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FRACTIONAL DARBOUX TRANSFORMATIONS FOR THE TIME-DEPENDENT SCHRÖDINGER EQUATION

Axel Schulze-Halberg

Department of Science
University of Colima
Bernal Diaz del Castillo 340, Col. Villas San Sebastian
Colima, Col., C.P. 28045, MEXICO
e-mail: xbat@ucol.mx

Abstract: We consider fractional Darboux transformations (FDT), recently defined for the stationary Schrödinger equation, and generalize them to a certain class of time-dependent Schrödinger equations (TDSE). The FDT is related to the well known Darboux transformation, but none of both transformations is a special case of the other one. We state the transformed solution of the TDSE and its transformed potential is closed form. Our results are finally generalized to TDSEs steming from Hamiltonians with linear terms in the momentum.

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1. Introduction

One of the long standing issues of mathematical physics is the search for exactly-solvable cases of quantum-mechanical equations, the most fundamental of which is the time-dependent Schrödinger equation (TDSE). It is well known that this equation admits exact solutions only for very few potentials, typical examples of which are the harmonic oscillator potential, see Pedrosa [9], the Coulomb potential, see Landau et al [8], the linear potential, see Feng [4] or a quadratic plus inverse quadratic potential, see Kaushal et al [7]. New exactly-solvable cases of the TDSE are usually obtained by application of local or non-local transforma-

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tions to a known exactly-solvable case. Local transformations, sometimes called point transformations or form-preserving transformations, consist in a change of the dependent and independent variables in the TDSE, preserving its form, but changing the potential term, see Ray [10] and Finkel et al [5]. The most famous example of a non-local transformation is the Darboux transformation, defined in Darboux [3] and Bagrov et al [1], which consists in applying a homogeneous differential operator to the solution of the TDSE, such that the transformed solution solves a TDSE different from the original one. The Darboux transformation is equivalent to the supersymmetric factorization method (SUSY), see Bagrov et al [2], and has been applied widely for generating exactly-solvable TDSEs.

Recently Humi [6] defined a new non-local transformation was found for the stationary Schrödinger equation. This new transformation is called fractional Darboux transformation (FDT), as it has the form of a ratio of two Darboux transformations. It is interesting to note that the Darboux transformation is not a special case of the FDT, both transformations are completely inequivalent, and therefore, new exactly-solvable cases of the stationary Schrödinger equation can be generated by means of the FDT. However, so far the FDT cannot be applied to the TDSE with fully time-dependent potentials, as it is defined only for stationary cases. In fact, existence of the FDT for the TDSE is not even guaranteed.

In the present note we show that a partial generalization is possible. For a certain class of TDSEs we are able to define an FDT and compute the transformed potential and the transformed solution in explicit form. In the preliminaries (Section 2) we summarize facts about the FDT for the stationary Schrödinger equation and shortly review local transformations for the TDSE. Section 3 contains a summary of our results, that are proved in Section 4. Section 5 contains an extension of the FDT to generalized TDSEs, steming from a Hamiltonian with linear terms in the momentum.

2. Preliminaries

The main purpose of this paper is to construct an FDT for a certain class of TDSEs. Our construction is mainly based on two recent results, which – for the sake of completeness – will be stated in the following sections. The first of these results taken from Humi [6] is the FDT for the stationary Schrödinger equation, while the second result from Finkel [5] establishes a connection between the stationary Schrödinger equation and certain TDSEs.

2.1. The FDT for the Stationary Schrödinger Equation

We shortly summarize form and basic properties of the FDT for the stationary Schrödinger equation. Afterwards, we show that the FDT can in general not be reduced to a DT.

2.1.1. Definition and Basic Properties of the FDT

We summarize results from Humi [6]. Consider the one-dimensional stationary Schrödinger equation

$$\frac{1}{2\ m}\ \psi'' + (E - V_0)\ \psi = 0, \tag{1}$$

where $\psi = \psi(x)$ is the wavefunction and $V_0 = V_0(x)$ represents the potential. Let h_1 and h_2 be solutions of the Schrödinger equation (1) for an energy $C \neq E$. Then the function

$$\hat{\psi} = \frac{-\frac{h_1'}{h_1}\ \psi + \psi'}{-\frac{h_2'}{h_2}\ \psi + \psi'}, \tag{2}$$

with $\hat{\psi} = \hat{\psi}(x)$, solves the stationary Schrödinger equation

$$\frac{1}{2\ m}\ \hat{\psi}'' + (E - V_1)\ \hat{\psi} = 0, \tag{3}$$

where the potential $V_1 = V_1(x)$ is given by

$$V_1 = V_0 + \frac{1}{2\ m}\left[\left(\frac{h_1'}{h_1}\right)^2 - 2\ \frac{h_1'\ h_2'}{h_1\ h_2} - \left(\frac{h_1'}{h_1}\right)'\ \right].$$

The transformation (2) is called FDT. The operator

$$D = \frac{-\frac{h_1'}{h_1} + \frac{d}{dx}}{-\frac{h_2'}{h_2} + \frac{d}{dx}},$$

the application of which is defined in (2), will be called fractional Darboux operator.

2.1.2. Relation Between FDT and DT

We will now show that the FDT and the DT are not related to each other, that is, we show that the DT is no special case of the FDT. Therefore, the particular form of the FDT as the ratio of two DTs is quite remarkable. Let us first recapitulate the definition of the DT for the stationary Schrödinger equation. It is well known, see Darboux [3] and Humi [6] that the first-order

DT for the stationary Schrödinger equation transforms the solution ψ of the stationary Schrödinger equation (1) according to

$$\hat{\psi} = L \left(-\frac{h_1'}{h_1} \; \psi + \psi' \right). \tag{4}$$

Here L is a constant, $h_1 = h_1(x)$ is a solution of the original equation (1), and $\hat{\psi} = \hat{\psi}(x)$ solves the stationary Schrödinger equation (3) with transformed potential

$$V_1 = V - \frac{1}{m} \left(\frac{h_1'}{h_1} \right)'.$$

Now let us compare the FDT (2) to the DT (4). We will now see that the FDT is not a special case of the DT, but a separate transformation. If the DT were a special case of the FDT, then in (2) there would be a choice for the function h_2, such that the denominator of (2) becomes a constant. Thus, we are looking for a solution h_2 of the equation

$$-\frac{h_2'}{h_2} \; \psi + \psi' = \frac{1}{L}, \tag{5}$$

where L is an arbitrary constant. We will now show that the latter equation can in general not hold. To this end, we make use of the fact that h_2 is a solution of the stationary Schrödinger equation:

$$\frac{1}{2\ m} \; h_2'' + (C - V_0) \; h_2 = 0. \tag{6}$$

Now by means of (5), we can compute h_2'':

$$\begin{aligned} h_2' &= -h_2 \left(-\frac{\psi}{\psi} + \frac{1}{L\ \psi} \right) \\ \Rightarrow h_2'' &= -h_2' \left(-\frac{\psi}{\psi} + \frac{1}{L\ \psi} \right) - h_2 \left(-\frac{\psi''}{\psi} + \left(\frac{\psi'}{\psi} \right)^2 - \frac{\psi'}{L\ \psi^2} \right). \end{aligned} \tag{7}$$

On inserting (7) into the latter expression, we get

$$h_2'' = h_2 \left(\frac{\psi''}{\psi} + \left(\frac{1}{L\ \psi} \right)^2 - \frac{\psi'}{L\ \psi^2} \right).$$

Since h_2 satisfies the stationary Schrödinger equation (6), we have

$$\frac{1}{2\ m} \left(\frac{\psi''}{\psi} + \left(\frac{1}{L\ \psi} \right)^2 - \frac{\psi'}{L\ \psi^2} \right) = V_0 - C.$$

But ψ solves the stationary Schrödinger equation (1), which gives, together with the last equation, the following constraint:

$$\frac{1}{2\ m}\left(2\ m\ (V_0 - E) + \left(\frac{1}{L\ \psi}\right)^2 - \frac{\psi'}{L\ \psi^2}\right) = V_0 - C$$
$$\Leftrightarrow \left(\frac{1}{L\ \psi}\right)^2 - \frac{\psi'}{L\ \psi^2} = 2\ m\ (E - C)$$
$$\Leftrightarrow \psi' + 2\ m\ L\ (E - C)\ \psi^2 - \frac{1}{L} = 0.$$

Derivation and division by ψ gives

$$\frac{\psi''}{\psi} = -2\ L\ (E - C)\ \psi'. \tag{8}$$

Again we make use of the fact that ψ solves (1), and convert equation (8) to

$$\psi = \frac{m}{L\ (C - E)} \int (V_0 - E)\ dx,$$

which, due to the linearity of the stationary Schrödinger equation implies that $\int (V_0 - E)\ dx$ also solves (1), which is in general not true. This finally implies that the FDT can in general not be reduced to a DT.

2.2. The Form-Preserving Transformation

The next tool that we will need is the FPT between TDSEs, the results about which we take from Finkel [5]. Let us consider the TDSE with constant mass m and potential $V = V(x, t)$:

$$i\ \Psi_t + \frac{1}{2\ m}\ \Psi_{xx} - V\ \Psi = 0. \tag{9}$$

We now state the most general change of variables, such that the TDSE (9) maintains its form except for the potential. For obvious reasons, such a change of variables is called FPT. Thus, we transform the solution Ψ of the TDSE (9) as follows:

$$\Psi(x, t) = \exp(f(x, t))\ \Phi(u(x, t), v(t)), \tag{10}$$

with

$$u = \sqrt{v'}\ x + k\,, \tag{11}$$
$$v = \text{arbitrary}\,, \tag{12}$$
$$f = -i\ m\ \left(\frac{v''}{4\ v'}\ x^2 + \frac{k'}{\sqrt{v'}}\ x\right) + i\ c, \tag{13}$$

where $v = v(t)$, $k = k(t)$, and $c = c(t)$ are arbitrary functions. After the transformation, the TDSE (9) is converted into a TDSE for Φ:

$$i\ \Phi_v + \frac{1}{2\ m}\ \Phi_{uu} - \hat{V}\ \Phi = 0, \tag{14}$$

where $\hat{V}$ is of the form

$$\hat{V} = \left[\frac{V}{v'} + G_2\ x^2 + G_1\ x + G_0\right]\Bigg|_{\substack{x = x(u,v)\\ t = t(v)}} . \tag{15}$$

Here $G_0 = G_0(t)$, $G_1 = G_1(t)$ and $G_2 = G_2(t)$ depend on m, k, and v:

$$G_2 = -\frac{m}{v'}\left(\frac{v'''}{4\ v'} - \frac{3\ (v'')^2}{8\ (v')^2}\right), \tag{16}$$

$$G_1 = -\frac{m\ k''}{(v')^{\frac{3}{2}}} + \frac{m\ v''\ k'}{(v')^{\frac{5}{2}}}, \tag{17}$$

$$G_0 = \frac{m}{2}\frac{k'}{(v')^2} + i\ \frac{v''}{4\ (v')^2} - c' . \tag{18}$$

Hence, under the FPT the original potential V in (9) is reproduced (up to a time-dependent function k added to the spatial variable), and a quadratic term is added. In Section 4.1 we will discuss a particular case of the FPT (10)-(13), such that the transformed potential (15) is stationary in the new coordinates u and v, that is, it does not depend on v.

2.3. Mapping the TDSE onto a Stationary Schrödinger Equation

The FPT (10)-(13) can be used to map time-dependent potentials onto stationary potentials and vice versa. There is a criterion on the solution of the TDSE, such that the latter is transformed into a TDSE with stationary potential, see Finkel [5]. Since the potential of a TDSE is determined by its solution, in the following we give a re-formulated version of the criterion in Finkel [5], imposing a constraint on the potential instead on the solution:

Criterion. Consider the TDSE

$$i\ \Psi_t + \frac{1}{2\ m}\ \Psi_{xx} - V\ \Psi = 0, \tag{19}$$

where the potential V is given by

$$V = \alpha\ x^2 + \beta\ x + \gamma + \exp\left(\frac{4}{m}\int A\ dt\right)\ G. \tag{20}$$

Here the following notation has been used:

$$\alpha = A' - \frac{2\ A^2}{m}, \quad \beta = B' - \frac{2\ A\ B}{m}, \quad \gamma = -\frac{B^2}{2\ m},$$

$$G = G\left[\exp\left(\frac{2}{m}\int A\ dt\right)\ x + \frac{1}{m}\int \exp\left(\frac{2}{m}\int^t A\ dt'\right)\ B\ dt\right] . \tag{21}$$

The functions $A = A(t)$, $B = B(t)$, and G are arbitrary. Then the FPT (10)-(13) with the following parameters

$$v = \int \exp\left(\frac{4}{m} \int^{t} A \; dt'\right) \; dt \,, \tag{22}$$

$$k = \frac{1}{m} \int \exp\left(\frac{2}{m} \int^{t} A \; dt'\right) \; B \; dt, \tag{23}$$

rendering (11) and (13) in the form

$$u = \exp\left(\frac{2}{m} \int A \; dt\right) \; x + \frac{1}{m} \int \exp\left(\frac{2}{m} \int^{t} A \; dt'\right) \; B \; dt \,, \tag{24}$$

$$f = -i \; \left(A \; x^2 + B \; x\right) - \frac{1}{m} \int A \; dt, \tag{25}$$

transforms the TDSE (19) with potential (20) into the TDSE with stationary potential

$$i \; \Phi_v + \frac{1}{2 \; m} \; \Phi_u - G \; \Phi = 0.$$

Thus, on setting

$$\Phi(u,t) = \exp(-i \; E \; t) \; \psi(u),$$

the function ψ solves the stationary Schrödinger equation

$$\frac{1}{2 \; m} \; \psi'' + (E - G) \; \psi = 0.$$

Thus, if the potential of the TDSE can be written in the form (20), then it can be mapped onto a stationary potential.

Proof. We prove this criterion by simply computing the transformed potential (15), employing the settings (22)-(25). Due to the length of the expressions involved, let us perform the calculation in several parts. We find

$$-\frac{m}{v'} \left(\frac{v'''}{4 \; v'} - \frac{3 \; (v'')^2}{8 \; (v')^2}\right) = \frac{2 \; A^2}{m} - A' \,,$$

$$-\frac{m \; k''}{(v')^{\frac{3}{2}}} + \frac{m \; v'' \; k'}{(v')^{\frac{5}{2}}} = \frac{2 \; A \; B}{m} - B' \,,$$

$$\frac{m}{2} \frac{k'}{(v')^2} + i \; \frac{v''}{4 \; (v')^2} \; - c' = \frac{B^2}{2 \; m} + i \; \frac{A}{m} - c'.$$

We remove the imaginary term in the last row by setting

$$c = \frac{i}{m} \int A \ dt.$$

On comparing the last results with the functions α, β and γ in (20), we find

$$-\frac{m}{v'} \left(\frac{v'''}{4 \ v'} - \frac{3 \ (v'')^2}{8 \ (v')^2} \right) = -\alpha \,, \tag{26}$$

$$-\frac{m \ k''}{(v')^{\frac{3}{2}}} + \frac{m \ v'' \ k'}{(v')^{\frac{5}{2}}} = -\beta \,, \tag{27}$$

$$\frac{m}{2} \frac{k'}{(v')^2} + i \ \frac{v''}{4 \ (v')^2} \ - i \ \frac{A}{m} = -\gamma. \tag{28}$$

Now we insert the original potential (20) into the transformed potential (15). On taking into account (26)-(28) and the definition of v in (22), we get

$$U = \frac{1}{v'} \ \exp\left(\frac{4}{m} \int A \ dt \right) \ G = G. \tag{29}$$

If we now write the latter potential in the new coordinate u, we infer from the definition of u in (24) and from (21) that

$$G = G \left[\exp\left(\frac{2}{m} \int A \ dt \right) \ x + \frac{1}{m} \int \exp\left(\frac{2}{m} \int A \ dt \right) \ B \ dt \right] = G(u).$$

Therefore, G depends only on the spatial variable u, but not on t. Hence, the potential (29) becomes a stationary potential and the proof of our criterion is complete. □

3. The FDT for the TDSE

In this section we summarize our results, which will be proved in Section 4. The FDT for the stationary Schrödinger equation will be generalized, such that it becomes applicable to TDSEs for a certain class of potentials.

3.1. The Original TDSE

Consider the TDSE in $(1+1)$ dimensions

$$i \ \Psi_t + \frac{1}{2 \ m} \ \Psi_{xx} - V \ \Psi = 0, \tag{30}$$

with constant mass m and potential $V = V(x,t)$. Assume that the potential V is of the following form:

$$V = \alpha\ x^2 + \beta\ x + \gamma + \exp\left(\frac{4}{m}\int A\ dt\right)\ G, \tag{31}$$

where the following notation has been used:

$$\begin{aligned}
\alpha &= A' - \frac{2\ A^2}{m}, \\
\beta &= B' - \frac{2\ A\ B}{m}, \\
\gamma &= -\frac{B^2}{2\ m}, \\
G &= G\left[\exp\left(\frac{2}{m}\int A\ dt\right)\ x + \frac{1}{m}\int \exp\left(\frac{2}{m}\int^t A\ dt'\right)\ B\ dt\right].
\end{aligned} \tag{32}$$

The functions $A = A(t)$, $B = B(t)$ and G are arbitrary. Furthermore, note that G is a function of the argument given in square brackets in (32). If the potential of a TDSE has the form (31), then we can define a FDT, as will be proved in Section 4.

3.2. The FDT and the Transformed TDSE

Let g_1 and g_2 be solutions of the TDSE (30) for a potential of the form (31). Define the FDT $\mathcal{D}(\Psi)$ for the solution Ψ of the TDSE (30) as

$$\mathcal{D}(\Psi) = \exp\left(-i\ \left(A\ x^2 + B\ x\right) - \frac{1}{m}\int A\ dt\right)\ \frac{-\frac{(g_1)_x}{g_1}\ \Psi + \Psi_x}{-\frac{(g_2)_x}{g_2}\ \Psi + \Psi_x}. \tag{33}$$

Then the function $\Phi = \mathcal{D}(\Psi)$ solves the TDSE

$$i\ \Phi_t + \frac{1}{2\ m}\ \Phi_{xx} - \hat{V}\ \Phi = 0,$$

with transformed potential

$$\begin{aligned}
\hat{V} = A'\ x^2 + B'\ x + \exp\left(\frac{4}{m}\int A\ dt\right)\ G - i\ \frac{A}{m} + \frac{1}{2\ m}\Bigg[\left(\frac{(g_1)_x}{g_1}\right)^2 \\
- \left(\frac{(g_1)_x}{g_1}\right)_x - 2\ \frac{(g_1)_x\ (g_2)_x}{g_1\ g_2} - 2\ i\ \frac{(g_2)_x}{g_2}\,(2\ A\ x + B)\Bigg].
\end{aligned} \tag{34}$$

As in the case of the stationary Schrödinger equation, the FDT for the TDSE does not contain the DT a a special case.

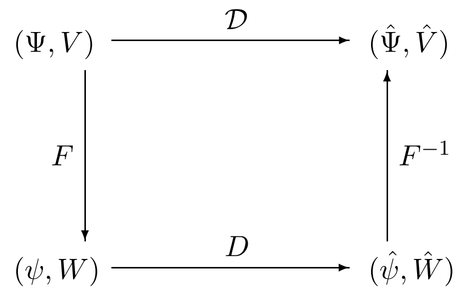

Figure 1: Definition of the fractional Darboux operator $\mathcal{D}$

4. Derivation of the FDT

We will now prove the results stated in the last section. To this end, we first give an outline of our construction, then discuss a special case of the FPT developed in Section 2.2, and finally compute the Darboux operator and the transformed potential.

4.1. Outline of the Construction

We now define the FDT for the TDSE by means of the construction displayed in the following diagram, which shall be explained now. First consider the four pairs in each corner of the diagram. The first component of a pair contains the solution of a Schrödinger equation, while the second component contains the corresponding potential. In (Ψ, V), the function Ψ solves the TDSE (19) for the potential V as given in (20). Our purpose is to define a fractional Darboux transformation $\mathcal{D}$ that takes (Ψ, V) onto $(\hat{\Psi}, \hat{V})$. To this end, we employ a function F that converts (Ψ, V) into a solution ψ of the stationary Schrödinger equation with potential W. To this stationary Schrödinger equation the conventional fractional Darboux transformation is applicable, yielding the transformed solution $\hat{\psi}$ and the transformed potential $\hat{W}$. Finally, the inverse F^{-1} converts $\hat{\psi}$ back into a solution of the TDSE, while the stationary potential $\hat{W}$ is mapped onto the time-dependent potential $\hat{V}$. In summary, we define the fractional Darboux operator $\mathcal{D}$ for the TDSE as

$$\mathcal{D} = F^{-1} \circ D \circ F. \tag{35}$$

Let us now evaluate the latter formula explicitly.

4.2. The Fractional Darboux Operator

First we note that (35) is well-defined, as F converts the TDSE into a stationary Schrödinger equation, which permits application of the fractional Darboux operator D. The function F is given by a combination of the FPT (10)-(13) with the parameters (22)-(24) and, as a replacement of (25), the following modified function f:

$$f = -i \left(A\ x^2 + B\ x\right) - \frac{1}{m} \int A\ dt + i\ E \int \exp\left(\frac{4}{m} \int^t A\ dt'\right) dt, \qquad (36)$$

where the last term is equivalent to $i\ E\ v$ with v as in (22). Note that this term was added to convert the TDSE with a stationary potential into a stationary Schrödinger equation. We now calculate the explicit form of the operator (35). To this end, let Ψ be a solution of the TDSE (19), and let us abbreviate $\hat{\psi} = D(\psi)$ and $\hat{\Psi} = F^{-1}(\hat{\psi})$. By means of (35) we then find

$$\mathcal{D}(\Psi) = F^{-1} \circ D \circ F(\Psi) = F^{-1} \circ D \left(\exp(-\tilde{f})\ \tilde{\psi}\right), \qquad (37)$$

where we set $f(x,t) = \tilde{f}(u,t)$ and $\psi(x,t) = \tilde{\psi}(u)$, because F involves the change of coordinates $x \rightarrow u(x)$ as given in (24). We evaluate the FDT in (37):

$$D \left(\exp(-\tilde{f})\ \tilde{\psi}\right) = \frac{-\frac{\tilde{h}_1'}{\tilde{h}_1}\ \tilde{\psi} + \tilde{\psi}'}{-\frac{\tilde{h}_2'}{\tilde{h}_2}\ \tilde{\psi} + \tilde{\psi}'}. \qquad (38)$$

We now rewrite all functions carrying a tilde in the original coordinates (x,t). The derivative changes as follows:

$$\tilde{\psi}' = \psi_x\ x_u + \psi_t \underbrace{t_u}_{=0} = \frac{1}{u_x}\ \psi_x = \exp\left(-\frac{2}{m} \int A\ dt\right)\ \psi_x, \qquad (39)$$

and in the same way the derivatives of $\tilde{h}_1$ and $\tilde{h}_2$ change. In the following we will not use the explicit form of u_x as given in (39), but instead just write u_x. On taking into account (39), we can rewrite (38) as

$$D \left(\exp(-\tilde{f})\ \tilde{\psi}\right) = \left(-\frac{\frac{1}{u_x}\ (h_1)_x}{h_1}\ \psi + \frac{1}{u_x}\ \psi_x\right) \left(-\frac{\frac{1}{u_x}\ (h_2)_x}{h_2}\ \psi + \frac{1}{u_x}\ (h_1)_x\right)^{-1}$$
$$= \frac{-\frac{(h_1)_x}{h_1}\ \psi + \psi_x}{-\frac{(h_2)_x}{h_2}\ \psi + \psi_x}. \qquad (40)$$

Now we continue the evaluation of $\mathcal{D}$ in (37) by applying F^{-1} to (40):

$$\mathcal{D}(\Psi) = F^{-1}\left(\frac{-\frac{(h_1)_x}{h_1}\ \psi + \psi_x}{-\frac{(h_2)_x}{h_2}\ \psi + \psi_x}\right) = \exp(f)\ \frac{-\frac{(h_1)_x}{h_1}\ \psi + \psi_x}{-\frac{(h_2)_x}{h_2}\ \psi + \psi_x}. \tag{41}$$

This is not the final form of the Darboux operator $\mathcal{D}$, because the functions h_1, h_2 and ψ are not solutions of the original TDSE (19), but solve a stationary Schrödinger equation. But the latter stationary Schrödinger equation and the original TDSE are related by the function F, that is,

$$\psi = \exp(-f)\ \Psi\,, \quad h_1 = \exp(-f)\ g_1\,, \quad h_2 = \exp(-f)\ g_2,$$

where g_1 and g_2 are solutions of the original TDSE (19). By substituting these three functions into (41), we obtain

$$\begin{aligned}\mathcal{D}(\Psi) &= \exp(f)\ \left(-\frac{(\exp(-f)\ g_1)_x}{\exp(-f)\ g_1}\ \exp(-f)\ \Psi + (\exp(-f)\ \Psi)_x\right)\\ &\times\left(-\frac{(\exp(-f)\ g_2)_x}{\exp(-f)\ g_2}\ \exp(-f)\ \Psi + (\exp(-f)\ \Psi)_x\right)^{-1} = \exp(f)\\ &\times\left(-\frac{-\exp(-f)f_x\ g_1 + \exp(-f)(g_1)_x}{\exp(-f)g_1}\ \exp(-f)\ \Psi - \exp(-f)f_x\ \Psi\right.\\ &\left.+\exp(-f)\Psi_x\right) \times \left(-\frac{-\exp(-f)f_x\ g_2 + \exp(-f)(g_2)_x}{\exp(-f)g_2}\ \exp(-f)\Psi\right.\\ &\left.-\exp(-f)f_x\ \Psi + \exp(-f)\Psi_x\right)^{-1} = \exp(f)\ \frac{-\frac{(g_1)_x}{g_1}\ \Psi + \Psi_x}{-\frac{(g_2)_x}{g_2}\ \Psi + \Psi_x}.\end{aligned} \tag{42}$$

This is the desired explicit form of the fractional Darboux operator, where the function f can be found in (36).

4.3. Calculation of the Transformed Potential

We now calculate the potential of the transformed TDSE. To this end, let V be the original potential in the TDSE (19), associated with the solution Ψ. The explicit form of V can be found in (20):

$$V = \alpha\ x^2 + \beta\ x + \gamma + \exp\left(4\ \int \frac{A}{m}\ dt\right)\ G.$$

Now, according to the criterion in Section 2.3 and Figure 1, after application of the transformation F, the potential V is transformed into the stationary potential W, given by

$$W = G.$$

Next, the FDT transforms the potential W into the potential $\hat{W}$, given by

$$\hat{W} = G + \frac{1}{2\,m}\left[\left(\frac{\tilde{h}_1'}{\tilde{h}_1}\right)^2 - 2\,\frac{\tilde{h}_1'\,\tilde{h}_2'}{\tilde{h}_1\,\tilde{h}_2} - \left(\frac{\tilde{h}_1'}{\tilde{h}_1}\right)'\right]. \tag{43}$$

Finally, application of the transformation F^{-1} yields the following potential $\hat{V}$:

$$\hat{V} = V + v'\,\hat{W} = V + v'\,\frac{1}{2\,m}\left[\left(\frac{\tilde{h}_1'}{\tilde{h}_1}\right)^2 - 2\,\frac{\tilde{h}_1'\,\tilde{h}_2'}{\tilde{h}_1\,\tilde{h}_2} - \left(\frac{\tilde{h}_1'}{\tilde{h}_1}\right)'\right]. \tag{44}$$

We need to rewrite this expression in two ways: the derivative of the functions $\tilde{h}_1$ and $\tilde{h}_2$ must be converted into a derivative with respect to the coordinates x, t; furthermore, $\tilde{h}_1$ and $\tilde{h}_2$ have to be represented by solutions of the original TDSE (19). We set $\tilde{h}_1(u) = h_1(x,t)$, $\tilde{h}_2(u) = h_2(x,t)$, and obtain for $j = 1, 2$:

$$\tilde{h}_j' = (h_j)_x\,x_u + (h_j)_t\,\underbrace{t_u}_{=0} = \frac{1}{u_x}\,(h_j)_x.$$

Consequently,

$$\frac{\tilde{h}_j'}{\tilde{h}_j} = \frac{(h_j)_x}{u_x\,h_j} \tag{45}$$

and

$$\left(\frac{\tilde{h}_j'}{\tilde{h}_j}\right)' = \frac{1}{u_x}\,\left(\frac{(h_j)_x}{u_x\,h_j}\right)_x. \tag{46}$$

Next, we know that a solution h_j of the stationary Schrödinger equation is related to a solution g_j of the original TDSE (19) via the transformation F, that is, for $j = 1, 2$ we have

$$h_j = \exp(-f)\,g_j. \tag{47}$$

By inserting the latter into (45) we obtain

$$\begin{aligned}\frac{\tilde{h}_j'}{\tilde{h}_j} &= \frac{(\exp(-f)\,g_j)_x}{u_x\,\exp(-f)\,g_j}\\ &= \frac{-\exp(-f)\,f_x\,g_j + \exp(-f)\,(g_j)_x}{u_x\,\exp(-f)\,g_j} = \frac{1}{u_x}\,\left(-f_x + \frac{(g_j)_x}{g_j}\right).\end{aligned} \tag{48}$$

We substitute (47) also in (46), giving

$$\left(\frac{\tilde{h}_j'}{\tilde{h}_j}\right)' = \frac{1}{u_x}\,\left(\frac{1}{u_x}\,\left(-f_x + \frac{(g_j)_x}{g_j}\right)\right)_x$$

$$= \frac{1}{u_x}\left[-\underbrace{\frac{u_{xx}}{u_x^2}}_{=0}\left(-f_x+\frac{(g_j)_x}{g_j}\right)+\frac{1}{u_x}\left(-f_{xx}+\left(\frac{(g_j)_x}{g_j}\right)_x\right)\right]$$

$$= -\frac{f_{xx}}{u_x^2}+\frac{1}{u_x^2}\left(\frac{(g_j)_x}{g_j}\right)_x. \tag{49}$$

Now we are ready to evaluate the transformed potential (44). On inserting (48) and (49) into (44), and taking into account that $v' = 1/u_x^2$ we arrive at

$$\hat{V} = V + v' \frac{1}{2\,m}\left[\frac{1}{u_x^2}\left(-f_x+\frac{(g_1)_x}{g_1}\right)^2 - 2\,\frac{1}{u_x^2}\left(-f_x+\frac{(g_1)_x}{g_1}\right)\right.$$
$$\times\left.\left(-f_x+\frac{(g_2)_x}{g_2}\right)+\frac{f_{xx}}{u_x^2}-\frac{1}{u_x^2}\left(\frac{(g_1)_x}{g_1}\right)_x\right] = V + \frac{1}{2\,m}\left[\left(-f_x+\frac{(g_1)_x}{g_1}\right)^2\right.$$
$$\left. -2\left(-f_x+\frac{(g_1)_x}{g_1}\right)\left(-f_x+\frac{(g_2)_x}{g_2}\right)+f_{xx}-\left(\frac{(g_1)_x}{g_1}\right)_x\right] = V + \frac{1}{2\,m}$$
$$\times\left[f_{xx}-f_x^2+2\,f_x\,\frac{(g_2)_x}{g_2}-2\,\frac{(g_1)_x\,(g_2)_x}{g_1\,g_2}+\left(\frac{(g_1)_x}{g_1}\right)^2-\left(\frac{(g_1)_x}{g_1}\right)_x\right]. \tag{50}$$

If we substitute the explicit forms of V and f as given in (20) and (25), respectively, we obtain

$$\hat{V} = A'\,x^2 + B'\,x + \exp\left(\frac{4}{m}\int A\,dt\right)\,G - i\,\frac{A}{m}+\frac{1}{2\,m}$$
$$\times\left[\left(\frac{(g_1)_x}{g_1}\right)^2-\left(\frac{(g_1)_x}{g_1}\right)_x-2\,\frac{(g_1)_x\,(g_2)_x}{g_1\,g_2}-2\,i\,\frac{(g_2)_x}{g_2}\,(2\,A\,x+B)\right]. \tag{51}$$

This is the final form of the transformed potential, coinciding with (34).

5. Generalized Hamiltonians

In this section we want to extend the definition of the FDT to TDSEs of a more general type, corresponding to Hamiltonians with additional linear terms. For such Hamiltonians, resp. for their corresponding TDSEs, Darboux transformations, see Klauder et al [12] and FPTs, see Schulze-Halberg [11], have already been defined.

5.1. The Hamiltonian with Linear Terms and its TDSE

A Hamiltonian with additional linear terms we define as follows:

$$H = \frac{1}{2\ m}\ (p + R)^2 + U, \tag{52}$$

where m is the constant mass, p is the momentum operator, $R = R(x,t)$ is an arbitrary, real-valued function, and $U = U(x,t)$ represents the potential. In three spatial dimensions, the function R can be seen as a vector potential, but since we work in one spatial dimension, this interpretation cannot be applied here. The time-dependent Schrödinger equation associated with the Hamiltonian (52) reads

$$i\ \chi_t - \left(\frac{1}{2\ m}\ (p + R)^2 + U \right)\ \chi = 0.$$

Here $\chi = \chi(x,t)$ denotes the wave function. Now, on substituting the momentum operator $p = -i\partial_x$ into the lattter equation and evaluating the derivatives, we finally arrive at the equation

$$i\ \chi_t + \frac{1}{2\ m}\ \chi_{xx} + i\ \frac{R}{m}\ \chi_x + \left(i\ \frac{R_x}{2\ m} - \frac{R^2}{2m} - U \right)\ \chi = 0. \tag{53}$$

This generalized TDSE is related to the conventional TDSE by the invertible transformation

$$\chi = \exp\left(-i \int R\ dx \right)\ \Psi, \tag{54}$$

that converts (53) into the following equation for $\Psi = \Psi(x,t)$:

$$i\ \Psi_t + \frac{1}{2\ m}\ \Phi_{xx} + \left(\int R_t\ dx - U \right)\ \Psi = 0, \tag{55}$$

which is the conventional form of a TDSE with potential

$$V = -\int R_t\ dx + U. \tag{56}$$

5.2. The Fractional Darboux Operator

In the last section we have defined the FDT for conventional TDSEs with potential (20). Therefore, the FDT is applicable to the TDSE (55) with potential

$$U \quad = \quad -\int R_t\ dx + \alpha\ x^2 + \beta\ x + \gamma + \exp\left(\frac{4}{m} \int A\ dt \right)\ G, \tag{57}$$

where the notation is the same as in (20). Hence, for the TDSE (53) with potential of the form (57) we can define an FDT as in the previous section. The corresponding fractional Darboux operator will be assigned the symbol

$\mathcal{D}_R$. Let us first consider the fractional Darboux operator (42) for conventional TDSEs:

$$\mathcal{D}(\Psi) = \exp(f) \; \frac{-\frac{(g_1)_x}{g_1} \; \Psi + \Psi_x}{-\frac{(g_2)_x}{g_2} \; \Psi + \Psi_x}. \tag{58}$$

We have to modify the following:

— The argument Ψ and the parameters g_1 and g_2 of $\mathcal{D}$ should be solutions of (53), whereas now they solve (55).

— The function $\mathcal{D}(\Psi)$ solves a TDSE of conventional form, whereas it is supposed to solve a TDSE of the form (53).

These two points we address as follows: we substitute

$$\begin{aligned} \Psi &= \exp\left(i \int R \; dx\right) \; \chi \,, \\ g_j &= \exp\left(i \int R \; dx\right) \; k_j, \quad j = 1, 2 \,, \\ \mathcal{D}_R(\chi) &= \exp\left(-i \int R \; dx\right) \; \mathcal{D}(\Psi). \end{aligned} \tag{59}$$

where $k_1 = k_1(x,t)$ and $k_2 = k_2(x,t)$, into (58) and simplify the resulting expression. Since the calculation is very similar to the calculation that led to (42), we just state the result:

$$\mathcal{D}_R(\chi) = \exp\left(-i \int R \; dx + f\right) \; \frac{-\frac{(k_1)_x}{k_1} \; \chi + \chi_x}{-\frac{(k_2)_x}{k_2} \; \chi + \chi_x}, \tag{60}$$

where f can be found in (36). Thus, (60) is the fractional Darboux operator for the TDSE (53).

5.3. The Transformed Potential

We come to the calculation of the transformed potential. As in the case of the solution, we take the transformed potential (51) resulting from the FDT for the conventional TDSE, and replace g_1, g_2 according to (59). Furthermore, the original potential V in (50) has to be replaced by (56). On using

$$-\frac{(g_j)_x}{g_j} \;\; = \;\; -i \; R - \frac{(k_j)_x}{k_j}, \quad j = 1, 2,$$

we obtain

$$\hat{U} = -\int R_t \; dx + A' \; x^2 + B' \; x + \exp\left(\frac{4}{m} \int A \; dt\right) \; G - i \; \frac{A}{m} +$$

$$+\frac{1}{2\ m}\Bigg[\left(\frac{(k_1)_x}{k_1}\right)^2-\left(\frac{(k_1)_x}{k_1}\right)_x-2\ \frac{(k_1)_x\ (k_2)_x}{k_1\ k_2}+R^2+2\ B\ R+4\ A\ R\ x- \\ -i\left(\frac{4\ A\ (k_2)_x}{k_2}\ x+\frac{2\ B\ (k_2)_x}{k_2}+\frac{2\ R\ (k_2)_x}{k_2}+R_x\right)\Bigg].$$

This is the transformed potential in its final form.

6. Concluding Remarks

We have generalized the non-local FDT to a certain class of TDSEs, thus providing a tool for the generation of new exactly-solvable cases of the TDSE. It should be pointed out that so far only the first-order FDT has been studied (first order refers to the order of the Darboux transformations in numerator and denominator of the FDT). Therefore, a few interesting questions on higher order FDTs arise naturally, for example, whether the iterated FDT can be given in closed form, as it is true for the Darboux transformation. Another open problem is the factorization properties of higher-order FDT, which will be investigated in forthcoming work.

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