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A NOTE ON CERTAIN INTEGRAL OPERATORS

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Abstract: Let $\mathcal{N}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in the open unit disk $\mathcal{E} = \{z : |z| < 1\}$. We define the subclasses $\mathcal{U}_k^\alpha(\lambda, \mu, \rho)$ and $\mathcal{V}_k^\alpha(\lambda, \mu, \rho)$ of $\mathcal{N}$ using certain integral operators and their inclusion properties are studied.

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1. Introduction

Let $\mathcal{N}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{E} = \{z : |z| < 1\}$.

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The convolution or Hadamard product of two functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.2}$$

Let $\mathcal{P}_k^{\alpha}(\rho)$ denote the class of functions $p(z)$ in the unit disk $\mathcal{E}$ with the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re\{e^{i\alpha} p(z) - \rho \cos\alpha\}}{1-\rho} \right| < k\pi \cos\alpha, \tag{1.3}$$

where $k \geq 2$, α is real, $|\alpha| < \frac{\pi}{2}$, $0 \leq \beta \leq 1$, $z = re^{i\theta}$ and $0 \leq r < 1$, [3]. For $\rho = 0$ and $\alpha = 0$, we obtain the class $\mathcal{P}_k$ defined by Pinchuk [8] and for $k = 2$, $\rho = 0$ and $\alpha = 0$, the class $\mathcal{P}$ of functions with positive real part [2]. And also for $k = 2$, and $\alpha = 0, \mathcal{P}_2(\rho) = \mathcal{P}(\rho)$, the class of functions with positive real part greater than ρ [2]. Further, if $p(z) \in \mathcal{P}_k^{\alpha}(\rho)$, we have

$$e^{i\alpha} p(z) = \frac{\cos\alpha}{2} \int_0^{2\pi} \frac{1 + (1-2\rho)ze^{-it}}{1 - ze^{-it}} dm(t) + i \sin\alpha\,, \tag{1.4}$$

where $m(t)$ is a real valued function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} dm(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |dm(t)| \leq k. \tag{1.5}$$

From (1.4) and (1.5) we can write, for $p(z) \in \mathcal{P}_k^{\alpha}(\rho)$

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{2} - \frac{1}{2}\right) p_2(z)\,, \tag{1.6}$$

where $p_1(z)$, $p_2(z) \in \mathcal{P}_k^{\alpha}(\rho)$.

Let

$$\mathcal{U}_k^{\alpha}(\rho) = \left\{ f(z) : f(z) \in \mathcal{N} \text{ and } \frac{zf'(z)}{f(z)} \in \mathcal{P}_k^{\alpha}(\rho), \quad z \in E, 0 \leq \rho < 1 \right\}.$$

Note that $\mathcal{U}_2(\rho) = \mathcal{S}^*(\rho)$, the class of star like functions of order ρ [2]. For $\rho = 0$, $\mathcal{U}_k^{\alpha}(0) = \mathcal{U}_k^{\alpha}$ the Moulis class [5] and $\mathcal{U}_k^{0}(0) = \mathcal{U}_k$ [7] and $\mathcal{U}_k^{\alpha}(\rho) = \mathcal{U}_k(\rho)$ [6]. And

$$\mathcal{V}_k^{\alpha}(\rho) = \left\{ f(z) : f(z) \in \mathcal{N} \text{and} \, \frac{(zf'(z))'}{f'(z)} \in \mathcal{P}_k^{\alpha}(\rho),\, z \in E, 0 \leq \rho < 1 \right\}.$$

For $\alpha = 0$, $\mathcal{V}_k^\alpha(\rho) = \mathcal{V}_k^0(\rho)$, class of functions studied by Padmanabhan and Parvatham [6]. Further, for $\alpha = 0, k = 2$, $V_k^\alpha(\rho) = V_2(\rho)$, class of convex functions of order ρ and $\rho = 0$ we obtain the class for $\mathcal{V}_k^\alpha$ [5] and for $\alpha = 0$, $\rho = 0, \mathcal{V}_k^0(0) = \mathcal{V}_k$ [7]. We note that $f(z) \in \mathcal{V}_k^\alpha(\rho)$ if and only if $zf'(z) \in \mathcal{U}_k^\alpha(\rho)$.

2. Preliminary Results

We define the integral operator $L_\lambda^\mu : \mathcal{N} \to \mathcal{N}$ for $\lambda > -1$, $\mu > 0$, $f(z) \in \mathcal{N}$ by

$$\begin{aligned} L_\lambda^\mu f(z) &= C_\lambda^{\lambda+\mu} \frac{\mu}{z^\lambda} \int_0^z t^{\lambda-1} \left(1 - \frac{t}{z}\right)^{\mu-1} f(t)\, dt \\ &= z + \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda+\mu+n)}\, a_n z^n\,, \end{aligned} \quad (2.1)$$

where Γ denotes the gamma function. From (2.1) we can obtain the well known generalized Bernard operator as follows:

$$\begin{aligned} I_\mu f(z) &= \frac{\mu+1}{z^\mu} \int_0^z t^{u-1} f(t)\, dt \\ &= z + \sum_{n=2}^{\infty} \frac{\mu+1}{\mu+n} a_n z^n, \mu > -1; f(z) \in \mathcal{N}. \end{aligned} \quad (2.2)$$

We define the following subclasses of $\mathcal{N}$ using the integral operator L_λ^μ.

Definition 2.1. Let $f(z) \in \mathcal{N}$. Then $f(z) \in \mathcal{U}_k^\alpha(\lambda, \mu, \rho)$ if and only if $L_\lambda^\mu f(z) \in \mathcal{U}_k^\alpha(\rho)$, for $z \in \mathcal{E}$.

Definition 2.2. Let $f(z) \in \mathcal{N}$. Then $f(z) \in \mathcal{V}_k^\alpha(\lambda, \mu, \rho)$ if and only if $L_\lambda^\mu f(z) \in \mathcal{V}_k^\alpha(\rho)$, for $z \in \mathcal{E}$.

Lemma 2.3. (see [4]) *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and Φ be a complex -valued function satisfying the conditions:*

1. *$\Phi(u, v)$ is continuous in a domain $\mathcal{D} \subset \mathbb{C}^2$,*
2. *$(1, 0) \in \mathcal{D}$ and $\Phi(1, 0) > 0$,*
3. *$\Re\{\Phi(iu_2, v_1)\} \le 0$, whenever $(iu_2, v_1) \in \mathcal{D}$ and $v_1 \le -\frac{1}{2}(1 + u_2^2)$.*

If $h(z) = 1 + \sum_{m=2}^{\infty} c_m z^m$ is a function analytic in $\mathcal{E}$ such that $(h(z), zh'(z)) \in \mathcal{D}$ and $\Re\{\Phi(h(z), zh'(z))\} > 0$ for $z \in \mathcal{E}$, then $\Re\{h(z)\} > 0$, in $\mathcal{E}$.

3. Main Results

We now prove some inclusion properties.

Theorem 3.1. *Let* $f(z) \in \mathcal{N}, \lambda > -1, \mu > 0$ *and* $\lambda + \mu > 0$. *Then,* $\mathcal{U}_k^\alpha(\lambda, \mu, o) \subset \mathcal{U}_k^\alpha(\lambda, \mu + 1, \rho)$, *where*

$$\rho = \frac{2}{(\beta + 1) + \sqrt{\beta^2 + 2\beta + 9}} \tag{3.1}$$

with $\beta = 2(\lambda + \mu)$.

Proof. Let $f(z) \in \mathcal{U}_k^\alpha(\lambda, \mu, 0)$ and

$$\frac{z\left(L_\lambda^{\mu+1} f(z)\right)'}{L_\lambda^{\mu+1} f(z)} = p(z) = \left(\frac{k}{2} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z)\,,$$

where $p(0) = 1$ and $p(z)$ is analytic in $\mathcal{E}$.

From (2.1) it can easily be seen that

$$z\left(L_\lambda^{\mu+1} f(z)\right)' = (\lambda + \mu + 1) L_\lambda^\mu f(z) - (\lambda + \mu) L_\lambda^{\mu+1} f(z), \tag{3.2}$$

which simplifies to

$$\frac{z\left(L_\lambda^\mu f(z)\right)'}{L_\lambda^\mu f(z)} = \left\{p(z) + \frac{zp'(z)}{p(z) + \lambda + \mu}\right\} \in P_k^\alpha, z \in \mathcal{E}\,.$$

Let

$$\Phi_{\lambda,\mu}(z) = \sum_{j=1}^{\infty} \frac{(\lambda + \mu) + j}{\lambda + \mu + 1} z^j = \left(\frac{\lambda + \mu}{\lambda + \mu + 1}\right) \frac{z}{1 - z} + \left(\frac{1}{\lambda + \mu + 1}\right) \frac{z}{(1 - z)^2}.$$

Then,

$$\begin{aligned}
p(z) * \Phi_{\lambda,\mu}(z) &= p(z) + \frac{zp'(z)}{p(z) + \lambda + \mu} \\
&= \left(\frac{k}{4} + \frac{1}{2}\right) [p_1(z) * \Phi_{\lambda,\mu}(z)] - \left(\frac{k}{4} - \frac{1}{2}\right) [p_2(z) * \Phi_{\lambda,\mu}(z)] \\
&= \left(\frac{k}{4} + \frac{1}{2}\right) \left[p_1(z) + \frac{zp_1'(z)}{p_1(z) + \lambda + \mu}\right] - \left(\frac{k}{4} - \frac{1}{2}\right) \left[p_2(z) + \frac{zp_2'(z)}{p_2(z) + \lambda + \mu}\right]
\end{aligned}$$

and this implies that

$$\left[p_i(z) + \frac{zp_i'(z)}{p_i(z)+\lambda+\mu}\right] \in \mathcal{P}, \quad z \in \mathcal{E}.$$

We prove that $p_i(z) \in \mathcal{P}(\rho)$, where ρ is given by (3.1) and this will prove that $p(z) \in \mathcal{P}_k^{\alpha}(\rho)$ for $z \in \mathcal{E}$.

Let

$$p_i(z) = (1-\rho)h_i(z) + \rho, \text{ for } i = 1, 2, 3, \dots .$$

We frame the functional $\Psi(u,v)$ by choosing $u = h_i(z), v = zh_i'(z)$. Thus,

$$\Psi(u,v) = (1-\rho)u + \rho + \frac{(1-\rho)v}{(1-\rho)u+\rho+\lambda+\mu}\,.$$

The first two conditions of Lemma 2.3 are clearly satisfied. We verify the condition (iii) as follows:

$$\Re\{\Psi(iu_2, v_1)\} = \frac{\rho + (1-\rho)(\rho+\lambda+\mu)v_1}{(\rho+\lambda+\mu)^2 + (1-\rho)^2u_2^2}\,.$$

By putting $v_1 \leq \dfrac{-(1+u_2^2)}{2}$, we obtain

$$\Re\{\Psi(iu_2, v_1)\} \leq \rho - \frac{1}{2}\left[\frac{(1-\rho)(\rho+\lambda+\mu)(1+u_2^2)}{2(\rho+\lambda+\mu)^2+(1-\rho)^2u_2^2}\right] =$$

$$\frac{2\rho(\rho+\lambda+\mu)^2 + 2\rho(1-\rho)^2u_2^2 - (1-\rho)(\rho+\lambda+\mu) - (1-\rho)(\rho+\lambda+\mu)u_2^2}{2[(\rho+\lambda+\mu)^2+(1-\rho)^2u_2^2]}$$

$$= \frac{A+Bu_2^2}{2C}\,,$$

where

$$\begin{aligned} A &= 2\rho(\rho+\lambda+\mu)^2 - (1-\rho)(\rho+\lambda+\mu)\,, \\ B &= 2\rho(1-\rho)^2 - (1-\rho)(\rho+\lambda+\mu)\,, \\ C &= (\rho+\lambda+\mu)^2 + (1-\rho)^2u_2^2 > 0\,. \end{aligned}$$

We note that $\Re\{\Psi(iu_2, v_1)\} \leq 0$ if and only if $A \leq 0$ and $B \leq 0$. From $A \leq 0$, we obtain ρ as given by (3.1) and $B \leq 0$ gives us $0 \leq \alpha < 1$. This completes the proof. □

Remark 3.2. For $\alpha = 0$ we obtain Theorem 2.1 in [2]:

Let $f(z) \in \mathcal{N}, \lambda > -1, \mu > 0$ and $\lambda + \mu > 0$. Then, $\mathcal{U}_k(\lambda, \mu, o) \subset \mathcal{U}_k(\lambda, \mu + 1, \rho)$, where

$$\rho = \frac{2}{(\beta + 1) + \sqrt{\beta^2 + 2\beta + 9}}$$

with $\beta = 2(\lambda + \mu)$.

Theorem 3.3. *For $\lambda > -1$, $\mu > 0$ and $\lambda + \mu > 0$:*

$$\mathcal{V}_k^{\alpha}(\lambda, \mu, 0) \subset \mathcal{V}_k^{\alpha}(\lambda, \mu + 1, \rho),$$

where ρ is given by (3.1).

Proof. Let $f(z) \in \mathcal{V}_k^{\alpha}(\lambda, \mu, 0)$. Then,

$$L_{\lambda}^{\mu}(f(z)) \in \mathcal{V}_k^{\alpha}(0) \quad \text{and} \quad z\left[L_{\lambda}^{\mu}(f(z))\right]' \in \mathcal{U}_k^{\alpha}(0) = \mathcal{U}_k^{\alpha}.$$

This implies, $L_{\lambda}^{\mu}(zf'(z)) \in \mathcal{U}_k^{\alpha}$. That is,

$$zf'(z) \in \mathcal{U}_k^{\alpha}(\lambda, \mu, 0) \subset \mathcal{U}_k^{\alpha}(\lambda, \mu + 1, \rho).$$

Consequently, $f(z) \in \mathcal{V}_k^{\alpha}(\lambda, \mu + 1, \rho)$, where ρ is given by (3.1). This completes the proof. □

Remark 3.4. For $\alpha = 0$ we get Theorem 2.2 in [2]:

For $\lambda > -1, \mu > 0$ and $\lambda + \mu > 0, \mathcal{V}_k(\lambda, \mu, 0) \subset \mathcal{V}_k(\lambda, \mu + 1, \rho)$, where

$$\rho = \frac{2}{(\beta + 1) + \sqrt{\beta^2 + 2\beta + 9}}$$

with $\beta = 2(\lambda + \mu)$.

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