to prove that for every $a,\ b\in\mathbb{R}$, $a<b$, we have
$$\int_a^b\left\langle T,\ \Psi_y^\alpha\right\rangle\phi(y)d\mu_\alpha(y)=\left\langle T,\ \int_a^b\Psi_y^\alpha(\cdot)\phi(y)d\mu_\alpha(y)\right\rangle. \tag{25}$$
For $y\geq a$ and ϕ in $\mathcal{S}(\mathbb{R})$, we put
$$H(y)=\left\langle T,\ \int_a^y\Psi_t^\alpha(\cdot)\phi(t)d\mu_\alpha(t)\right\rangle.$$
Using Lemma 3.7, we deduce that H is a differentiable function on $[a,\ +\infty[$ and we have
$$\forall\ y\geq a,\quad H'(y)=\left\langle T,\ \Psi_y^\alpha(\cdot)\phi(y)\frac{|y|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}\right\rangle.$$
Hence we obtain
$$H(y)\ =\ \int_a^y\left\langle T,\ \Psi_t^\alpha(\cdot)\phi(t)\frac{|t|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}\right\rangle dt\ =\ \int_a^y\left\langle T,\ \Psi_t^\alpha\right\rangle\phi(t)d\mu_\alpha(t).$$
Thus the proof is completed. □

Proposition 3.9. *Let T be in $\mathcal{H}'_\chi$. If for all $y\in\mathbb{R}$, $\mathcal{F}_\alpha(T)(y)=0$, then $T=0$.*

Proof. The result follows immediately from Proposition 3.8 and the fact that $\mathcal{S}(\mathbb{R})$ is a dense subspace of $\mathcal{H}_\chi$. □

Proposition 3.10. *Let T be in $\mathcal{H}'_\chi$. For all $r>0$ and $\delta<0$ the distribution given by the function $x\mapsto\int_\delta^r\mathcal{F}_\alpha(T)(y)\Psi_{-y}^\alpha(x)d\mu_\alpha(y)$, belongs to $\mathcal{S}'(\mathbb{R})$ and we have*
$$T=\lim_{\substack{r\to+\infty\\ \delta\to-\infty}}\int_\delta^r\mathcal{F}_\alpha(T)(y)\Psi_{-y}^\alpha(\cdot)d\mu_\alpha(y), \tag{26}$$
weakly in $\mathcal{S}'(\mathbb{R})$.

Proof. We put
$$\forall x\in\mathbb{R},\quad F_{r,\delta}(x)=\int_\delta^r\mathcal{F}_\alpha(T)(y)\Psi_{-y}^\alpha(x)d\mu_\alpha(y).$$
Indeed by virtue of Proposition 3.6, there exists a polynomial Q such that
$$\forall x\in\mathbb{R},\quad |F_{r,\delta}(x)|\leq\int_\delta^r|Q(y)|\,d\mu_\alpha(y)=C_{r,\delta}.$$

Then the function $F_{r,\delta}$ is in $L^{\infty}(d\mu_\alpha)$ and the distribution given by this function belongs to $\mathcal{S}'(\mathbb{R})$.

Let ϕ be in $\mathcal{S}(\mathbb{R})$, using Fubini's Theorem, we obtain

$$\int_{\mathbb{R}} F_{r,\delta}(x)\phi(x)d\mu_\alpha(x) = \int_{\delta}^{r} \mathcal{F}_\alpha(T)(y)\left[\int_{\mathbb{R}} \phi(x)\Psi_{-y}^{\alpha}(x)d\mu_\alpha(x)\right] d\mu_\alpha(y).$$

Then from (21), we deduce that

$$\int_{\mathbb{R}} F_{r,\delta}(x)\phi(x)d\mu_\alpha(x) = \langle T, \int_{\delta}^{r} \mathcal{F}_\alpha(\phi)(-y)\Psi_{y}^{\alpha}(\cdot)d\mu_\alpha(y)\rangle. \tag{27}$$

But from (8) and Proposition 3.3, the function $x \longmapsto \int_\delta^r \mathcal{F}_\alpha(\phi)(-y)\Psi_y^\alpha(x)$ $d\mu_\alpha(y)$ belongs to $\mathcal{H}_\chi$. Moreover for all $k \in \mathbb{N}$, we have

$$q_k\left[\int_{r}^{+\infty} \mathcal{F}_\alpha(\phi)(-y)\Psi_{y}^{\alpha}(x)d\mu_\alpha(y)\right] \leq \int_{r}^{+\infty} |\mathcal{F}_\alpha(\phi)(-y)|\,|y|^k\, d\mu_\alpha(y), \tag{28}$$

and

$$q_k\left[\int_{-\infty}^{\delta} \mathcal{F}_\alpha(\phi)(-y)\Psi_{y}^{\alpha}(x)d\mu_\alpha(y)\right] \leq \int_{-\infty}^{\delta} |\mathcal{F}_\alpha(\phi)(-y)|\,|y|^k\, d\mu_\alpha(y). \tag{29}$$

Therefore, from (14), we obtain

$$\lim_{\substack{r \to +\infty \\ \delta \to -\infty}} \int_{\delta}^{r} \mathcal{F}_\alpha(\phi)(-y)\Psi_{y}^{\alpha}(x)d\mu_\alpha(y) = \int_{\mathbb{R}} \mathcal{F}_\alpha(\phi)(y)\Psi_{-y}^{\alpha}(x)d\mu_\alpha(y) = \phi(x),$$

in the sense of convergence in H_χ.

From (27), (28), and (29), we deduce that

$$T = \lim_{\substack{r \to +\infty \\ \delta \to -\infty}} \int_{\delta}^{r} \mathcal{F}_\alpha(T)(y)\Psi_{-y}^{\alpha}(\cdot)d\mu_\alpha(y),$$

weakly in $\mathcal{S}'(\mathbb{R})$. □

4. Application

We now study an operational calculus for the generalized Dunkl transform and a distributional differential-difference equation is solved in H'_χ.

We will denote by Λ_α^* the formal adjoint of the operator Λ_α. That is, for every $T \in H'_\chi$, $\Lambda_\alpha^* T$ is the element of H'_χ given by

$$\langle \Lambda_\alpha^* T, \phi\rangle = \langle T, \Lambda_\alpha \phi\rangle, \quad \phi \in H_\chi\,.$$

Since Λ_α is a continuous linear mapping from H_χ into itself, from well-known results of analysis it is deduced that Λ_α^* defines a continuous linear mapping from H'_χ into itself, when we consider on H'_χ either the weak or the strong topology.

Proposition 4.1. *Let T be in H'_χ. Then*
$$\forall\,\lambda \in \mathbb{R},\ \mathcal{F}_\alpha(\Lambda^*_\alpha T)(\lambda) = -i\lambda\mathcal{F}_\alpha(T)(\lambda).$$

Proof. It is sufficient to note that $\Lambda_\alpha\psi^\alpha_\lambda(x) = -i\lambda\psi^\alpha_\lambda(x)$, $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. □

Proposition 4.2. *The following differential-difference equation*
$$P(\Lambda^*_\alpha)T = S, \tag{30}$$
where $S \in H'_\chi$ and P is a polynomial such that $P(iy) \neq 0$, $y \in \mathbb{R}$, has a solution in $S'(\mathbb{R})$.

Proof. We apply the Dunkl transform to (30). By Proposition 4.1 we deduce that
$$\forall\,\lambda \in \mathbb{R},\ P(i\lambda)F(\lambda) = G(\lambda), \tag{31}$$
where F and G being the Dunkl transform of T and S, respectively.

From Proposition 3.10 and (31), we can write
$$\langle T,\phi\rangle = \lim_{\substack{s\to+\infty\\ \delta\to+\infty}} \langle \int_{-\delta}^{s} \frac{G(y)}{P(-iy)}\psi_y(.)d\mu_\alpha(y),\phi\rangle,\ \phi \in S(\mathbb{R}).$$
We will see that the last limit exists. Let $m,n \in \mathbb{N}$, $m < n$. For every polynomial $Q = \sum_0^p a_jX^j$ such that $Q(iy) \neq 0$, for all $y \in \mathbb{R}$, we have
$$Q(\Lambda_\alpha)\int_{-n}^{-m} \frac{G(y)}{P(-iy)Q(-iy)}\psi_y(x)d\mu_\alpha(y) = \int_{-n}^{-m} \frac{G(y)}{P(-iy)}\psi_y(x)d\mu_\alpha(y),\quad x \in \mathbb{R}. \tag{32}$$
From Proposition 3.6, we can choose Q such that
$$|\frac{G(y)}{P(iy)Q(iy)}| = O(|y|^{-t}),\ \text{as}\,|y| \to \infty, \tag{33}$$
with $t > 2\alpha + 4$.

From (12), we see that for $\phi \in S(\mathbb{R})$, we have
$$\int_{\mathbb{R}}(\Lambda_\alpha)_x\Big(\int_{-n}^{-m} \frac{G(y)}{P(-iy)Q(-iy)}\psi_y(x)d\mu_\alpha(y)\Big)\phi(x)A_\alpha(x)dx$$
$$= -\int_{\mathbb{R}}\int_{-n}^{-m} \frac{G(y)}{P(-iy)Q(-iy)}\psi^\alpha_y(x)d\mu_\alpha(y) \times \Lambda_\alpha\phi(x)A_\alpha(x)dx.$$
Hence, from (32) one infers
$$\langle\int_{-n}^{-m} \frac{G(y)}{P(-iy)}\psi^\alpha_y(.)d\mu_\alpha(y),\phi\rangle$$
$$= \langle\int_{-n}^{-m} \frac{G(y)}{P(-iy)Q(-iy)}\psi^\alpha_y(.)d\mu_\alpha(y), Q'(\Lambda_\alpha)\phi\rangle\,,$$

where $Q' = \sum_0^p (-1)^j a_j X^j$.

According again to (33), there exist a positive constant C such that
$$|\langle \int_{-n}^{-m} \frac{G(y)}{P(-iy)} \psi_y^\alpha(.) d\mu_\alpha(y), \phi(x)\rangle| \leq C \int_{-n}^{-m} |\frac{G(y)}{P(-iy)Q(-iy)}| d\mu_\alpha(y)\,.$$
Thus we conclude that for $s > 0$, the sequence
$$\left\{ \int_{-n}^{s} \frac{G(y)}{P(-iy)} \psi_y^\alpha(.) d\mu_\alpha(y) \right\}_{n\in\mathbb{N}}$$
is a Cauchy sequence in the weak topology of $S'(\mathbb{R})$. In the same way we prove that for $\delta > 0$, the sequence
$$\left\{ \int_{-\delta}^{n} \frac{G(y)}{P(-iy)} \psi_y^\alpha(.) d\mu_\alpha(y) \right\}_{n\in\mathbb{N}}$$
is also a Cauchy sequence in the weak topology of $S'(\mathbb{R})$. Then there exists $T \in S'(\mathbb{R})$ such that
$$\langle T, \phi\rangle = \lim_{\substack{n\to+\infty \\ m\to+\infty}} \langle \int_{-m}^{n} \frac{G(y)}{P(-iy)} \psi_y^\alpha(.) d\mu_\alpha(y), \phi\rangle,\ \phi \in S(\mathbb{R}).$$
Moreover, Proposition 3.10 leads to
$$\begin{aligned}\langle P(\Lambda_\alpha^*)T, \phi\rangle = \langle T, P(\Lambda_\alpha)\phi\rangle &= \lim_{m,n\to+\infty} \langle \int_{-m}^{n} \frac{G(y)}{P(-iy)} \psi_y^\alpha(.) d\mu_\alpha(y), P(\Lambda_\alpha)\phi\rangle, \\ &= \lim_{m,n\to+\infty} \langle P(\Lambda_\alpha) \int_{-m}^{n} \frac{G(y)}{P(-iy)} \psi_y^\alpha(.) d\mu_\alpha(y), \phi\rangle, \\ &= \lim_{m,n\to+\infty} \langle \int_{-m}^{n} G(y) \psi_y^\alpha(.) d\mu_\alpha(y), \phi\rangle = \langle S, \phi\rangle, \quad \phi \in S(\mathbb{R}).\end{aligned}$$
Hence $T \in S'(\mathbb{R})$ is a solution of (31).

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