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MAPS BETWEEN POLYSTABLE VECTOR BUNDLES ON CURVES WITH ARITHMETIC GENUS ONE

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Abstract: Let C be an integral projective curve such that $p_a(C) = 1$ and E, F polystable vector bundles on C such that $\mu(F) < \mu(E)$. Fix a general $f \in \mathrm{Hom}(F, E)$. Set $G := \mathrm{Im}(f)$. Then:

(a) If $\mathrm{rank}(F) \geq \mathrm{rank}(E)$, then $\mathrm{rank}(G) = \mathrm{rank}(E)$.

(b) If $\mathrm{rank}(F) \leq \mathrm{rank}(E)$, then f is injective.

(c) If $\mathrm{rank}(F) > \mathrm{rank}(E)$, then f is surjective.

(d) If $\mathrm{rank}(F) < \mathrm{rank}(F)$, then f in injective and E/G is locally free.

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Let C be an integral complex projective curve such that $p_a(C) = 1$. Hence either C is a smooth elliptic curve or $\mathbf{P}^1$ is the normalization of C and C has a unique singular point which is either an ordinary node or an ordinary cusp. As in the smooth case for all coprime integers m, d and all $L \in \mathrm{Pic}^d(C)$ there is a unique rank m stable vector bundle E on C such that $\det(E) \cong L$ ([2], Theorem 5.1). Hence for all integers r, d such that $r \geq 1$ there is a rank r polystable vector bundle E on C such that the indecomposable factors of E are pairwise non-isomorphic. These are the vector bundles on C which are easier to work with. Notice that $\mathrm{Hom}(A, B) = 0$ if A and B are polystable and either $\mu(A) > \mu(B)$ of $\mu(A) = \mu(B)$ and no indecomposable factor of A is isomorphic

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to an indecomposable factor of B. Here we will prove the following result.

Theorem 1. *Let E, F polystable vector bundles on C such that $\mu(F) < \mu(E)$. Fix a general $f \in \mathrm{Hom}(F, E)$. Set $G := \mathrm{Im}(f)$.*

(a) If $\mathrm{rank}(F) \geq \mathrm{rank}(E)$, then $\mathrm{rank}(G) = \mathrm{rank}(E)$.

(b) If $\mathrm{rank}(F) \leq \mathrm{rank}(E)$, then f is injective.

(c) If $\mathrm{rank}(F) > \mathrm{rank}(E)$, then f is surjective.

(d) If $\mathrm{rank}(F) < \mathrm{rank}(F)$, then f in injective and E/G is locally free.

Remark 1. Let A be a vector bundle on C and G, F torsion free sheaves on C. Both $\mathrm{Hom}(G, A)$ and $\mathrm{Hom}(A, F)$ are torsion free. Obviously, $\deg(\mathrm{Hom}(A, F)) = \mathrm{rank}(A) \cdot \mathrm{rank}(F)(\mu(F) - \mu(A))$. Since C is Gorenstein, every torsion free sheaf on C is reflexive. Hence we also get $\deg(\mathrm{Hom}(G, A)) = \mathrm{rank}(A) \cdot \mathrm{rank}(G)(\mu(A) - \mu(G))$.

Remark 2. Let A, B polystable vector bundles on C such that $\mu(B) > \mu(A)$. By [2], Proposition 8.2, $\mathrm{Hom}(A, B)$ is semistable. Hence $h^1(C, \mathrm{Hom}(A, B)) = 0$. Thus Riemann-Roch gives $h^0(C, \mathrm{Hom}(A, B)) = \mathrm{rank}(A) \cdot \mathrm{rank}(B)(\mu(B) - \mu(A))$.

Proof of Theorem 1. Assertions (a) and (b) (resp. (c) and (d)) are duals. We will prove simultaneously all these assertions by induction on the integer $\mathrm{rank}(F) + \mathrm{rank}(G)$. The starting point $\mathrm{rank}(F) = \mathrm{rank}(G) = 1$ of the induction is trivial.

(i) Here we will show how to modify the proof of [1], Proposition 4.3, to get part (a). Let $\widetilde{G}$ be a polystable vector bundle whose indecomposable factors are pairwise non-isomorphic and with the same rank and degree as G. In order to obtain a contradiction we assume $\mathrm{rank}(G) < \mathrm{rank}(F)$. By the inductive assumption there is a surjection $u : F \to \widetilde{G}$ and an injection with locally free cokernel $v : \widetilde{G} \to E$. We apply Remark 1 and the generality of f to get that G is locally free. With this observation we may copy the proof of [1], Proposition 4.3.

(ii) Now we assume $\mathrm{rank}(F) > \mathrm{rank}(E)$ and we will prove part (c). By part (a) we may assume $\mathrm{rank}(G) = \mathrm{rank}(F)$. Assume $G \neq E$, as in part (a) we may even obtain the local freenes of G. Let x be the only integer such that $x \leq \deg(E)$ and $(x-1)/\mathrm{rank}(E) \leq \mu(F)$. Let A any polystable vector bundle on C with degree x and rank $\mathrm{rank}(E)$. Fix a general $h \in H^0(C, \mathrm{Hom}(F, A))$ and set $B := \mathrm{Im}(h)$. Since F is polystable, every indecomposable factor of B has slope $\leq \mu(F)$ and the strict inequality holds if it is not isomorphic to a direct summand of F. Since A is polystable, every indecomposable factor of B has slope $\leq \mu(F)$ and the strict inequality holds if it is not isomorphic to a direct summand of A. The choice of the integer x and Remark 2 gives the surjectivity of h. Hence part (c) is true if $\deg(E) = x$. Now assume $\deg(E) > x$. We use induction on the integer $\deg(E) - x$. By the inductive assumption a general morphism $c : U \to V$ is injective for all polystable vector bundles U, V such that $\text{rank}(U) = \text{rank}(V) = \text{rank}(E)$. Remark 1 allows us to copy the proof of [1], Proposition 2.3. □

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