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ON THE VERTEX-DISTINGUISHING TOTAL COLORING OF $P_m \vee F_n$

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Abstract: Let $G(V, E)$ be a simple graph, f be a mapping from $V(G) \cup E(G)$ to $\{1, 2, \cdots, k\}$. Let $C_f(v) = \{f(v)\} \cup \{f(vw)|w \in V(G), vw \in E(G)\}$ for every $v \in E(G)$. If f is a k-proper-total-coloring, and for $\forall u, v \in V(G)$, we have $C_f(u) \neq C_f(v)$, then f is called the k-vertex-distinguishing total coloring (k-VDEC for short). Let $\chi'_{vt}(G) = \min\{k|$G has a k-vertex-distinguishing total coloring$\}$. Then $\chi'_{vt}(G)$ is called the vertex-distinguishing total chromatic number. The total chromatic numbers on $P_m \vee F_n$ are presented in this paper.

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The graphs considered in this paper are connected, finite, undirected and simple graph.

A k-proper-total-coloring of a graph G is a mapping f from $V(G) \cup E(G)$ to $\{1, 2, \cdots, k\}$ such that:

1) $\forall uv \in E(G)$, then $f(u) \neq f(v)$, $f(u) \neq f(uv) \neq f(v)$;
2) $\forall uv, uw \in E(G), v \neq w$, then $f(uv) \neq f(uw)$;
3) $\forall u, v \in V(G)$, then $C(u) \neq C(v)$.

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Let f be a k-proper-total coloring of G. Let $C_f(u) = \{f(u)\} \cup \{f(uw)|w \in V(G), uw \in E(G)\}$ and $\overline{C}_f(u) = \{1, 2, \cdots, k\} - C_f(u)$ for every $u \in V(G)$. If $\forall u, v \in V(G)$, we have $C_f(u) \neq C_f(v)$, i.e. $\overline{C}_f(u) \neq \overline{C}_f(v)$, then f is called a k-vertex-distinguishing total coloring (k-VDTC for short).

The number $\min\{k|G$ has a k-vertex-distinguishing total coloring$\}$ is called the vertex-distinguishing total chromatic number and denoted by $\chi'_{vt}(G)$.

Definition 1. Suppose G and H are two simple graphs with $V(G) \cap V(H) = \emptyset$,

$$V(G \vee H) = V(G) \cup V(H),$$
$$E(G \vee H) = E(G) \cup E(H) \cup \{uv|u \in V(G), v \in V(H),$$

then $G \vee H$ is called join-graph of G and H.

Definition 2. For a graph G, n_i denotes the number of vertex which has degree i, δ, Δ denote the minimum, maximum degree of G respectively, then

$$\mu(G) = \max\{k|k = \min\{\lambda|\tbinom{\lambda}{d_i+1} \geq n_i, \delta \leq d_i \leq \Delta\}\}$$

is called total combinatorial degree of G.

For vertex-distinguishing total chromatic number, we will give a conjecture as follow:

Conjecture 1. *For every graph G with at least order* 2*, we have*

$$\mu(G) \leq \chi'_{vt}(G) \leq \mu(G) + 1.$$

In this paper, we obtain the vertex-distinguishing chromatic number of $P_m \vee F_n$. For the graph-theoretic terminology the reader is referred to [1], [2].

Lemma 1. *For join-graph of* $P_m \vee F_n$*,*

$$\mu(G) = \begin{cases} 7, & m = 3,\ n = 2; \\ m+4, & m > 3,\ n = 2; \\ 8, & m = 3,\ n = 3; \\ n+5, & m = 3,\ n > 3; \\ m+5, & m > 3,\ n = 3; \\ m+n+1, & m > 3,\ n > 3. \end{cases}$$

Theorem 1. *Let* P_m *be a path with order m and* F_n *be a fan with order* n*, then*

$$\chi'_{vt}(P_m \vee F_n) = \mu(G).$$

Proof. Let $V(P_m) = \{u_1, u_2, \cdots, u_m\}, V(F_n) = \{w\} \cup \{v_1, v_2, \cdots, v_n\}$, there are six cases to be discussed also.

Case 1. When $m = 3, n = 2$, according to $F_2 = C_3$, so let $V(P_3) = \{u_1, u_2, u_3\}$ and $V(F_2) = \{v_1, v_2, v_3\}$. $\mu(P_3 \vee F_2) = 7$ by Lemma 1, in order to prove the conclusion is true, we only need to prove that $P_3 \vee F_2$ exists 7-VDEC. So we construct a map f from $V(P_3 \vee F_2) \cup E(P_3 \vee F_2)$ to $C = \{1, 2, 3, 4, 5, 6, 0\}$:

$$f(u_1v_j) = j, j = 1, 2, 3;\ f(u_iv_j) = i + j,\ i = 2, 3;\ j = 1, 2, 3;\ f(u_1u_2) = 6;$$

$$f(u_2u_3) = 0;\ f(v_1v_2) = 0;\ f(v_2v_3) = 1;\ f(v_1v_3) = 2;\ f(u_1) = 4;\ f(u_2) = 2;$$

$$f(u_3) = 1;\ f(v_1) = 5;\ f(v_2) = 6;\ f(v_3) = 0.$$

So we have

$$\overline{C}(u_1) = \{0, 5\};\ \overline{C}(u_2) = \{1\};\ \overline{C}(u_3) = \{2, 3\};$$

$$\overline{C}(v_1) = \{6\};\ \overline{C}(v_2) = \{3\};\ \overline{C}(v_3) = \{4\}.$$

Obviously, f is a 7-VDTC of $P_3 \vee F_2$, the conclusion is true.

Case 2. When $m > 3, n = 2$, $\mu(P_m \vee F_2) = m + 4$ by Lemma 1. In order to prove the conclusion is true, we only need to prove $P_m \vee F_2$ exists $(m + 4)$-VDEC. So we construct a map f from $V(P_m \vee F_2) \cup E(P_m \vee F_2)$ to $C = \{1, 2, \cdots, m + 3, 0\}$:

$$\begin{aligned}
&f(u_1v_j) = j,\ j = 1, 2, 3; \quad f(u_iv_j) = i + j,\ i = 2, 3 \cdots, m;\ j = 1, 2, 3;\\
&f(u_1u_2) = m + 2;\ f(u_iu_{i+1}) = i - 2,\ i = 2, 3, \cdots, m - 1;\ f(v_1v_2) = 0;\\
&f(v_2v_3) = 1;\ f(v_1v_3) = 2;\ f(u_1) = m + 1;\ f(u_i) = i,\ i = 2, 3, \cdots, m;\\
&f(v_1) = m + 2;\ f(v_2) = m + 3;\ f(v_3) = 0.
\end{aligned}$$

So we have

$$C(u_1) = \{m + 1, m + 2, 1, 2, 3\}; \quad C(u_2) = \{m + 2, 0, 2, 3, 4, 5\};$$

$$C(u_i) = \{i - 3, i - 2, i, i + 1, i + 2, i + 3\}, \quad i = 3, 4, \cdots, m - 1;$$

$$C(u_m) = \{m - 3, m, m + 1, m + 2, m + 3\};$$

$$\overline{C}(v_1) = \{m + 3\};\ \overline{C}(v_2) = \{3\};\ \overline{C}(v_3) = \{4\}.$$

Obviously, f is a (m+4)-VDTC of $P_m \vee F_2$, the conclusion is true.

Case 3. When $m = n = 3$, $\mu(P_3 \vee F_3) = 8$ by Lemma 1. In order to prove the conclusion is true, we only need to prove $P_3 \vee F_3$ exists 8-VDEC. So we construct a map f from $V(P_3 \vee F_3) \cup E(P_3 \vee F_3)$ to $C = \{1, 2, \cdots, 7, 0\}$:

$$f(w) = 3;\ f(wu_i) = i - 1,\ i = 1, 2, 3;$$

$$f(u_iv_j) = i + j - 1,\ i = 1, 2, 3,\ j = 1, 2, 3;$$

$$f(u_i) = i + 3, i = 1, 2, 3;\ f(u_iu_{(i+1)}) = i + 5(\text{mod } 8), i = 1, 2;$$

$$f(wv_j) = j + 3,\ j =, 2, 3;$$

$$f(v_j) = j - 1,\ j = 1, 2, 3;\ f(v_jv_{(j+1)}) = j + 6(\text{mod } 8), i = 1, 2.$$

So we have

$$\overline{C}(w) = \{7\};\ \overline{C}(u_1) = \{5, 7\};\ \overline{C}(u_2) = \{0\};\ \overline{C}(u_3) = \{0, 1\};$$

$$\overline{C}(v_1) = \{5, 6\};\ \overline{C}(v_2) = \{6\};\ \overline{C}(v_3) = \{1, 7\}.$$

Obviously, f is a 8-VDTC of $P_3 \vee F_3$, the conclusion is true.

Case 4. When $m = 3, n > 3$, $\mu(P_3 \vee F_n) = n + 5$ by Lemma 1. In order to prove the conclusion is true, we only need to prove $P_3 \vee F_n$ exists $(n+5)$-VDEC. So we construct a map f from $V(P_3 \vee F_n) \cup E(P_3 \vee F_n)$ to $C = \{1, 2, \cdots, n+4, 0\}$:

$$f(w) = m;\ f(wu_i) = i - 1,\ i = 1, 2, 3;$$

$$f(u_iv_j) = i + j - 1,\ i = 1, 2, 3,\ j = 1, 2, \cdots, n;$$

$$f(u_i) = n + i,\ i = 1, 2, 3;\ f(u_iu_{(i+1)}) = n + i + 2(\text{mod } n + 5),\ i = 1, 2;$$

$$f(wv_j) = m + j,\ j = 1, 2, \cdots, n;$$

$$f(v_j) = m + j + 1,\ j = 1, 2, \cdots, n;$$

$$f(v_jv_{(j+1)}) = m + j + 3(\text{mod } n + 5),\ j = 1, 2, \cdots, n - 1.$$

So, we have

$$\overline{C}(w) = \{n + 4\};\ \overline{C}(u_1) = \{n + 2, n + 4\};$$

$$\overline{C}(u_2) = \{0\};\ \overline{C}(u_3) = \{0, 1\};$$

$$\overline{C}(v_1) = \{0, 6, 8\};$$

$$\overline{C}(v_i) = \{0, 1, \cdots, i - 1, i + 7, \cdots, n + 4, i = 2, 3, \cdots, n - 2\};$$

$$\overline{C}(v_{n-1}) = \{1, 2, \cdots, n - 2\};\ \overline{C}(v_3) = \{1, 7\};$$

$$\overline{C}(v_n) = \{1, 2, \cdots, n-1\}.$$

Obviously, f is a (n+5)-VDTC of $P_3 \vee F_n$, the conclusion is true.

Case 5. When $m > 3, n = 3$, $\mu(P_m \vee F_3) = m + 5$ by Lemma 1. In order to prove the conclusion is true, we only need to prove $P_m \vee F_3$ exists $(m+5)$-VDEC. So we construct a map f from $V(P_m \vee F_3) \cup E(P_m \vee F_3)$ to $C = \{1, 2, \cdots, m+4, 0\}$:

$$f(w) = m;\ f(wu_i) = i - 1,\ i = 1, 2, \cdots, m;$$

$$f(u_iv_j) = i + j - 1,\ i = 1, 2, \cdots, m,\ j = 1, 2, 3;$$

$$f(u_i) = n + i,\ i = 1, 2, \cdots, m;$$

$$f(u_iu_{(i+1)}) = n + i + 2,\ i = 1, 2, \cdots, m-1;$$

$$f(wv_j) = m + j,\ j = 1, 2, 3;$$

$$f(v_j) = j - 1,\ j = 1, 2, 3;\ f(v_jv_{(j+1)}) = m + j + 2(\mathrm{mod} m + 5),\ j = 1, 2.$$

So, we have:

$$\overline{C}(w) = \{m+4\};\ \overline{C}(u_1) = \{m, m+2, m+3, m+4\};$$

$$\overline{C}(u_i) = \{0, 1, \cdots, i-2, i+6, i+7, \cdots, m+4\};$$

$$\overline{C}(v_1) = \{m+2, m+4\};\ \overline{C}(v_2) = \{0\};$$

$$\overline{C}(v_3) = \{0, 1\}.$$

Obviously, f is a (m+5)-VDTC of $P_m \vee F_3$, the conclusion is true.

Case 6. When $m > 3, n > 3$, $\mu(P_m \vee F_3) = m + n + 1$ by Lemma 1. In order to prove the conclusion is true, we only need to prove $P_m \vee F_n$ exists $(m+n+1)$-VDEC. So we construct a map f from $V(P_m \vee F_3) \cup E(P_m \vee F_3)$ to $C = \{1, 2, \cdots, m+n, 0\}$:

$$f(w) = m;\ f(wu_i) = i - 1,\ i = 1, 2, \cdots, m;$$

$$f(u_iv_j) = i + j - 1,\ i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n;$$

$$f(u_i) = n + i(\mathrm{mod}\ m + n + 1),\ i = 1, 2, \cdots, m;$$

$$f(u_iu_{(i+1)}) = n + i + 2(\mathrm{mod}\ m + n + 1),\ i = 1, 2, \cdots, m-1;$$

$$f(wv_j) = m + j,\ j = 1, 2, \cdots, n;$$

$$f(v_j) = m + j + 1,\ j = 1, 2, \cdots, n;$$

$$f(v_j v_{(j+1)}) = m + j + 3(\text{mod } m + n + 1),\ j = 1, 2, \cdots, n - 1.$$

So, we have:

$$C(w) = \{0, 1, \cdots, m + n\};\ C(u_1) = \{0, 1, \cdots, n + 1, n + 3\};$$

$$C(u_i) = \{i - 1, i, \cdots, n + i + 2(\text{mod } m + n + 1)\}, i = 2, 3, \cdots, m - 1\};$$

$$C(u_m) = \{m - 1, m, \cdots, 0\};$$

$$C(v_1) = \{1, 2 \cdots, m + 2, m + 4\};$$

$$C(v_i) = \{i, i + 1, \cdots, m + i + 3(\text{mod } m + n + 1), i = 2, 3, \cdots, n - 1\};$$

$$C(v_n) = \{n, n + 1, \cdots, m + n + 2(\text{mod } m + n + 1)\};$$

Obviously, f is a $(m + 5)$-VDTC of $P_m \vee F_3$, the conclusion is true. □
From all of above, the conclusion is true.

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References

[1] J.A. Bondy, U.S.R. Marty, *Graph Theory with Applications*, The Macmillan Press, New York (1976).

[2] P. Hansen, O. Marcotte, *Graph Coloring and Application*, AMS Providence, Rhode, Island, USA (1999).

[3] Zhang Zhongfu, Wang Jianfang, A summary of the progress on total coloring of graphs, *Adv. Math.* (1992).