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A NOTE ON THE SECOND SOLUTION OF CHEBYSHEV'S EQUATION

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Abstract: In spite of that Chebyshev equation is very similar to Legendre equation, in the sense that their first solution span an orthogonal basis in $[-1, 1]$, their second solution is very different in nature, namely, in the case of Legendre equation the functions Q_n have a singularity at ± 1 while Chebyshev ones are well behaved in all the interval. Regarding the second solution in $[1, \infty)$, the situation is more dramatic since Q_n are still singular at 1 and goes to zero at infinity, while Chebyshev second solution is well behaved at 1 but diverges at infinity. However, certain physical applications demand that Chebyshev equation second solution behaves as Q_n when the argument is large. In such a case, the only possibility to get a second solution of the equation consists in finding a Frobenius series representation. In this work we discuss the properties of the second solution of Chebyshev equation in both, $[-1, 1]$ and $[1, \infty)$, a matter that, to our knowledge has not been discussed nor in textbooks or current literature.

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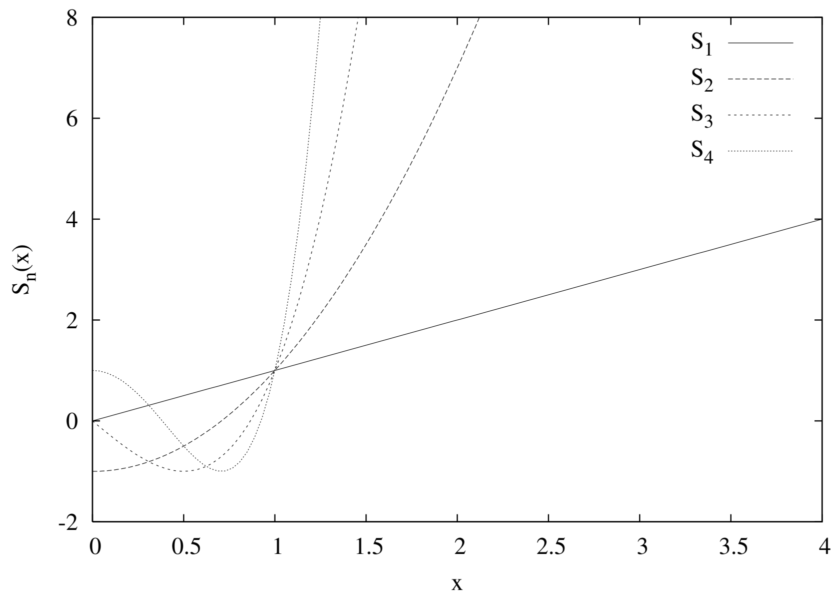

Figure 1: Plot of functions $S_n(x) = \cosh(n \cosh^{-1} x)$ for $n = 1, 2, 3, 4$.

1. Introduction

The need to solve second order differential equations is common in physics both at the classical or quantum mechanical level, these equations appears ordinary as a result of the separation of Laplace or Schrödinger equations in a given coordinate system. Most of the resulting equations are well presented in both textbooks on mathematical physics and current literature involving research problems. One of the most common is the Legendre equation which appears in electrostatic problems of spherical symmetry or in the theory of angular momentum, its first solution provides an orthogonal basis set of functions which generates a infinite dimensional space of the Hilbert type in $[-1, 1]$, a very important property because they can be used to represent any continuous function as an infinite linear combination of these basis vectors, see [5].

In spite of that the second solution is irregular in ± 1, some problems demand its use to have a complete solution, and in this sense are of relative importance also. Other important equation is Chebyshev type I equation, which is very similar in structure to Legendre equation; indeed, its first solution, the Chebyshev type I polynomials, also span an infinite dimensional Hilbert-like space. The main difference between both solutions rests in that, while Legendre oper-

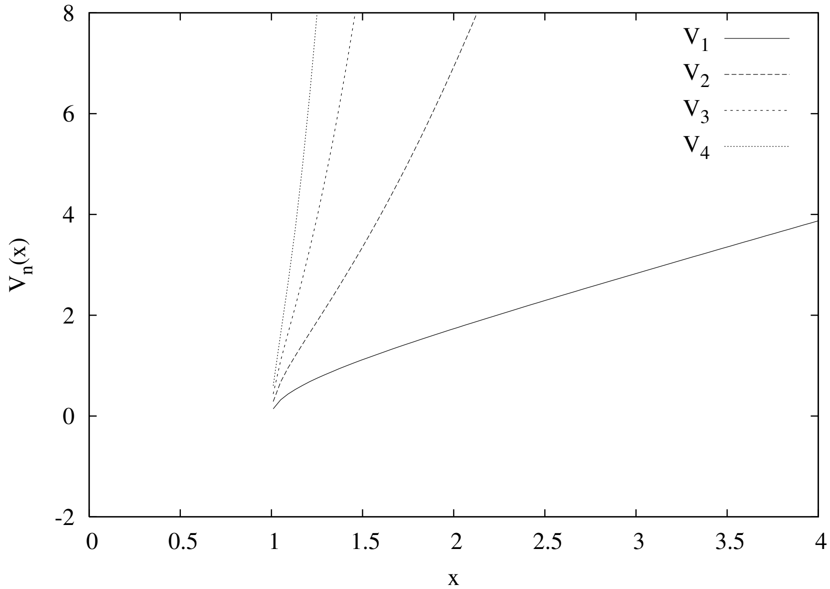

Figure 2: Plot of functions $V_n(x) = \sinh(n \cosh^{-1} x)$ for $n = 1, 2, 3, 4$.

ator is naturally self-adjoint, Chebyshev operator is not at first glance, but it can be casted in such a form and this matter reflects on the orthogonality condition since Legendre polynomials are orthogonal with a weight function equal 1, while Chebyshev polynomials are also orthogonal but with a weight function $(1 - x^2)^{-1/2}$, see [1].

As in the case of the Legendre equation, some problems demand the knowledge of the second solution of Chebyshev equation when the interval $[1, \infty)$ is involved. Unfortunately, to our knowledge, the second solution of this equation in this interval was not reported or studied in standard textbooks or literature on this subject. Therefore, it seems to be needed to do so, however, on the contrary of the second solution of Legendre equation, the analytical second solution of Chebyshev equation seems not to be useful *per se*. That is why, in some physical problems, one is compelled to construct a Frobenius series representation that guarantee the proper behavior of the complete solution of the problem when the argument is large.

The aim of this work is to report the behavior of both, analytical and series solution of Chebyshev equation and their properties such as orthogonality of the first solutions in $[-1, 1]$, divergence of the second solution as $x \to \infty$, and the Frobenius series representation for the second solution. Also, we show that

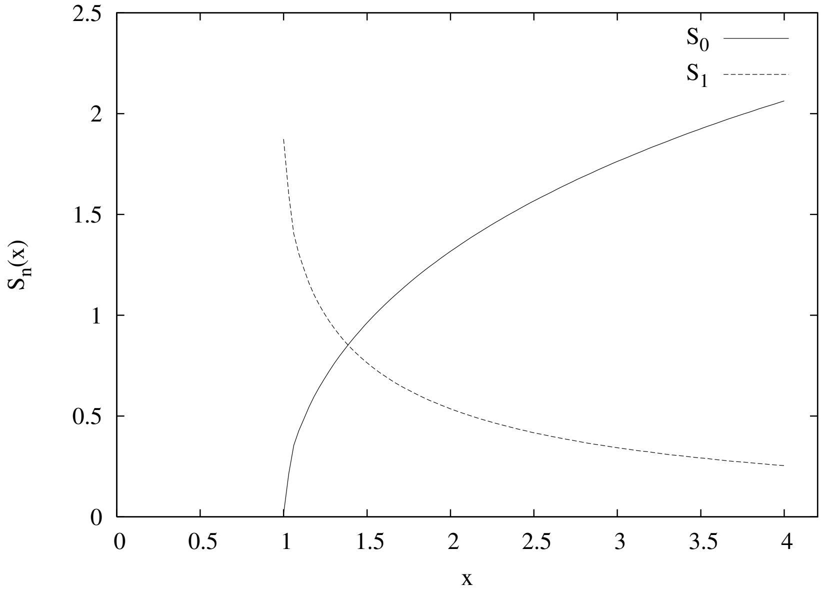

Figure 3: Plot of functions $S_n(x)$ for $n = 0, 1$.

the latter is specially useful for some physical problems which involve Laplace or Schrödinger equations in elliptic coordinates and in these applications rests its value, see [4, 3].

2. Chebyshev Equation

Chebyshev type I equation is commonly written as in [2]:

$$\left\{(1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx} + n^2\right\} F(x) = 0 \ , \ n = 0, 1, 2, \cdots \ , \tag{1}$$

if $x \in [-1, 1]$, or

$$\left\{(x^2-1)\frac{d^2}{dx^2} + x\frac{d}{dx} - n^2\right\} F(x) = 0 \ , \ n = 0, 1, 2, \cdots \ , \tag{2}$$

if $x \in [1, \infty)$.

If we make the change $x = \cos u$ in (1) or $x = \cosh v$ in (2), we obtain

$$\left\{\frac{d^2}{du^2} + n^2\right\} F(u) = 0, \quad u \in [0, \pi], \tag{3}$$

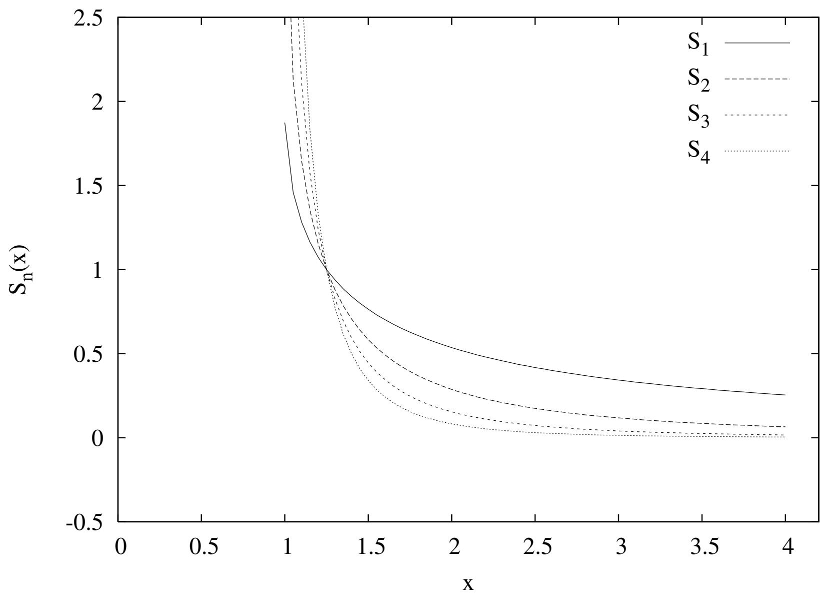

Figure 4: Plot of functions $S_n(x)$ for $n = 1, 2, 3, 4$.

and

$$\left\{\frac{d^2}{dv^2} - n^2\right\} F(v) = 0, \quad v \in [0, \infty], \tag{4}$$

and the solution can be simply written as

$$F_n \begin{pmatrix} u \\ v \end{pmatrix} = \left\{ \begin{array}{c} A\cos(nu) + B\sin(nu),\ u \in [0, \pi]\,, \\ C\cosh(nv) + D\sinh(nv),\ v \in\ [0, \infty)\,, \end{array} \right. \tag{5}$$

where $A, B, C,$ and D are constants.

Notice that, as the cosine and the sine (circular or hyperbolic) are linearly independent, the choice of one of the constants in the pair (A, B) or (C, D), equal to zero gives the first or second solutions, respectively.

2.1. First and Second Solutions

From the well known properties of circular functions, it is clear that both solutions in $[-1, 1]$ (or $[0, \pi]$) have the property

$$\int\limits_{-1}^{1} \frac{F_n(x)F_m(x)}{\sqrt{1-x^2}} dx = \int\limits_{o}^{\pi} F_n(u)F_m(u) du = a_n \delta_{nm}\,, \tag{6}$$

that is, each set of functions is an orthogonal basis which spans an infinite dimensional space and both basis are, in addition, orthogonal. Here a_n is a normalization constant, which is given by

$$a_n = \begin{cases} \dfrac{2}{\pi}\ , & n > 0 \\ \dfrac{1}{\pi} & n = 0 \end{cases} \quad 1-\text{st soln., and } a_n = \begin{cases} \dfrac{2}{\pi}\ , & n > 0 \\ 0\ , & n = 0 \end{cases} \quad 2-\text{nd soln.} \tag{7}$$

Indeed, the first solution to this equation is $\cos(nu) = \cos(n\cos^{-1}x) \equiv T_n(x)$, the first class Chebyshev polynomials while the second solution is $\sin(nu) = \sin(n\cos^{-1}x) \equiv \sqrt{1-x^2}U_{n-1}(x)$, the second class Chebyshev polynomials. The properties of both class of polynomials are very well described in the literature. Of course, outside this interval, namely, in $[1,\infty]$ both solutions are well behaved at 1 both diverge as $x \to \infty$.

In this case, the solutions are hyperbolic functions which can be named as $\cosh(nv) = \cosh(n\cosh^{-1}x) \equiv S_n(x)$ and $\sinh(nv) = \sinh(n\cosh^{-1}x) \equiv V_n(x)$, now S_n and V_n are only linearly independent but not longer orthogonal. Both are well behaved at $x = 1$ but diverge as $x \to \infty$. In Figures 1 and 2, the behavior of this functions is drawn. From these figures, an opposite behavior of these functions, as compared with the second solution of Legendre equation, $Q_n(x)$, can be observed, that is the latter are irregular at $x = 1$, while the former are regular, the contrary occurs when $x \to \infty$.

2.2. Frobenius Series Representation for the Second Solution in $[1,\infty)$

As we mention at the beginning of this work, some physical problems which involve Chebyshev equation, demand that part of the solution goes to 0 as $x \to \infty$ (see for instance [6]), and it is clear that the solutions presented before do not have such a property, we have still the recourse of construct a Frobenius, series representation for the desired solution, in inverse powers of x. Note first that when $n = 0$, the solution of (2) is of the form

$$S_0 = \ln[x + \sqrt{x^2-1}], \tag{8}$$

which can be immediately obtained by direct integration. Of course, this function is irregular at $x = \infty$, as is the potential of an infinite line of charge in polar or elliptic coordinates. If we are compelled to solve Laplace equation in the latter coordinate system, this part of the solution is necessary in a physical problem involving such a charge distributions, but the rest $(n \neq 0)$, must goes to 0 as $x \to \infty$. With this fact in mind, a Frobenius series representation of the form

$$S_n(x) = \sum_{l=0}^{\infty} a_{-l} x^{k-l} \tag{9}$$

is proposed. Making the substitution of this expression and its derivatives in (2), we will have

$$\sum_{l=0}^{\infty} \left[(k-l)^2 - n^2\right] a_{-l} x^{k-l} - \sum_{l=2}^{\infty} (k+l+2)(k+l+1) a_{-l+2} x^{k-l} = 0, \tag{10}$$

and from it, the recurrence relations for the coefficients will appear:

$$\begin{gathered} a_0(k^2 - n^2) = 0\,, \quad a_{-1}\left[(k-1)^2 - n^2\right] = 0\,, \\ a_{-l}\left[(k-l)^2 - n^2\right] = (k+l+2)(k+l+1) a_{-l+2}\,, \quad \text{for } l \geq 2. \end{gathered} \tag{11}$$

From the secular equation (11) we will find the values of k; if we assume that

$$a_0 \neq 0,\ a_1 = 0,\ \text{then } k^2 - n^2 = 0 \text{ or } k = \pm n. \tag{12}$$

But we require that the function $S_n(x)$ vanishes as $x \to \infty$

$$k = -n,\ \text{with } n > 0.$$

Using (11), we can derive a compact expression for the coefficients:

$$a_{-2l} = \frac{n \cdot \prod_{s=l+1}^{2l-1} (n+s)}{2^{2l} l!} a_0,\ \text{with } l = 1, 2, 3, \cdots . \tag{13}$$

Functions of well-defined parity will then be built with the aid of these coefficients:

$$S_n^{\pm}(\xi) = a_0 x^{-n} \left\{ 1 + n \cdot \sum_{l=1}^{\infty} \frac{\prod_{s=l+1}^{2l-1} (n+s)}{2^{2l} l!} x^{-2l} \right\}, \tag{14}$$

where $+$ stands for n even and $-$ for n odd. As we mentioned before, in the special case when $n = 0$, the function $S_0(x)$ is solution to the differential equation

$$(x^2-1)^{1/2} \frac{d}{dx} \left[(x^2-1)^{1/2} \frac{dS_0(x)}{dx} \right] = 0, \tag{15}$$

where we have put it in self-adjoint form; this can be solved by direct integration and yields the function

$$S_0(x) = C \ln\left(x + \sqrt{x^2 - 1} \right). \tag{16}$$

We call then this set of functions the Chebyshev functions of 2-nd class that are solution of (2), and are defined by

$$S_n(x) = \begin{cases} a_0 \ln\left(x + \sqrt{x^2 - 1}\right), \text{ for } n = 0\,, \\ a_0 x^{-n}\left[1 + n \cdot \sum_{l=1}^{\infty} \frac{\Gamma(n+2l)(2x)^{-2l}}{\Gamma(n+l+1)\Gamma(l+1)}\right], \text{ for } n \geq 1\,. \end{cases} \tag{17}$$

In Figures 3 and 4 we show graphs of those functions for values of the index $n = 0, 1, 2, 3, 4$.

By using Gauss' test (see [2], p. 245) and (11) we can easily show that the series (14) converges at $x = 1$.

Finally, we consider necessary to point out that this method to obtain Chebyshev functions of 2-nd class is not unique; an alternative way to build those functions would involve the direct evaluation of the Wronskian and the Chebyshev polynomials of the 1-st class, as discussed by Arfken for the Legendre polynomials, see [2]. Both representations are compatible when calculated for $x > 1 + \varepsilon$, but the series form of the functions $S_n(x)$ is easier to implement in a numerical calculation, as that of Green's function in elliptic coordinates.

3. Final Remarks

In this work we have presented the second solution of Chebyshev equation in the intervals $[-1, 1]$ and $[1,\infty)$. In the former, both solutions constitute an orthogonal basis set which spans an infinite dimensional space of the Hilbert type, this property allows any continuous function in $[-1, 1]$ to be expressed as an infinite linear combination of these basis vector, this is a quite similar property a that of the Legendre polynomials. In general for the solutions of a Sturm-Liouville problem in which the second order differential operator can be casted in self-adjoint form. On the contrary of Legendre differential equation, the second solution in $[1, \infty)$ is not mentioned in standard textbooks on special functions perhaps because there were no need to use such a solution, but indeed, we have previously shown that some problems in electrostatics involving elliptic coordinates leads to Chebyshev-like equation both in $[-1, 1]$ and $[1, \infty)$, so the second solution like that presented here-in is needed to solve the problem. Moreover, in problems involving infinite lines of charge, part of the exterior solution needs to behave in a way that the analytic solution in no adequate and we need to construct a Frobenius series representation to account adequately the physics of the problem. There are, however, still a few more cases which in turn need such a solution, work is in progress to study such a problems and the results will be presented elsewhere.

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