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STABILITY ANALYSIS OF A NON-AUTONOMOUS MODEL IN QUANTUM OPTICS

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Abstract: Periodic stability regions are identified for a non-autonomous system of differential equations modelling the interaction of a single 2-level atom with a coherent field and bathed in a non-classical (squeezed vacuum) reservoir.

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1. Introduction

In the field of quantum optics, the system of differential equations (called Bloch equations) modelling the interaction of a single 2-level atom with a coherent laser field in the presence of a non-classical squeezed vacuum (SV) reservoir environment reads the following [3], [7], [11]:

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$$\dot{R} = -\left(\Gamma + m\cos\left(\phi + qt\right)\right)R + \left(\Delta - m\sin\left(\phi + qt\right)\right)S + \Omega Z\,, \qquad (1.1)$$
$$\dot{S} = -\left(\Delta + m\sin\left(\phi + qt\right)\right)R - \left(\Gamma - m\cos\left(\phi + qt\right)\right)S\,,$$
$$\dot{Z} = -\Omega R - 2\Gamma Z - \frac{\gamma}{2}\,,$$

where $\dot{R} = \frac{dR}{dt}$, etc.

The notations are the following:

The real quantities R and S are the in-phase and out of-phase component of the quantum averaged atomic polarization, Z is the real quantum averaged atomic inversion $\left(|Z| \leq \frac{1}{2}\right)$, $\Gamma = \frac{\gamma}{2}\left(1 + 2N\right)$, where γ is the Einstein-A (damping) coefficient. The complex parameter $M = |M|\,e^{i\phi}$ and the real parameter N are the SV parameters (N = average photon number, M = degree of squeezing) such that $m = |M| \leq \sqrt{N\left(N+1\right)}$, $\Delta = \omega_o - \omega_L$ is the frequency detuning parameter (ω_o = atomic frequency transition, ω_L = laser oscillating frequency), Ω is the real Rabi laser frequency associated with the 2-level atom, q is the SV frequency detuning parameter (frequency mismatch between the central frequency of the SV field ω_S and ω_L). For details of derivation see [4] and references therein.

Few comments are noteworthy regarding the non-autonomous linear system (1.1).

1. For $q = 0$ (resonant SV field) the system is autonomous and exactly solvable. The solution is oscillatory damped and has non-trivial steady state ($t \to \infty$) solution due to the non-homogeneous term $\left(-\frac{\gamma}{2}\right)$ in the last equation for Z. Quantal analysis of the system regarding the fluorescence and absorption spectra were examined by many authors [5], [6], [7], [8], [9].

2. For $q \neq 0$ (off-resonant SV field) the system has no exact analytic solution. It can be analyzed using Fourier series expansion and continued fraction method [1], [10] and one then resorts to computation approach to analyze the various harmonic components of the atomic polarization and inversion. Also, analytic solutions of system (1.1) have been obtained in the strong field limit ($\Omega \gg \gamma$; secular approximation Joshi and Hassan [2], Hassan, Joshi and Frege [3] and Joshi and Hassan [11] and in the weak field limit ($\Omega \ll \gamma$ Hassan, Joshi and Frege [3]).

3. Note that for $q \neq 0$ system (1.1), according to Floquet's Theorem, see Hale [12], has a solution (which is damped in our case) with oscillatory nature at all harmonics of q and the presence of the non-homogeneous term $\left(-\frac{\gamma}{2}\right)$ leads to steady ($t \to \infty$) oscillatory behavior.

In the present paper, we are analyzing the model equation (1.1) regarding its stability in both cases of resonant ($q = 0$) and off-resonant ($q \neq 0$) SV fields in Sections 2 and 3 respectively, followed by a conclusion in Section 4.

2. Autonomous System

For $q = 0$ system (1.1) is an autonomous system of non-homogeneous linear differential equations which can be put in the matrix form

$$\dot{X} = AX + B\,, \tag{2.1}$$

where

$$X = \begin{pmatrix} R \\ S \\ Z \end{pmatrix}, \qquad A = \begin{pmatrix} -(\Gamma + m\cos\phi) & \Delta - m\sin\phi & \Omega \\ -(\Delta + m\sin\phi) & -(\Gamma - m\cos\phi) & 0 \\ -\Omega & 0 & -2\Gamma \end{pmatrix},$$

$$B = \begin{pmatrix} 0 \\ 0 \\ -\frac{\gamma}{2} \end{pmatrix}.$$

It is convenient to examine the stability of the system via the eigenvalues of the constant matrix A. In the steady state, where $\dot{X} = 0$, the non-zero equilibrium solution of (2.1) is $\overline{X} = -A^{-1}B$. To study the stability of $\overline{X}$ we first consider the homogeneous equation

$$\dot{X} = AX \tag{2.2}$$

and use the fact that (e.g. Perko [13]) every solution of (2.1) has the form $X(t) = U(t) + V(t)$, where $U(t)$ is a particular solution of (2.1) and $V(t)$ is a solution of (2.2). Therefore, $\lim\limits_{t\to\infty} X(t) = \overline{X} + \lim\limits_{t\to\infty} V(t)$ and the stability of the origin for the homogeneous system (2.2) leads to information about the stability of $\overline{X}$ for the non-homogeneous system (2.1). Although the eigenvalues of A can be obtained explicitly for some choices of the parameters Δ and ϕ, we discuss the general case. The eigenvalues of A satisfy the characteristic equation

$$\lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3 = 0\,, \tag{2.3}$$

where $\alpha_1 = 4\Gamma$, $\alpha_2 = \Omega^2 + 5\Gamma^2 + \Delta^2 - m^2$ and $\alpha_3 = \Omega^2\Gamma - \Omega^2 m\cos\phi + 2\Gamma\Delta^2 - 2\Gamma m^2 + 2\Gamma^3$. Let $H = \Omega^2\Gamma + \Omega^2 m + 2\Gamma\Delta^2 - 2\Gamma m^2 + 2\Gamma^3$, then $\alpha_3 \leq H$. Now $\alpha_1\alpha_2 = 4\Omega^2\Gamma + 20\Gamma^3 + 4\Gamma\Delta^2 - 4\Gamma m^2 = 2\Omega^2(\Gamma - M) + 16\Gamma^3 + 2H > H \geq \alpha_3$. Using the Routh-Hurwitz criterion, all the eigenvalues have negative real parts. So the origin is a sink and consequently $\overline{X}$ is asymptotically stable.

3. Non-Autonomous System

For the general system in (1.1) we first consider the non-homogeneous system

$$\begin{aligned}\dot{R} &= -\left(\Gamma + m\cos\left(\phi + qt\right)\right)R + \left(\Delta - m\sin\left(\phi + qt\right)\right)S + \Omega Z\,, \\ \dot{S} &= -\left(\Delta + m\sin\left(\phi + qt\right)\right)R - \left(\Gamma - m\cos\left(\phi + qt\right)\right)S\,, \\ \dot{Z} &= -\Omega R - 2\Gamma Z\,.\end{aligned} \tag{3.1}$$

The origin is the only equilibrium point of (3.1). To study its stability we will use the Liapunov method Perko [13] and Fong and De Kee [14]. We use the following Liapunov function

$$V\left(R,S,Z\right) = R^2 + \frac{1}{2}S^2 + Z^2\,.$$

Then $V\left(R,S,Z\right) \geq 0$ on R^3 and $V\left(R,S,Z\right) = 0$ iff $\left(R,S,Z\right) = (0,0,0)$. The derivatives of V along trajectories of (3.1) is given by

$$\begin{aligned}\dot{V} = -2\left(\Gamma + m\cos\left(\phi + qt\right)\right)R^2 - \left(\Gamma - m\cos\left(\phi + qt\right)\right)S^2 - 4\Gamma Z^2 \\ + \left(\Delta - 3m\sin\left(\phi + qt\right)\right)RS\,.\end{aligned}$$

The coefficients of R^2 and S^2 are negative for all t since $\Gamma > m$, therefore if there is a region on which $\left(\Delta - 3m\sin\left(\phi + qt\right)\right)RS \leq 0$, then $\dot{V} < 0$ for all t and the origin will be asymptotically stable in that region. So we have the following theorem.

Theorem 1. *Consider the system (3.1). Then the origin is asymptotically stable in the region* $\mathbf{R}_1 = \{(R,S,Z) : RS \geq 0\}$ *if* $\Delta \leq -3m$, *and in the region* $\mathbf{R}_2 = \{(R,S,Z) : RS \leq 0\}$ *if* $\Delta \geq 3m$.

Now to study the stability of system (1.1) we introduce a new variable $\Psi = -\frac{\gamma}{2} + \eta$ to transform the non-homogeneous system (1.1) to the following homogeneous system (e.g. Thompson and Stewart [15])

$$\begin{aligned}\dot{R} &= -\left(\Gamma + m\cos\left(\phi + qt\right)\right)R + \left(\Delta - m\sin\left(\phi + qt\right)\right)S + \Omega Z\,, \\ \dot{S} &= -\left(\Delta + m\sin\left(\phi + qt\right)\right)R - \left(\Gamma - m\cos\left(\phi + qt\right)\right)S\,, \\ \dot{Z} &= -\Omega R - 2\Gamma Z + \Psi\,, \\ \dot{\Psi} &= 0\,.\end{aligned} \tag{3.2}$$

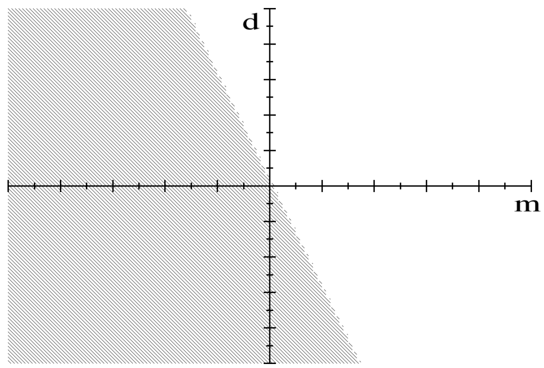

Figure 1: Condition of stability in region R3

Physically, the constant η represents a pumping parameter for the atomic inversion (Z).

To study the stability of system (3.2) we will use the following Liapunov function

$$V\left(R,S,Z,\Psi\right)=R^2+\frac{1}{2}S^2+Z^2+\Psi^2\,.$$

Then $V\left(R,S,Z,\Psi\right)\geq 0$ on R^4 and $V\left(R,S,Z,\Psi\right)=0$ iff $\left(R,S,Z,\Psi\right)=(0,0,0,0)$. The derivatives of V along trajectories of (3.2) is given by

$$\begin{aligned}\dot{V}&=-2\left(\Gamma+m\cos\left(\phi+qt\right)\right)R^2-\left(\Gamma-m\cos\left(\phi+qt\right)\right)S^2-4\Gamma Z^2\\&\quad+\left(\Delta-3m\sin\left(\phi+qt\right)\right)RS+2Z\Psi\,.\end{aligned}$$

Therefore if there is a region on which $\left(\Delta-3m\sin\left(\phi+qt\right)\right)RS\leq 0$ and $Z\Psi\leq 0$ then $\dot{V}<0$ for all t and the origin will be asymptotically stable in that region. So we have the following theorem.

Theorem 2. *Consider the system (3.2). Then the origin is asymptotically stable in the region* $\mathbf{R}_3=\{(R,S,Z,\Psi):RS\geq 0,Z\Psi\leq 0\}$ *if* $\Delta\leq -3m$, *and in the region* $\mathbf{R}_4=\{(R,S,Z,\Psi):RS\leq 0,Z\Psi\leq 0\}$ *if* $\Delta\geq 3m$.

The conditions of stability in Theorem 2 are demonstrated in Figures 1 and 2.

Finally, we note that the regions of stability (as compared with the homogeneous case of Theorem 1) are now constrained by the extra conditions:

1. $Z\leq 0$, $\Psi\geq 0$ and then $\eta\geq\frac{\gamma}{2}$.
2. $Z\geq 0$, $\Psi\leq 0$ and then $\eta\leq\frac{\gamma}{2}$.

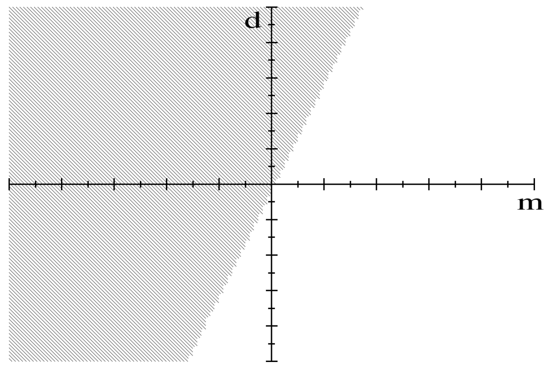

Figure 2: Condition of stability in region R4

4. Conclusion

The non-autonomous system of differential equations modelling the interaction of a single 2-level atom with a laser field and in contact with a squeezed vacuum(SV) reservoir has been analyzed regarding its stability. Using Liapunov method, the origin of the system is found to be asymptotically and periodically stable within certain intervals depending on the atomic and SV detuning parameters (Δ, q) and the degree of squeezing and its phase (m, ϕ).

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