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NUMERICAL APPROXIMATIONS OF THE SOLUTION FOR ONE-DIMENSIONAL COMPRESSIBLE VISCOUS MICROPOLAR FLUID MODEL

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Abstract: We consider the model for nonstationary 1-D flow of a compressible viscous heat-conducting micropolar fluid which is thermodynamically perfect and polytropic. The model is in the Lagrangean description and in the form of four nonlinear partial differential equations. Using simple initial and boundary conditions for this equation system we develop numerical method for obtaining state function of described fluid on $[0,1]\times[0,T]$, $T>0$.

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1. Introduction

In this paper we consider nonstationary 1-D flow of a compressible viscous and

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heat-conducting micropolar fluid. The equations of motion for this fluid are derived from the integral form of conservation laws for polar fluids, under a number of supplementary assumptions such as politropy, Fourier's law, Boyle's law and selection of constitutive equations (Mujaković [6]). A corresponding initial-boundary value problem has a unique strong solution on $]0,1[\times]0,T[$ for each $T>0$ (Mujaković [7]) which analytic form is not known. For sufficiently small T this solution is a limit of approximate solutions which we get by the Faedo-Galerkin method (Mujaković [6]). Using *Wolfram Mathematica* we construct numerical approximations of these solutions for simple initial and boundary conditions and analyze their convergence.

2. Mathematical Model

Let ρ, v, ω and θ denote, respectively, the mass density, velocity, microrotation velocity and temperature of the fluid in the Lagrangean description. In that case the problem which we consider has the formulation as follows (Mujaković [6]):

$$\frac{\partial \rho}{\partial t}+\rho^2\frac{\partial v}{\partial x}=0, \tag{1a}$$

$$\frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\left(\rho\frac{\partial v}{\partial x}\right)-K\frac{\partial}{\partial x}\left(\rho\theta\right), \tag{1b}$$

$$\rho\frac{\partial \omega}{\partial t}=A\left[\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial \omega}{\partial x}\right)-\omega\right], \tag{1c}$$

$$\begin{aligned}\rho\frac{\partial \theta}{\partial t}=&-K\rho^2\theta\frac{\partial v}{\partial x}+\rho^2\left(\frac{\partial v}{\partial x}\right)^2+\rho^2\left(\frac{\partial \omega}{\partial x}\right)^2\\&+\omega^2+D\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial \theta}{\partial x}\right)\end{aligned} \tag{1d}$$

in $]0,1[\times]0,T[, T>0$, where K, A and D are positive constants. Equations (1) are, respectively, local forms of the conservation laws for the mass, momentum, momentum moment and energy. We take the homogeneous boundary conditions:

$$v(0,t)=v(1,t)=0, \tag{2a}$$

$$\omega(0,t)=\omega(1,t)=0, \tag{2b}$$

$$\frac{\partial \theta}{\partial x}(0,t)=\frac{\partial \theta}{\partial x}(1,t)=0 \tag{2c}$$

for $t \in]0,T[$ and non-homogeneous initial conditions:

$$\rho(x,0) = \rho_0(x), \tag{3a}$$

$$v(x,0) = v_0(x), \tag{3b}$$

$$\omega(x,0) = \omega_0(x), \tag{3c}$$

$$\theta(x,0) = \theta_0(x) \tag{3d}$$

for $x \in \Omega =]0,1[$, where $\rho_0(x)$, $v_0(x)$, $\omega_0(x)$ and $\theta_0(x)$ are given functions. We assume that the functions ρ_0 and θ_0 are strictly positive and bounded:

$$0 < m \leq \rho_0(x) \leq M, \tag{4a}$$

$$m \leq \theta_0(x) \leq M \tag{4b}$$

for $x \in \Omega$.

The problem (1)-(4) is well posed (Mujaković [7]) and for

$$\rho_0, \theta_0 \in H^1(\Omega), \tag{5a}$$

$$v_0, \omega_0 \in H_0^1(\Omega) \tag{5b}$$

it has a unique solution

$$(x,t) \mapsto (\rho, v, \omega, \theta)(x,t),\ \ (x,t) \in Q_T = \Omega \times]0,T[\tag{6}$$

for each $T > 0$, with the properties:

$$\rho \in L^\infty(0,T;H^1(\Omega)) \cap H^1(Q_T), \tag{7a}$$

$$v, \omega, \theta \in L^\infty(0,T;H^1(\Omega)) \cap H^1(Q_T) \cap L^2(0,T;H^2(\Omega)), \tag{7b}$$

$$\rho > 0\ ,\ \theta > 0 \ \text{ on } \ \bar{Q}_T \tag{7c}$$

From embedding and interpolation theorems (Dautray et al [3] and Dautray et al [4]) one can conclude that from (7) what follows is:

$$\rho \in L^\infty(0,T;C(\bar{\Omega})) \cap C([0,T];L^2(\Omega)), \tag{8a}$$

$$v, \omega, \theta \in L^2(0,T;C^1(\bar{\Omega})) \cap C([0,T],H^1(\Omega)), \tag{8b}$$

$$\rho, v, \omega, \theta \in C(\bar{Q}_T). \tag{8c}$$

3. Approximate Solutions

In Mujaković [6] are introduced, similarly as in Antontsev et al [2], the partial sums:

$$v_0^n(x) = \sum_{i=1}^{n} v_{0i} \sin(\pi i x), \tag{9a}$$

$$\omega_0^n(x) = \sum_{j=1}^{n} \omega_{0j} \sin(\pi j x), \tag{9b}$$

$$\theta_0^n(x) = \sum_{k=0}^{n} \theta_{0k} \cos(\pi k x) \tag{9c}$$

of the Fourier series for the initial functions v_0, ω_0 and θ_0 and a sequence $\{(\rho^n, v^n, \omega^n, \theta^n),\ n \in \mathbf{N}\}$ of approximate solutions of the problem (1)-(4) is defined by

$$\rho^n(x,t) = \rho_0(x)(1 + \rho_0(x) \sum_{i=1}^{n} (\pi i) \cos(\pi i x) z_i^n(t))^{-1}, \tag{10a}$$

$$v^n(x,t) = \sum_{i=1}^{n} v_i^n(t) \sin(\pi i x), \tag{10b}$$

$$\omega^n(x,t) = \sum_{j=1}^{n} \omega_j^n(t) \sin(\pi j x), \tag{10c}$$

$$\theta^n(x,t) = \sum_{k=0}^{n} \theta_k^n(t) \cos(\pi k x), \tag{10d}$$

where

$$\{(v_i^n, \omega_j^n, \theta_k^n, z_r^n); i, j, r = 1, \ldots, n, k = 0, \ldots, n\} \tag{11}$$

is a solution to the following Cauchy problem:

$$\dot{v}_i^n(t) = \Phi_i^n(t, v_1^n, \ldots v_n^n, \omega_1^n, \ldots, \omega_n^n, \theta_0^n, \theta_1^n, \ldots, \theta_n^n, z_1^n, \ldots, z_n^n), \tag{12a}$$

$$\dot{\omega}_j^n(t) = \Psi_j^n(t, v_1^n, \ldots v_n^n, \omega_1^n, \ldots, \omega_n^n, \theta_0^n, \theta_1^n, \ldots, \theta_n^n, z_1^n, \ldots, z_n^n), \tag{12b}$$

$$\dot{\theta}_k^n(t) = \lambda_k \Pi_k^n(t, v_1^n, \ldots v_n^n, \omega_1^n, \ldots, \omega_n^n, \theta_0^n, \theta_1^n, \ldots, \theta_n^n, z_1^n, \ldots, z_n^n), \tag{12c}$$

$$\dot{z}_r^n(t) = v_r^n(t), \tag{12d}$$

$$v_i^n(0) = v_{0i},\ \omega_j^n(0) = \omega_{0j},\ \theta_k^n(0) = \theta_{0k},\ z_r^n(0) = 0 \tag{12e}$$

with

$$\lambda_0 = 1;\ \lambda_k = 2,\ k = 1, 2, \ldots, n \tag{13}$$

and

$$\Phi_i^n = 2\int_0^1 \left[\frac{\partial}{\partial x}\left(\rho^n \frac{\partial v^n}{\partial x}\right) - K\frac{\partial}{\partial x}(\rho^n \theta^n)\right] \sin(\pi i x) dx, \tag{14a}$$

$$\Psi_j^n = 2\int_0^1 A\left[\frac{\partial}{\partial x}\left(\rho^n \frac{\partial \omega^n}{\partial x}\right) - \frac{\omega^n}{\rho^n}\right] \sin(\pi j x) dx, \tag{14b}$$

$$\begin{aligned} \Pi_k^n = \int_0^1 \Bigg[& -K\rho^n\theta^n \frac{\partial v^n}{\partial x} + \rho^n \left(\frac{\partial v^n}{\partial x}\right)^2 + \rho^n \left(\frac{\partial \omega^n}{\partial x}\right)^2 \\ & + \frac{(\omega^n)^2}{\rho^n} + D\frac{\partial}{\partial x}\left(\rho^n \frac{\partial \theta^n}{\partial x}\right)\Bigg] \cos(\pi k x) dx. \end{aligned} \tag{14c}$$

In Mujaković [6] is also proved that there exists $T \in \mathbf{R}^+$ such that for each $n \in \mathbf{N}$ the problem (12) has a unique solution on $[0, T]$ and for the sequences $\{\rho^n\}$, and $\{(v^n, \omega^n, \theta^n)\}$ defined by (10) we have the following convergencies:

$$\rho^n \to \rho \text{ strongly in } C(\bar{Q}_T), \tag{15a}$$

$$(v^n, \omega^n, \theta^n) \to (v, \omega, \theta) \text{ strongly in } (L^2(Q_T))^3, \tag{15b}$$

where $(\rho, v, \omega, \theta)$ is a solution of the problem (1)-(4).

Because of simplicity here we take

$$\rho_0(x) = 1, \tag{16a}$$

$$v_0(x) = \sin(\pi x), \tag{16b}$$

$$\omega_0(x) = \sin(2\pi x), \tag{16c}$$

$$\theta_0(x) = 2 + \cos(\pi x) \tag{16d}$$

and

$$K = A = D = 1. \tag{17}$$

Inserting (10), (16) and (17) into (14), the Cauchy problem (12) becomes:

$$\dot{v}_i^n(t) = 2\pi \int_0^1 \left[\frac{\cos(\pi i x)\theta_0^n(t)}{1 + \sum_{j=1}^n \pi j \cos(\pi j x) z_j^n(t)} + \frac{\cos(\pi i x) \sum_{j=1}^n (\theta_j^n(t) - \pi j v_j^n(t)) \cos(\pi j x)}{1 + \sum_{j=1}^n \pi j \cos(\pi j x) z_j^n(t)} \right] dx, \tag{18a}$$

$$\begin{aligned} \dot{\omega}_i^n(t) = {} & -2\pi i \sum_{j=1}^n \pi j \omega_j^n(t) \int_0^1 \frac{\cos(\pi i x)\cos(\pi j x)}{1 + \sum_{k=1}^n \pi k \cos(\pi k x) z_k^n(t)} dx \\ & -2 \sum_{j=1}^n \omega_j^n(t) \times \int_0^1 \frac{\sin(\pi i x)\sin(\pi j x)}{\left[\sum_{k=1}^n \pi k \cos(\pi k x) z_k^n(t)\right]^{-1}} dx \\ & -2 \sum_{j=1}^n \omega_j^n(t) \int_0^1 \sin(\pi i x)\sin(\pi j x) dx, \end{aligned} \tag{18b}$$

$$\begin{aligned}\dot{\theta}_i^n(t) \quad = \quad & \lambda_i \int_0^1 \left[\frac{\left[\sum_{j=0}^{n} \theta_j^n(t)\cos(\pi j x)\right] \cdot \left[\sum_{j=1}^{n} \pi j \cos(\pi j x) v_j^n(t)\right]}{-1 - \sum_{j=1}^{n} \pi j \cos(\pi j x) z_j^n(t)} \right. \\ & + \frac{\left[\sum_{j=1}^{n} \pi j \cos(\pi j x) v_j^n(t)\right]^2 + \left[\sum_{j=1}^{n} \pi j \cos(\pi j x) \omega_j^n(t)\right]^2}{1 + \sum_{j=1}^{n} \pi j \cos(\pi j x) z_j^n(t)} \\ & + \left[1 + \sum_{j=1}^{n} \pi j \cos(\pi j x) z_j^n(t)\right] \cdot \left[\sum_{j=1}^{n} \sin(\pi j x) \omega_j^n(t)\right]^2 \\ & \left. + \pi i \frac{\left[\sum_{j=1}^{n} \pi j \sin(\pi j x) \theta_j^n(t)\right] \sin(\pi i x)}{1 + \sum_{j=1}^{n} \pi j \cos(\pi j x) z_j^n(t)} \right] \cos(\pi i x) dx, \end{aligned} \tag{18c}$$

$$\dot{z}_i^n(t) = v_i^n(t) \tag{18d}$$

with the following initial conditions:

$$v_1^n(0) = 1; v_i^n(0) = 0, i \neq 1, \tag{19a}$$

$$\omega_2^n(0) = 1; \omega_i^n(0) = 0, i \neq 2, \tag{19b}$$

$$\theta_0^n(0) = 2, \theta_1^n(0) = 1; \theta_i^n(0) = 0, i \neq 1, 2, \tag{19c}$$

$$z_i^n(0) = 0. \tag{19d}$$

For numerical calculations of the equations (18) we implement the Gaussian-Legendre quadrature formula (Plato [9]). Let x_l and w_l be, respectively, notes for support abscissas and weights derived from Legendre polynomial. Now, instead of (18) we have:

$$\dot{v}_i^n(t) = 2\pi \sum_{l=1}^{m} w_l \left[\frac{\cos(\pi i x_l)\theta_0^n(t)}{1 + \sum_{j=1}^{n} \pi j \cos(\pi j x_l) z_j^n(t)} \right.$$

$$+\left.\frac{\cos(\pi i x_l)\sum_{j=1}^{n}(\theta_j^n(t)-\pi j v_j^n(t))\cos(\pi j x_l)}{1+\sum_{j=1}^{n}\pi j\cos(\pi j x_l)z_j^n(t)}\right], \tag{20a}$$

$$\dot{\omega}_i^n(t) = -2\pi i\sum_{j=1}^{n}\pi j\omega_j^n(t)\sum_{l=1}^{m} w_l\frac{\cos(\pi i x_l)\cos(\pi j x_l)}{1+\sum_{k=1}^{n}\pi k\cos(\pi k x_l)z_k^n(t)}$$

$$-2\sum_{j=1}^{n}\omega_j^n(t)\sum_{l=1}^{m} w_l\frac{\sin(\pi i x_l)\sin(\pi j x_l)}{\left[1+\sum_{k=1}^{n}\pi k\cos(\pi k x_l)z_k^n(t)\right]^{-1}}, \tag{20b}$$

$$\dot{\theta}_i^n(t) = \lambda_i\sum_{l=1}^{m} w_l\cos(\pi i x_l)\left[\pi i\frac{\sin(\pi i x_l)\sum_{j=1}^{n}\pi j\sin(\pi j x_l)\theta_j^n(t)}{1+\sum_{j=1}^{n}\pi j\cos(\pi j x_l)z_j^n(t)}\right.$$

$$+\frac{\left[\sum_{j=1}^{n}\pi j\cos(\pi j x_l)v_j^n(t)\right]^2+\left[\sum_{j=1}^{n}\pi j\cos(\pi j x_l)\omega_j^n(t)\right]^2}{1+\sum_{j=1}^{n}\pi j\cos(\pi j x_l)z_j^n(t)}$$

$$+\left[1+\sum_{j=1}^{n}\pi j\cos(\pi j x_l)z_j^n(t)\right]\cdot\left[\sum_{j=1}^{n}\sin(\pi j x_l)\omega_j^n(t)\right]^2$$

$$\left.-\frac{\left[\sum_{j=0}^{n}\theta_j^n(t)\cos(\pi j x_l)\right]\cdot\left[\sum_{j=1}^{n}\pi j\cos(\pi j x_l)v_j^n(t)\right]}{1+\sum_{j=1}^{n}\pi j\cos(\pi j x_l)z_j^n(t)}\right], \tag{20c}$$

$$\dot{z}_i^n(t) = v_i^n(t). \tag{20d}$$

This system of $4n+1$ ordinary differential equations has the same initial conditions as (18).

Using *Wolfram Mathematica* we solve system (20) and its solution is a

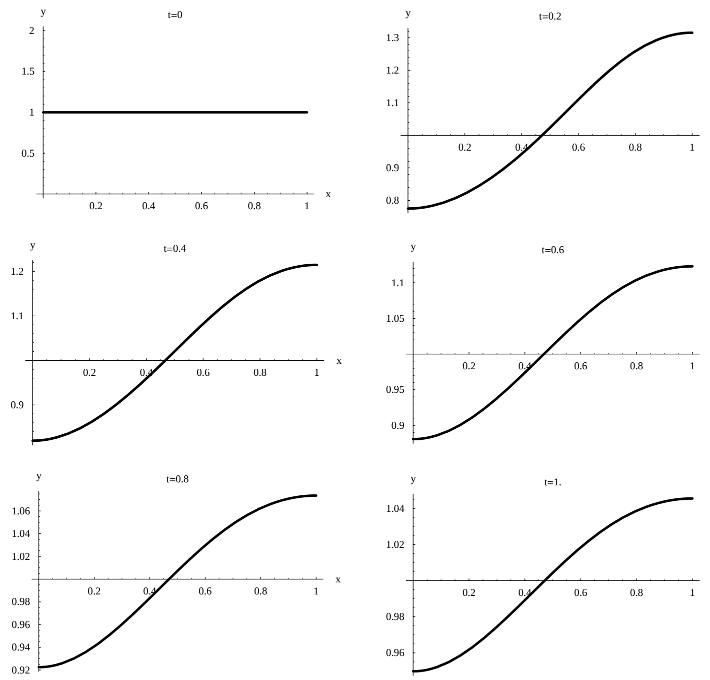

Figure 1: Graph ($n = 8,\ m = 20$) for the approximation ρ^n of the function $\rho(x,t)$ for some values of t.

numerical solution of the Cauchy problem (18)-(19).

4. Numerical Experiments

To get the numerical solutions of the system (18) we use *Wolfram Mathematica 5.2*. The procedure is as follows:

1. We use the package *NumericalMath'GaussianQuadrature'* and function *GaussianQuadratureWeights* to get the support abscissas x_l and weights w_l for Gauss-Legendre quadrature formula (Fukuda et al [5]). This way we get m pairs (x_l, w_l) and then form the system (20).

2. To solve the system (20) we use the function *NDSolve*. By using this solver it is possible to choose the method for solving the system of ordinary differential equations but we have decided to let the solver to choose appropriate method automatically and allow him to change it during the process to achieve

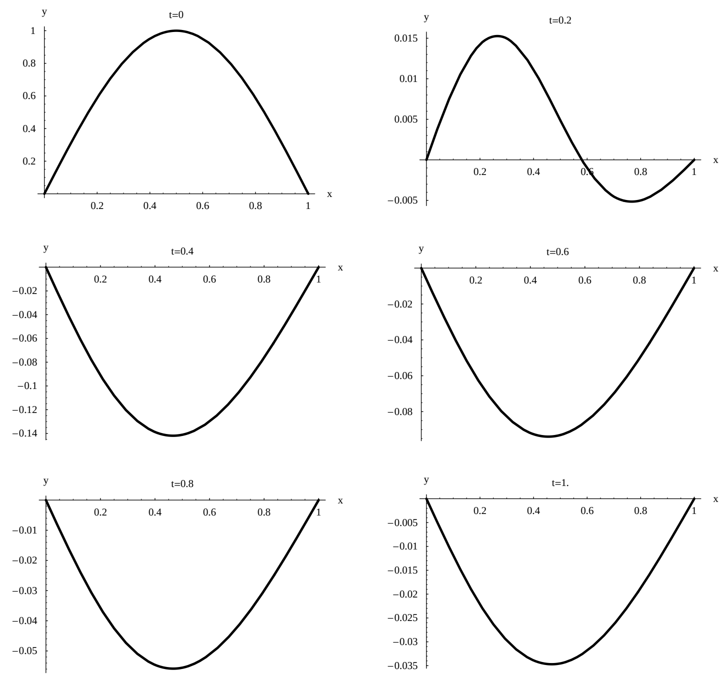

Figure 2: Graph ($n = 8,\ m = 20$) for the approximation v^n of the function $v(x,t)$ for some values of t.

better accuracy (Abel et al [1]).

The exact analytic solution of the problem (1)-(4) is not known so we could not test the errors of our calculations directly. The physical measurements for testing this model or other numerical calculations were not made so there are no possibilities to compare our results with other authors. The only way to test our calculations is to test the convergence of the sequence of the numerical solutions when $n \to \infty$.

The numerical analysis of the convergence is made in L_∞ norm and we expect the following results:

$$\|\rho^{n+1} - \rho^n\|_\infty \to 0, \tag{21}$$

$$\|v^{n+1} - v^n\|_\infty \to 0, \tag{22}$$

$$\|\theta^{n+1} - \theta^n\|_\infty \to 0, \tag{23}$$

$$\|\omega^{n+1} - \omega^n\|_\infty \to 0 \tag{24}$$

and the obtained results of this analysis are shown in Table 1. Our calculations

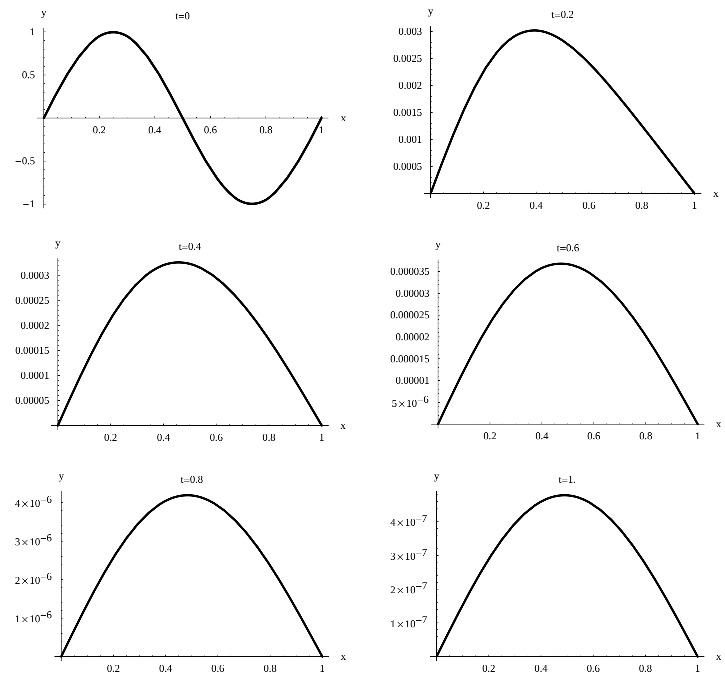

Figure 3: Graph ($n = 8,\ m = 20$) for the approximation ω^n of the function $\omega(x,t)$ for some values of t.

n	$\|\rho^n - \rho^{n-1}\|_\infty$	$\|v^n - v^{n-1}\|_\infty$	$\|\omega^n - \omega^{n-1}\|_\infty$	$\|\theta^n - \theta^{n-1}\|_\infty$
3	0.00212	0.00014	0.00193	0.00220
4	0.00128	0.00030	0.00008	0.00035
5	0.00019	0.00004	$3.41310 \cdot 10^{-6}$	0.00001
6	0.00002	$5.72610 \cdot 10^{-6}$	$3.98094 \cdot 10^{-6}$	$7.46962 \cdot 10^{-6}$
7	$1.15960 \cdot 10^{-6}$	$1.94703 \cdot 10^{-7}$	$3.53974 \cdot 10^{-7}$	$1.26289 \cdot 10^{-6}$
8	$2.76128 \cdot 10^{-7}$	$2.16108 \cdot 10^{-8}$	$2.19735 \cdot 10^{-8}$	$9.09979 \cdot 10^{-8}$
9	$1.74394 \cdot 10^{-8}$	$2.18109 \cdot 10^{-8}$	$3.36934 \cdot 10^{-9}$	$1.81073 \cdot 10^{-8}$
10	$7.30050 \cdot 10^{-9}$	$9.89589 \cdot 10^{-9}$	$5.04436 \cdot 10^{-9}$	$1.12634 \cdot 10^{-9}$

Table 1: Analysis of convergence in L_∞-norm for $m = 20$.

were done by taking the time variable t from $[0, 1]$.

Approximative functions ρ^n, v^n, ω^n and θ^n are presented in Figures 1, 2, 3 and 4 for some values of variable t.

By increasing the values of t we must expect our results in accordance

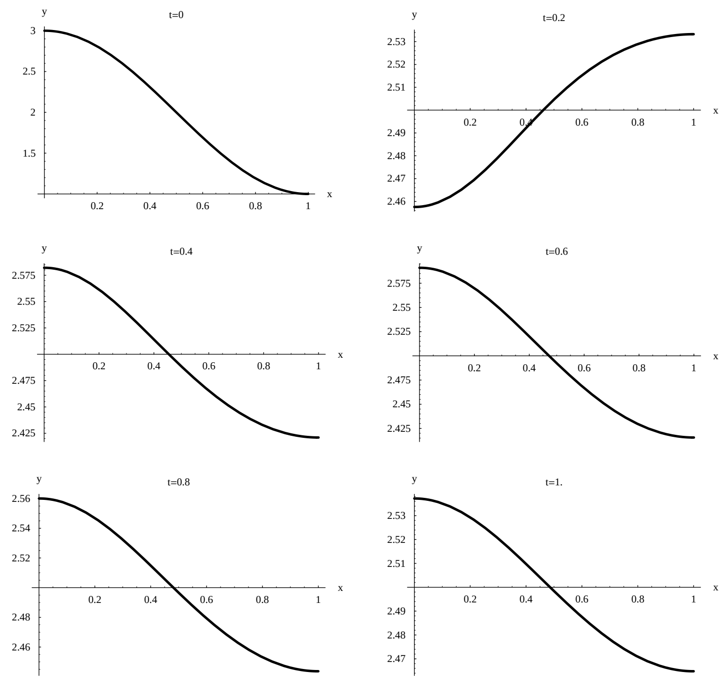

Figure 4: Graph ($n = 8,\ m = 20$) for the approximation θ^n of the function $\theta(x,t)$ for some values of t.

with the results from Mujaković [8]. In this article it has been proven that, when $t \to \infty$, the solution $(\rho, v, \omega, \theta)$ of the problem (1)-(4) converges to the stationary solution $(\alpha^{-1}, 0, 0, E_1)$ in the space $(H^1(\Omega))^4$. The constants α and E_1 are defined by

$$\alpha = \int_0^1 \frac{1}{\rho_0(x)} dx,\ E_1 = \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} + \frac{1}{2A}\|\omega_0\|^2_{L^2(\Omega)} + \|\theta_0\|_{L^1(\Omega)}. \tag{25}$$

For the initial conditions (16) we have

$$\alpha = 1,\ E_1 = 2,5. \tag{26}$$

In Figure 1 we could see that the density oscillates around the initial condition $\rho_0 = 1$, for $x \in \Omega$, and is getting nearer to the initial functions with the increasing of time, as we expected.

By increasing time the velocity and microrotation tend to zero (Figure 2, Figure 3) and temperature tends towards 2.5 (Figure 4) which is in accordance with the results in Mujaković [8].

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