

MATHEMATICAL MODEL FOR FRACTAL MANIFOLD

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Abstract: We have built a new kind of manifolds which leads to an alternative new geometrical space. The study of the nowhere differentiable functions via a family of mean functions leads to a new characterization of this category of functions. A fluctuant manifold has been built with an appearance of a new structure on it at every scale, and we embedded into it an internal structure to transform its fluctuant geometry to a new fixed geometry. This approach leads us to what we call fractal manifold. The elements of this kind of manifold appear locally as tiny double strings, with an appearance of new structure at every step of approximation. We have obtained a variable dimensional space which is locally neither a continuum nor a discrete, but a mixture of both. Space acquires a variable geometry, it becomes explicitly dependent on the approximation process, and the geometry on it is assumed to be characterized not only by curvature, but also by the appearance of new structure at every step of approximation.

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1. Introduction

Up to now there is no real understanding of the nature of the space of an elementary particle, and the current space-time is not satisfactory for particle theory [1]. Moreover, Feynman has demonstrated that the typical quantum

mechanical paths (that contribute in a dominant way to the path integral) are continuous but nowhere differentiable, and can be characterized by a fractal dimension 2 [8, 9, 20]. A fractal¹ space time was suggested in 1983 by G. Ord and L. Nottale [18, 19], but until now there is no mathematical model of such space. Given up the differentiability² to describe the space-time in quantum mechanics was suggested early by Einstein without pursuit, and suggested in 1993 by Nottale [17], which leads to the scale relativity. Giving up the differentiability means that we have to deal with the nowhere differentiable functions, and the difficulties encountered are as follows:

i) There are no much tools nor analytic characterization for the nowhere differentiable functions [4, 21] other than the notion of fractal dimension.

ii) There is no definition of nowhere differentiable manifolds to deal with it, and this is a big challenge. Do we need a class of manifolds with no smooth structure on it? Or a class of manifolds with a smooth structure on it? Or any other kind of mathematical object which gives us a variable geometrical space?

A non differentiable manifold with no smooth structure on it is an old problem at the time of Milnor, unsolved because of its difficulty. Our purpose in the present paper is to investigate the possibility to construct a manifold that presents a variable local geometry. For this purpose our plan is as follows:

a) We approximate the nowhere differentiable function by a mean function which follows the geometry of the approximated function.

b) We introduce a new characterization of nowhere differentiable functions, and we define a double space called (ε -manifolds) using graphs.

c) We introduce a model of fractal manifold using family of graphs.

e) At the end, using the process mean of the mean we prove that elements of the fractal manifold are double strings with appearance of new structures on it at every step of approximation. The fractal manifolds obtained are locally neither a continuum nor a discrete, but a mixture of both.

¹The word *fractal* is a new term introduced by Mandelbrot [14, 15] to represent shapes or phenomena having no characteristic length.

²In a letter to Pauli [7], Einstein suggested that a true understanding of quantum physics could imply to give up differentiability.

2. Basic Concepts and Tools

2.1. Scale and Resolution

Let $[a, b]$, $a < b$, be an interval included in \mathbb{R} , with length $l = b - a$. It is possible to construct a family of subintervals of length $l = \frac{b-a}{n}$, $\forall n \geq 1$. This kind of subdivision is countable and the subintervals obtained depend on the real numbers a and b . For a non countable subdivision of a unit interval, let us consider the length $l = \frac{1}{\beta}$, $\beta > 0$ real number and we introduce the following definitions.

Definition 1. We call *resolution* the quotient of the unit by a real number $\beta > 0$ and we denote it by $\varepsilon = \frac{1}{\beta}$, for all $\beta > 0$.

This resolution ε takes an infinity of values between 0 and $+\infty$, and we can extend it by continuity on zero. We admit that the resolution ε is equal to zero as β tends to infinity and we have $\forall \beta > 0, \varepsilon \in [0, +\infty[$.

Definition 2. We call *scale* and we denote it by \mathcal{E} the inverse of the resolution, such that $\mathcal{E}\varepsilon = 1$.

Let us consider a continuous and nowhere differentiable function $f(x)$, in a given interval $\mathcal{I} \subset \mathbb{R}$. Since $\forall x \in \mathcal{I}$ we can not find $\frac{df(x)}{dx}$, it is impossible to draw the graph of this kind of functions. However, it is possible to approximate it through differentiable functions. We replace the nowhere differentiable function f by the following mean function given by

$$f(x, \varepsilon) = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(t)dt, \quad \forall x \in \mathcal{I}. \tag{1}$$

2.2. Approximation Domain

For all $\varepsilon > 0$, the derivative of the mean function (1) is given by $f'(x, \varepsilon) = \frac{f(x+\varepsilon)-f(x-\varepsilon)}{2\varepsilon}$. For $\varepsilon = 0$ we have $f(x, 0) = \lim_{\varepsilon \rightarrow 0} f(x, \varepsilon) = f(x)$ which is nowhere differentiable, and $f(x, \varepsilon)$ is always differentiable for all $\varepsilon \in]0, +\infty[$. We call domain of approximation of the function f the set

$$\mathcal{Q}_f = \{ \varepsilon \in \mathbb{R}^+ / f(x, \varepsilon) \text{ is differentiable on } \mathcal{I} \}, \tag{2}$$

Definition 3. We call small resolution domain, and we denote it by \mathcal{R}_f , the intersection between the domain of approximation \mathcal{Q}_f and $[0, \alpha]$, $0 < \alpha \ll 1$

$$\mathcal{R}_f = \{ \varepsilon \in \mathbb{R}^+ / f(x, \varepsilon) \text{ is differentiable on } \mathcal{I} \} \cap [0, \alpha]. \tag{3}$$

2.3. Rate of Change

Let us consider a function $f(x)$, $x \in \mathcal{I} \subset \mathbb{R}$ an open interval, where the only information we have is given by the following:

- 1) The function f is continuous on \mathcal{I} .
- 2) The function f is nowhere differentiable on \mathcal{I} .
- 3) The function f is unknown.
- 4) The mean function given by (1) is defined for all $\varepsilon \in \mathcal{Q}_f$.

What kind of information can we obtain about the unknown function f ? In this case, we cannot say that the main information we have is given by the best approximation, and this for the simple reason that we cannot determine the error of approximation if the function is unknown. One can say by experience that the best approximation is given by using the smallest resolution, but the problem here is as follows: there is no smallest value of $\varepsilon \in]0, 1]$ (no minimal resolution in $]0, 1]$). We propose the following lemma.

Lemma 4. 1) Let f be a continuous and nowhere differentiable function. If the mean forward $F^+(x, \varepsilon) = \frac{1}{\varepsilon} \int_x^{x+\varepsilon} f(t)dt$, and the mean-backward $F^-(x, \varepsilon) = \frac{1}{\varepsilon} \int_{x-\varepsilon}^x f(t)dt$, $\forall (x, \varepsilon) \in I \times \mathcal{R}_f$, then

$$f(x) = F^\sigma(x, \varepsilon) + \varepsilon \left(\frac{\partial F^\sigma(x, \varepsilon)}{\partial \varepsilon} - \sigma \frac{\partial F^\sigma(x, \varepsilon)}{\partial x} \right), \quad \sigma = \pm. \quad (4)$$

- 2) Let f be a continuous and nowhere differentiable function, if

$$F^\sigma(x, \delta_1, \dots, \delta_n) = \frac{(\sigma 1)^n}{\prod_{i=1}^n \delta_i} \int_x^{x+\sigma\delta_n} \dots \int_{t_1}^{t_1+\sigma\delta_1} f(t_0) dt_0 \dots dt_{n-1}, \quad \sigma = \pm. \quad (5)$$

$\forall \delta_2, \dots, \delta_n \in [0, \alpha]$, $0 < \alpha \ll 1$, $\forall \delta_1 \in \mathcal{R}_f$, $x \in \mathcal{I}$, then $\forall n \geq 1$, $\sigma = \pm$

$$f(x) = F^\sigma(x, \delta_1, \dots, \delta_n) + \sum_{i=1}^n \delta_i \left(\frac{\partial F^\sigma(x, \delta_1, \dots, \delta_i)}{\partial \delta_i} - \sigma \frac{\partial F^\sigma(x, \delta_1, \dots, \delta_i)}{\partial x} \right). \quad (6)$$

Proof. 1) It is not difficult to see that

$$\frac{\partial F^+(x, \varepsilon)}{\partial x} = \frac{f(x+\varepsilon) - f(x)}{\varepsilon}, \quad \frac{\partial F^+(x, \varepsilon)}{\partial \varepsilon} = \frac{-1}{\varepsilon} F^+(x, \varepsilon) + \frac{f(x+\varepsilon)}{\varepsilon}. \quad (7)$$

From the equations (7), one can find

$$f(x) - F(x, \varepsilon) = \varepsilon \left(\frac{\partial F(x, \varepsilon)}{\partial \varepsilon} - \frac{\partial F(x, \varepsilon)}{\partial x} \right).$$

Similar computation for $\sigma = -$ allow us to conclude.

- 2) Using formula (4) and by induction over n we find the result. \square

Remark. If we have the graphs of the mean forward function or the mean backward function for all $(x, \varepsilon) \in I \times \mathcal{R}_f$ of a nowhere differentiable function f , then it is sufficient to study the variation of the mean forward function or the

mean backward function for all $(x, \varepsilon) \in I \times \mathcal{R}_f$ to obtain the unknown function f . It means that we must consider not only one approximation but all the approximations given for all $\varepsilon \in \mathcal{R}_f$. We know that the position of an electron and his momentum are subject to Heisenberg's uncertainly principle. It states that the position and momentum of a particle at the quantum microscopic level, cannot be measured simultaneously with complete accuracy. Hence, this means in our case that it is impossible to determine our unknown function by one approximation (even the best), actually, we cannot obtain a real description of the function f if we consider only one approximation³ out of all.

2.4. Double and Multiple Additions

Let us consider the function given by the formula (1) for a given continuous and nowhere differentiable function $f(x)$, $x \in \mathcal{I}$. This mean function can be written as sum of two mean functions

$$f(x, \varepsilon) = \frac{1}{2} \left[f\left(x - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) + f\left(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) \right], \tag{8}$$

where the functions $f\left(x - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$ and $f\left(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$ represent an approximation of the function f at the resolution $\frac{\varepsilon}{2}$. The graph of the function $f\left(x - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$ denoted by Γ_ε^- is a copy of the graph of the function $f\left(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$ denoted by Γ_ε^+ , where Γ_ε^+ is obtained by horizontal translation from the graph Γ_ε^- . If we superimpose the two graphs Γ_ε^- and Γ_ε^+ , we obtain two identical graphs of the function f with gap. This procedure can be repeated n times and we have the following lemma.

Lemma 5. *Let f be a continuous function on \mathcal{I} . For all $(x, \varepsilon) \in (\mathcal{I} \times \mathcal{R}_f)$, we have*

$$f(x, \varepsilon) = \frac{1}{2^n} \sum_{i=1}^{2^{n-1}} f\left(x - \frac{(2i-1)\varepsilon}{2^n}, \frac{\varepsilon}{2^n}\right) + f\left(x + \frac{(2i-1)\varepsilon}{2^n}, \frac{\varepsilon}{2^n}\right), \quad \forall n \geq 1. \tag{9}$$

Proof. For $n = 1$, the formula (9) is verified (addition property of integral) and we have $f(x, \varepsilon) = \frac{1}{2}(f(x - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}) + f(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2}))$. We suppose that we have the formula (9) until the order n , and let us prove it at the order $n + 1$. We have

$$f(x, \varepsilon) = \frac{1}{2^n} \sum_{i=1}^{2^{n-1}} f\left(x - \frac{(2i-1)\varepsilon}{2^n}, \frac{\varepsilon}{2^n}\right) + f\left(x + \frac{(2i-1)\varepsilon}{2^n}, \frac{\varepsilon}{2^n}\right)$$

³If we can draw the graph of one mean function F_1 , then we have a measurement of the position $(x, F_1(x))$ and the momentum simultaneously (since we have $\frac{dF_1}{dx}$).

and using the addition given by the formula (8) of the functions

$$f\left(x - \frac{(2i-1)}{2^n}\varepsilon, \frac{\varepsilon}{2^n}\right) \quad \text{and} \quad f\left(x + \frac{(2i-1)}{2^n}\varepsilon, \frac{\varepsilon}{2^n}\right), \quad \text{we obtain}$$

$$\begin{aligned} f(x, \varepsilon) &= \frac{1}{2^{n+1}} \sum_{i=1}^{2^{n-1}} f\left(x - \frac{(2i-1)}{2^n}\varepsilon - \frac{\varepsilon}{2^{n+1}}, \frac{\varepsilon}{2^{n+1}}\right) \\ &\quad + f\left(x - \frac{(2i-1)}{2^n}\varepsilon + \frac{\varepsilon}{2^{n+1}}, \frac{\varepsilon}{2^{n+1}}\right) \\ &+ f\left(x + \frac{(2i-1)}{2^n}\varepsilon - \frac{\varepsilon}{2^{n+1}}, \frac{\varepsilon}{2^{n+1}}\right) + f\left(x + \frac{(2i-1)}{2^n}\varepsilon + \frac{\varepsilon}{2^{n+1}}, \frac{\varepsilon}{2^{n+1}}\right), \quad \text{then} \\ f(x, \varepsilon) &= \frac{1}{2^{n+1}} \sum_{i=1}^{2^{n-1}} f\left(x - \frac{(4i-1)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right) + f\left(x - \frac{(4i-3)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right) \\ &\quad + f\left(x + \frac{(4i-3)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right) + f\left(x + \frac{(4i-1)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right). \end{aligned}$$

With an adequate rearrangement, our sum can be written as

$$\begin{aligned} f(x, \varepsilon) &= \frac{1}{2^{n+1}} \sum_{i=1}^{2^{n-1}} f\left(x - \frac{(4i-1)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right) + f\left(x + \frac{(4i-1)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right) \\ &\quad + \frac{1}{2^{n+1}} \sum_{i=1}^{2^{n-1}} f\left(x + \frac{(4i-3)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right) + f\left(x - \frac{(4i-3)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right). \end{aligned}$$

For simplicity, we put $f(x, \varepsilon) = E + O$ with

$$E = \frac{1}{2^{n+1}} \sum_{i=1}^{2^{n-1}} f\left(x - \frac{(4i-1)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right) + f\left(x + \frac{(4i-1)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right) \quad (10)$$

$$O = \frac{1}{2^{n+1}} \sum_{i=1}^{2^{n-1}} f\left(x + \frac{(4i-3)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right) + f\left(x - \frac{(4i-3)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right). \quad (11)$$

Now, if we put $j = 2i$ for the formula (10), and $j = 2i - 1$ for the formula (11), we obtain the following sums:

a) Sum over even numbers:

$$E = \frac{1}{2^{n+1}} \sum_{j=2}^{2^n} f\left(x - \frac{(2j-1)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right) + f\left(x + \frac{(2j-1)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right) \quad (12)$$

b) Sum over odd numbers:

$$O = \frac{1}{2^{n+1}} \sum_{j=1}^{2^{n-1}} f\left(x + \frac{(2j-1)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right) + f\left(x - \frac{(2j-1)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right). \quad (13)$$

Finally we see that we have

$$\begin{aligned}
 f(x, \varepsilon) &= E + O \\
 &= \frac{1}{2^{n+1}} \sum_{i=1}^{2^n} f\left(x - \frac{(2i-1)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right) + f\left(x + \frac{(2i-1)}{2^{n+1}}\varepsilon, \frac{\varepsilon}{2^{n+1}}\right), \quad (14)
 \end{aligned}$$

and then the result $\forall n \geq 1$. □

Remark. The simple addition given by the formula (8) can be repeated n times to obtain the formula (9), where $f(x - \frac{(2i-1)}{2^n}\varepsilon, \frac{\varepsilon}{2^n})$ and $f(x + \frac{(2i-1)}{2^n}\varepsilon, \frac{\varepsilon}{2^n})$ for all $n \geq 1, i = 1, 2, \dots, 2^{n-1}$, are the mean functions of f at the resolution $\frac{\varepsilon}{2^n}$. As an example of nowhere differentiable function we mention the Weierstrass function given by $W_\alpha(x) = \sum_{m=1}^{+\infty} q^{-\alpha m} \exp(iq^m x)$.

For $n = 1$, we obtain 2 identical graphs of approximation (see Figure 1).

For $n = 2$, we obtain 2^2 identical graphs of approximation (see Figure 2).

For $n = 3$, we obtain 2^3 identical graphs of approximation (see Figure 3).

For a given integer n , we obtain 2^n multiple identical graphs of approximation. If we write $f(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ as an addition of two identical mean functions, we will obtain an odd number of non-identical graphs (see Figure 4).

2.5. New Characterization

Let consider, for $\varepsilon \in \mathcal{R}_f, \Gamma_\varepsilon = \{(x, y) \in \mathbb{R}^2 / y = f(x, \varepsilon), x \in \mathcal{I}\}$ the graph associated to the mean function given by (1).

Let $\Gamma_f = \{(x, y) \in \mathbb{R}^2 / y = f(x), x \in \mathcal{I}\}$ be the graph associated to the nowhere differentiable function f . Let (\mathbb{R}^2, ρ) be a metric space, and let us consider the Hausdorff (see [10]) measure given by

$$d_h(\Gamma_\varepsilon, \Gamma_f) = \text{Max} \left[\sup_{A \in \Gamma_\varepsilon} \inf_{B \in \Gamma_f} \rho(A, B), \sup_{A \in \Gamma_f} \inf_{B \in \Gamma_\varepsilon} \rho(A, B) \right].$$

The mean function given by (1) converges uniformly to the nowhere differentiable function f . We say that the curve Γ_ε converges to the graph Γ_f if $d_h(\Gamma_\varepsilon, \Gamma_f)$ tends to 0 as ε tends to 0, and that two curves Γ_1 and Γ_2 are disjoint if $d_h(\Gamma_1, \Gamma_2) > 0$. We have the following criterion.

Lemma 6. *Let f be a continuous function on $\mathcal{I} \subset \mathbb{R}$, then:*

- i) f is nowhere differentiable on $\mathcal{I} \iff \forall \varepsilon \in \mathcal{R}_f, d_h(\Gamma_\varepsilon^-, \Gamma_\varepsilon^+) > 0$.
- ii) f is differentiable on $\mathcal{I} \iff \exists \varepsilon \in \mathcal{R}_f$ such that $d_h(\Gamma_\varepsilon^-, \Gamma_\varepsilon^+) = 0$.

Proof. For

$$d_h(\Gamma_\varepsilon^+, \Gamma_\varepsilon^-) = \text{Max} \left[\sup_{A \in \Gamma_\varepsilon^+} \inf_{B \in \Gamma_\varepsilon^-} \rho(A, B), \sup_{A \in \Gamma_\varepsilon^-} \inf_{B \in \Gamma_\varepsilon^+} \rho(A, B) \right], \quad \text{where}$$

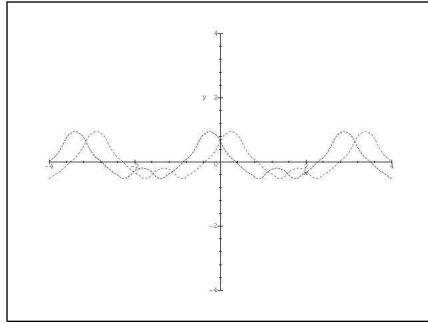


Figure 1: Double addition of a Weierstrass mean function, $\varepsilon = \frac{1}{2}$, $q = 2$, $\alpha = 0.75$.

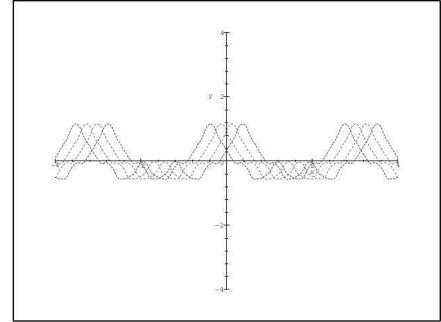


Figure 2: 2^2 additions of a Weierstrass mean function, $\varepsilon = \frac{1}{2}$, $q = 2$, $\alpha = 0.75$.

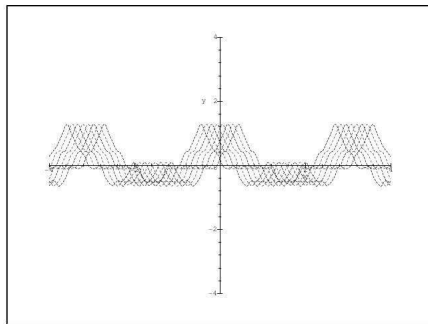


Figure 3: 2^3 additions of a Weierstrass mean function, $\varepsilon = \frac{1}{2}$, $q = 2$, $\alpha = 0.75$.

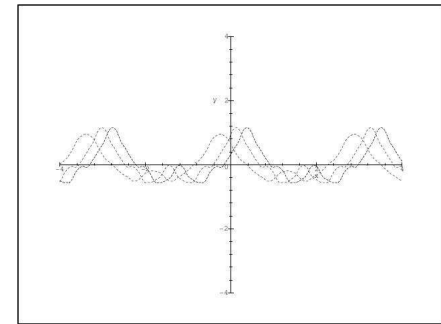


Figure 4: 3 additions of a Weierstrass mean function, $\varepsilon = \frac{1}{2}$, $q = 2$, $\alpha = 0.75$.

$$A = (x, y) \in \Gamma_\varepsilon^+, B = (x', y') \in \Gamma_\varepsilon^-, \text{ and } \rho(A, B) = \sqrt{(x - x')^2 + (y - y')^2}$$

$$= \sqrt{(x - x')^2 + (f(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2}) - f(x' - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}))^2}.$$

For $x' = x + \varepsilon$ we have $f(x' - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}) = f(x + \varepsilon - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}) = f(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, then we have $\rho(A, B) = \sqrt{\varepsilon^2} = \varepsilon$ and $d_h(\Gamma_\varepsilon^+, \Gamma_\varepsilon^-) = \varepsilon$, and following the definition of \mathcal{R}_f we can conclude. \square

The last property can be considered as a criterion of nowhere differentiability.

ity for functions.

3. Scale Manifold

We have seen that the geometry of the nowhere differentiable functions could be determined by the study of the variations of mean functions (1), we have seen also that the nowhere differentiability can be characterized by considering the superposition of the graph of the forward-backward mean functions. In this section we want to construct manifolds, which present locally an appearance of new structure at different scales (using differentiable objects), and we want to know if it is possible to construct a differentiable geometry on it. We will call it “fractal manifold”. This new object will allow us to construct a new space. The graphs of the family of mean functions $(f(x, \varepsilon))_{\varepsilon>0}$ could be considered globally as a fractal manifold of dimension 1 (the graphs have appearance of new structure at different resolutions), however, locally it does not present any new structure for different resolutions since it is homeomorphic to \mathbb{R} . The new idea is to consider the mixture of the family of mean functions and the local information given by the superposition of the forward-backward mean function that characterizes the nowhere differentiable functions.

3.1. Introduction of the ε -Manifolds

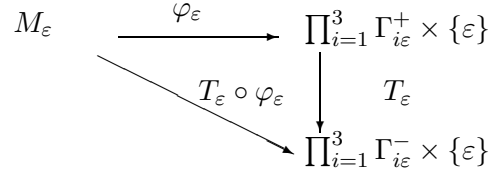
Let us consider f_1, f_2 and f_3 three continuous and nowhere differentiable functions, defined on the open interval \mathcal{I} . The graphs of these functions are given by $\Gamma_i(\mathcal{I}) = \left\{ (x, y) \in \mathbb{R}^2 / y = f_i(x), x \in \mathcal{I} \right\}$, $i = 1, 2, 3$. Let us consider the backward-forward mean functions given by

$$f_i\left(x - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) = \frac{1}{\varepsilon} \int_{x-\varepsilon}^x f_i(t) dt, \quad \text{and} \quad f_i\left(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) = \frac{1}{\varepsilon} \int_x^{x+\varepsilon} f_i(t) dt, \text{ for}$$

$i = 1, 2, 3$. We denote respectively their associated graphs: $\Gamma_{1,\varepsilon}^\sigma, \Gamma_{2,\varepsilon}^\sigma, \Gamma_{3,\varepsilon}^\sigma$ where $\sigma = \pm$ for the forward-backward. We see that the graphs $\Gamma_{1,\varepsilon}^\sigma, \Gamma_{2,\varepsilon}^\sigma, \Gamma_{3,\varepsilon}^\sigma$ converge to the graphs Γ_1, Γ_2 and Γ_3 as ε tends to 0 in a Hausdorff metric space (\mathbb{R}^2, d_h) . By definition of the set \mathcal{R}_f , we have $\mathcal{R}_{f_1} = \mathcal{R}_{f_2} = \mathcal{R}_{f_3}$. The set $\Gamma_{1,\varepsilon}^\sigma \times \Gamma_{2,\varepsilon}^\sigma \times \Gamma_{3,\varepsilon}^\sigma$ is included in \mathbb{R}^6 . Let us consider the metric space (\mathbb{R}^6, d_h) , then following the proof of Lemma 6, it is easy to find

$$d_h\left(\Gamma_{1,\varepsilon}^- \times \Gamma_{2,\varepsilon}^- \times \Gamma_{3,\varepsilon}^- \quad \Gamma_{1,\varepsilon}^+ \times \Gamma_{2,\varepsilon}^+ \times \Gamma_{3,\varepsilon}^+\right) = \sqrt{3}\varepsilon. \tag{15}$$

which means that the Hausdorff distance between $\Gamma_{1,\varepsilon}^\sigma \times \Gamma_{2,\varepsilon}^\sigma \times \Gamma_{3,\varepsilon}^\sigma$, $\sigma = \pm$, increases or decreases as a line with slope $\sqrt{3}$. We introduce what is called

Figure 5: Diagram of ε -manifold

ε -manifold or a local double space at the scale ε :

Definition 7. Let ε be in \mathcal{R}_f and let us consider the application

$$T_\varepsilon : \prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\} \longrightarrow \prod_{i=1}^3 \Gamma_{i\varepsilon}^- \times \{\varepsilon\} \quad \text{defined by}$$

$T_\varepsilon((a_1, b_1), (a_2, b_2), (a_3, b_3)) = ((a_1 + \varepsilon, b_1), (a_2 + \varepsilon, b_2), (a_3 + \varepsilon, b_3))$, where $(a_i, b_i) \in \Gamma_{i\varepsilon}^+$, that is to say $b_i = f_i(a_i + \frac{\varepsilon}{2}, \frac{\varepsilon}{2}) = \frac{1}{\varepsilon} \int_{a_i}^{a_i + \varepsilon} f_i(t) dt$, for $i = 1, 2, 3$.

Proposition 8. For all $\varepsilon \in \mathcal{R}_f$, we have the following properties:

- 1) the application T_ε is well defined.
- 2) T_ε^{-1} exists.
- 3) T_ε is an homeomorphism.
- 4) T_ε and T_ε^{-1} are differentiable.

Proof. 1) Let $((a_1, b_1), (a_2, b_2), (a_3, b_3)) \in \prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\}$, then for $i = 1, 2, 3$, we have $b_i = f_i(a_i + \frac{\varepsilon}{2}, \frac{\varepsilon}{2}) = f_i((a_i + \varepsilon) - \frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ then $(a_i + \varepsilon, b_i) \in \Gamma_{i\varepsilon}^-$, and $((a_1 + \varepsilon, b_1), (a_2 + \varepsilon, b_2), (a_3 + \varepsilon, b_3)) \in \prod_{i=1}^3 \Gamma_{i\varepsilon}^- \times \{\varepsilon\}$.

2) $(T_\varepsilon)^{-1}((a_1, b_1), (a_2, b_2), (a_3, b_3)) = ((a_1 - \varepsilon, b_1), (a_2 - \varepsilon, b_2), (a_3 - \varepsilon, b_3))$

3) and 4) T_ε and $(T_\varepsilon)^{-1}$ are continuous, differentiable as composite functions of continuous and differentiable functions. \square

Definition 9. Let ε be in \mathcal{R}_f , and M_ε be an Hausdorff topological space. We say that M_ε is an ε -manifold if for every point $x \in M_\varepsilon$, there exist a neighborhood Ω_ε of x in M_ε , a map φ_ε , and two open sets V_ε^+ of $\prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\}$ and V_ε^- of $\prod_{i=1}^3 \Gamma_{i\varepsilon}^- \times \{\varepsilon\}$ such that $\varphi_\varepsilon : \Omega_\varepsilon \longrightarrow V_\varepsilon^+$, and $T_\varepsilon \circ \varphi_\varepsilon : \Omega_\varepsilon \longrightarrow V_\varepsilon^-$ are two homeomorphisms.

One can just say that M_ε is an ε -manifold if M_ε is homeomorphic to $\prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\}$, and we have automatically the second homeomorphism $T_\varepsilon \circ \varphi_\varepsilon$.

Definition 10. A local chart on M_ε , for a given $\varepsilon \in \mathcal{R}_f$, is a triplet $(\Omega_\varepsilon, \varphi_\varepsilon, T_\varepsilon \circ \varphi_\varepsilon)$, where Ω_ε is an open set of M_ε , φ_ε is an homeomorphism from Ω_ε to an open set V_ε^+ of $\prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\}$ and $T_\varepsilon \circ \varphi_\varepsilon$ is an homeomorphism from

Ω_ε to an open set V_ε^- of $\prod_{i=1}^3 \Gamma_{i\varepsilon}^- \times \{\varepsilon\}$.

A collection $(\Omega_{\varepsilon,i}, \varphi_{\varepsilon,i}, \psi_{\varepsilon,i})_{i \in J}$ of local charts such that $\cup_{i \in J} \Omega_{\varepsilon,i} = M_\varepsilon$ is called an atlas. The coordinates of $x \in \Omega_\varepsilon$ related to the local chart $(\Omega_\varepsilon, \varphi_\varepsilon, T_\varepsilon \circ \varphi_\varepsilon)$ are the coordinates of the point $\varphi_\varepsilon(x)$ in $\prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\}$, and of the point $T_\varepsilon \circ \varphi_\varepsilon(x)$ in $\prod_{i=1}^3 \Gamma_{i\varepsilon}^- \times \{\varepsilon\}$.

Definition 11. An atlas of class \mathcal{C}^1 , on M_ε is an atlas for which all changes of charts are \mathcal{C}^1 . That is to say, if $(\Omega_{\varepsilon,i}, \varphi_{\varepsilon,i}, T_\varepsilon \circ \varphi_{\varepsilon,i})$, and $(\Omega_{\varepsilon,j}, \varphi_{\varepsilon,j}, T_\varepsilon \circ \varphi_{\varepsilon,j})$ are two local charts with $\Omega_{\varepsilon,i} \cap \Omega_{\varepsilon,j} \neq \emptyset$, then the map change of charts $\varphi_{\varepsilon,ij} = \varphi_{\varepsilon,j} \circ (\varphi_{\varepsilon,i})^{-1}$ from $\varphi_{\varepsilon,i}(\Omega_{\varepsilon,i} \cap \Omega_{\varepsilon,j})$ to $\varphi_{\varepsilon,j}(\Omega_{\varepsilon,i} \cap \Omega_{\varepsilon,j})$ is a diffeomorphism of class \mathcal{C}^1 , and the map change of charts $(T_\varepsilon \circ \varphi_{\varepsilon,j}) \circ (T_\varepsilon \circ \varphi_{\varepsilon,i})^{-1}$ from $T_\varepsilon \circ \varphi_{\varepsilon,i}(\Omega_{\varepsilon,i} \cap \Omega_{\varepsilon,j})$ to $T_\varepsilon \circ \varphi_{\varepsilon,j}(\Omega_{\varepsilon,i} \cap \Omega_{\varepsilon,j})$ is a diffeomorphism of class \mathcal{C}^1 .

We consider the following relation of equivalence between atlases of class \mathcal{C}^1 on M_ε : two atlases $(\Omega_{\varepsilon,i}, \varphi_{\varepsilon,i}, T_\varepsilon \circ \varphi_{\varepsilon,i})_{i \in I}$, and $(\Omega_{\varepsilon,j}, \varphi_{\varepsilon,j}, T_\varepsilon \circ \varphi_{\varepsilon,j})_{j \in J}$ of class \mathcal{C}^1 are said to be equivalent if their union is an atlas of class \mathcal{C}^1 . That is to say that $\varphi_{\varepsilon,i} \circ (\varphi_{\varepsilon,j})^{-1}$ is \mathcal{C}^1 on $\varphi_{\varepsilon,j}(\Omega_{\varepsilon,i} \cap \Omega_{\varepsilon,j})$ and $T_\varepsilon \circ \varphi_{\varepsilon,i} \circ (T_\varepsilon \circ \varphi_{\varepsilon,j})^{-1}$ is \mathcal{C}^1 on $T_\varepsilon \circ \varphi_{\varepsilon,j}(\Omega_{\varepsilon,i} \cap \Omega_{\varepsilon,j})$ when $\Omega_{\varepsilon,i} \cap \Omega_{\varepsilon,j} \neq \emptyset$.

Definition 12. A differentiable ε -manifold of class \mathcal{C}^1 , $\varepsilon \in \mathcal{R}_f$, is an ε -manifold together with an equivalence class \mathcal{C}^1 atlases.

Examples of Atlas. 1) The basic example is given by the space $\prod_{i=1}^3 \Gamma_{i,\varepsilon}^-$, for $\varepsilon \in \mathcal{R}_f$. The set of one element $\{ (\prod_{i=1}^3 \Gamma_{i,\varepsilon}^-, id, T_\varepsilon^-) \}$ is an atlas of class \mathcal{C}^1 , where

$$T_\varepsilon^- : \prod_{i=1}^3 \Gamma_{i,\varepsilon}^- \longrightarrow \prod_{i=1}^3 \Gamma_{i,\varepsilon}^+$$

$$\left((x_1, y_1), (x_2, y_2), (x_3, y_3) \right) \longmapsto \left((x_1 - \varepsilon, y_1), (x_2 - \varepsilon, y_2), (x_3 - \varepsilon, y_3) \right),$$

and id is the identity of $\prod_{i=1}^3 \Gamma_{i,\varepsilon}^-$.

2) Similarly, an example is given by the space $\prod_{i=1}^3 \Gamma_{i,\varepsilon}^+$, for $\varepsilon \in \mathbb{R}^{*+}$. The set of one element $\{ (\prod_{i=1}^3 \Gamma_{i,\varepsilon}^+, id, T_\varepsilon^+) \}$, where

$$T_\varepsilon^+ : \prod_{i=1}^3 \Gamma_{i,\varepsilon}^+ \longrightarrow \prod_{i=1}^3 \Gamma_{i,\varepsilon}^-$$

$$\left((x_1, y_1), (x_2, y_2), (x_3, y_3) \right) \longmapsto \left((x_1 + \varepsilon, y_1), (x_2 + \varepsilon, y_2), (x_3 + \varepsilon, y_3) \right)$$

and, where id is the identity of $\prod_{i=1}^3 \Gamma_{i,\varepsilon}^+$, is an atlas of class \mathcal{C}^1 .

3.2. Fractal Manifolds: Prototype

3.2.1. Diagonal Topology

Definition 13. Let $A = \bigcup_{\varepsilon \in I} A_\varepsilon$ and $B = \bigcup_{\varepsilon \in I} B_\varepsilon$ be two subsets of $E = \bigcup_{\varepsilon \in I} E_\varepsilon$ union of Hausdorff topological spaces, $I \subset \mathbb{R}^+$ an interval, such that $\forall \varepsilon \in I$, A_ε and B_ε are subsets of E_ε . We call diagonal intersection between A and B the set

$$A \tilde{\cap} B = \bigcup_{\varepsilon \in I} (A_\varepsilon \cap B_\varepsilon). \quad (16)$$

Definition 14. Let $I \subset \mathbb{R}^+$ be an interval. A diagonal topology $\mathcal{T}_d \subset \mathcal{P}(E)$ of a set $E = \bigcup_{\varepsilon \in I} E_\varepsilon$ union of Hausdorff topological spaces, where the E_ε are all disjoint or all the same, consists of subsets of E that verify the following axioms:

- (i) $\phi \in \mathcal{T}_d$, and $E \in \mathcal{T}_d$,
- (ii) $\omega_1 \in \mathcal{T}_d, \omega_2 \in \mathcal{T}_d \Rightarrow \omega_1 \tilde{\cap} \omega_2 \in \mathcal{T}_d$,
- (iii) $\omega_i \in \mathcal{T}_d, \forall i \in J \Rightarrow \bigcup_{i \in J} \omega_i \in \mathcal{T}_d$.

The elements $\omega_i \in \mathcal{T}_d$ are called open sets, (E, \mathcal{T}_d) is called diagonal topological space.

Proposition 15. Let $E = \bigcup_{\varepsilon \in I} E_\varepsilon$ be an union of Hausdorff topological spaces all disjoint or all the same, and let

$$\mathcal{T}_d = \{ \Omega = \bigcup_{\varepsilon \in I} \Omega_\varepsilon \subset E / \forall \varepsilon \in I, \Omega_\varepsilon \in E_\varepsilon \}.$$

Then \mathcal{T}_d is a diagonal topology on E .

Proof. (i) is obvious.

(ii) If $\omega_1 = \bigcup_{\varepsilon \in I} \Omega_{1\varepsilon} \in \mathcal{T}_d$, and $\omega_2 = \bigcup_{\varepsilon \in I} \Omega_{2\varepsilon} \in \mathcal{T}_d$, then we have

$$\omega_1 \tilde{\cap} \omega_2 = \bigcup_{\varepsilon \in I} (\Omega_{1\varepsilon} \cap \Omega_{2\varepsilon}) \in \mathcal{T}_d.$$

(iii) For $\omega_i = \bigcup_{\varepsilon \in I} \Omega_{i\varepsilon} \in \mathcal{T}_d, \forall i \in J$, we have

$$\bigcup_{i \in J} \omega_i = \bigcup_{i \in J} \bigcup_{\varepsilon \in \mathcal{R}_f} \Omega_{i\varepsilon} = \bigcup_{\varepsilon \in \mathcal{R}_f} \bigcup_{i \in J} \Omega_{i\varepsilon},$$

since E_ε is a topological space then we have $\bigcup_{i \in J} \omega_i \in \mathcal{T}_d$ □

Definition 16. Let $E = \bigcup_{\varepsilon \in I} E_\varepsilon$ be an union of Hausdorff topological spaces all disjoint or all the same. Let $x_\varepsilon \in E_\varepsilon \forall \varepsilon \in I$. A subset $\Omega \subset E$ is called a diagonal neighborhood of $x_\varepsilon, \forall \varepsilon \in I$, if there exist $\omega \in \mathcal{T}_d$ such that

$\omega \subset \Omega$ and $\forall \varepsilon \in I, x_\varepsilon \in \omega$.

3.2.2. Prototype

Let us consider a family of Hausdorff topological spaces $E_\varepsilon, \forall \varepsilon \in I, I \subset \mathbb{R}^+$, all disjoint or all the same, and $E = \bigcup_{\varepsilon \in I} E_\varepsilon$. Let $x : I \rightarrow E$ be a continuous path on E . If $\forall \varepsilon \in I, \Omega_\varepsilon$ is an open neighborhood of $x(\varepsilon)$ in E_ε , then the set $\Omega(\text{Range}(x)) = \bigcup_{\varepsilon \in I} \Omega_\varepsilon$ is a diagonal neighborhood of the set $\text{Range}(x) = \bigcup_{\varepsilon \in I} \{x(\varepsilon)\}$ in E .

Definition 17. Let $E = \bigcup_{\varepsilon \in I} E_\varepsilon$ union of Hausdorff topological spaces, where the Hausdorff topological spaces E_ε are all disjoint or all the same⁴. We say that E admits an internal structure x on $P \in E$, if there exists a \mathcal{C}^0 parametric path

$$\begin{aligned} x : I &\longrightarrow \bigcup_{\varepsilon \in I} E_\varepsilon, \\ \varepsilon &\longmapsto x(\varepsilon) \in E_\varepsilon, \end{aligned} \tag{17}$$

such that $\forall \varepsilon \in I, \text{Range}(x) \cap E_\varepsilon = \{x(\varepsilon)\}$, and $\exists \varepsilon' \in I$ such that $P = x(\varepsilon') \in E_{\varepsilon'}$.

Remark. The continuity of the internal structure is defined by the diagonal topology \mathcal{T}_d (ie. $\forall \Omega \in \mathcal{T}_d, x^{-1}(\Omega)$ is an open set of I).

Definition 18. Let $E = \bigcup_{\varepsilon \in I} E_\varepsilon$ be an union of Hausdorff topological spaces all disjoint or all the same. Let x, y be two internal structures on it. We say that $x \sim y \Leftrightarrow$

- i) $\exists \varepsilon' \in \mathcal{R}_f$ such that $x(\varepsilon') = y(\varepsilon')$.
- ii) $\exists \theta : I \rightarrow I$ diffeomorphism such that $x = y \circ \theta$.

Proposition 19. 1) The relation “ \sim ” is an equivalence relation.
 2) If $x \sim y$ then $x = y$.

Proof. 1) Using the definition of internal structure, it is not difficult to see that “ \sim ” is an equivalence relation.

2) Let us consider two paths of class \mathcal{C}^0 x and y such that $x \sim y$, we have then $\forall \varepsilon \in I, x(\varepsilon) \in E_\varepsilon$ and $y(\varepsilon) \in E_\varepsilon$, and we have $x(\varepsilon) = y \circ \theta(\varepsilon), \forall \varepsilon \in I$. If we suppose that $\theta \neq id$, then there exists $\varepsilon_0 \in I$ such that $\theta(\varepsilon_0) \neq \varepsilon_0$, we put $\theta(\varepsilon_0) = \varepsilon'$. We have $x(\varepsilon_0) = y \circ \theta(\varepsilon_0)$ then $x(\varepsilon_0) = y(\varepsilon')$ which yields $x(\varepsilon_0) \in E_{\varepsilon_0}$ and $x(\varepsilon_0) \in E_{\varepsilon'}$, impossible then $\theta = id$ and $\forall \varepsilon \in I, x(\varepsilon) = y(\varepsilon)$. \square

Definition 20. Let $E = \bigcup_{\varepsilon \in I} E_\varepsilon$ be an union of Hausdorff topological

⁴Either $E = \bigcup_{\varepsilon \in I} E_\varepsilon$ is a disjoint union, or $\forall \varepsilon \in I, E_\varepsilon = E_0$ and then $E = E_0$.

spaces all disjoint or all the same. Two points of E are equivalent if only if their internal structure are equal.

Definition 21. Let $E = \bigcup_{\varepsilon \in I} E_\varepsilon$ be an union of Hausdorff topological spaces all disjoint or all the same. Let $x : I \subset \mathbb{R} \longrightarrow \bigcup_{\varepsilon \in I} E_\varepsilon$ be an internal structure on it. We call object of E the set $\text{Range}(x)$.

Definition 22. A diagonal topological space (M, \mathcal{T}_d) is called fractal manifold if $M = \bigcup_{\varepsilon \in \mathcal{R}_f} M_\varepsilon$, where for all $\varepsilon \in \mathcal{R}_f$, M_ε is an ε -manifolds, and if $\forall P \in M$, M admits an internal structure x on P such that there exist a neighborhood $\Omega(\text{Range}(x)) = \bigcup_{\varepsilon \in \mathcal{R}_f} \Omega_\varepsilon$, with Ω_ε a neighborhood of $x(\varepsilon)$ in M_ε , two open sets $V^+ = \bigcup_{\varepsilon \in \mathcal{R}_f} V_\varepsilon^+$ and $V^- = \bigcup_{\varepsilon \in \mathcal{R}_f} V_\varepsilon^-$, where V_ε^σ is an open set in $\prod_{i=1}^3 \Gamma_{i\varepsilon}^\sigma \times \{\varepsilon\}$ for $\sigma = \pm$, and there exist two families of maps $(\varphi_\varepsilon)_{\varepsilon \in \mathcal{R}_f}$ and $(T_\varepsilon \circ \varphi_\varepsilon)_{\varepsilon \in \mathcal{R}_f}$ such that $\varphi_\varepsilon : \Omega_\varepsilon \longrightarrow V_\varepsilon^+$ and $T_\varepsilon \circ \varphi_\varepsilon : \Omega_\varepsilon \longrightarrow V_\varepsilon^-$ are homeomorphisms for all $\varepsilon \in \mathcal{R}_f$.

Remark. 1) Of course, if the family $(\varphi_\varepsilon)_{\varepsilon \in \mathcal{R}_f}$ exists, then the family $(T_\varepsilon \circ \varphi_\varepsilon)_{\varepsilon \in \mathcal{R}_f}$ exists automatically.

2) According to the Proposition 19, we can associate for each $P \in M$ only one path, and two points of M are equivalent if only if they are on the same path.

Definition 23. A local chart on fractal manifold M is a triplet $(\Omega, \varphi, T \circ \varphi)$, where $\Omega = \bigcup_{\varepsilon \in \mathcal{R}_f} \Omega_\varepsilon$ is an open set of M , φ is a family of homeomorphisms φ_ε from Ω_ε onto an open set V_ε^+ of $\prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\}$ and $T \circ \varphi$ is a family of homeomorphisms $T_\varepsilon \circ \varphi_\varepsilon$ from Ω_ε onto an open set V_ε^- of $\prod_{i=1}^3 \Gamma_{i\varepsilon}^- \times \{\varepsilon\}$ for all $\varepsilon \in \mathcal{R}_f$. A collection $(\Omega_i, \varphi_i, (T \circ \varphi)_i)_{i \in J}$ of local charts on the fractal manifold M such that $\bigcup_{i \in J} \Omega_i = \bigcup_{i \in J} \bigcup_{\varepsilon \in \mathcal{R}_f} \Omega_{i,\varepsilon} = \bigcup_{\varepsilon \in \mathcal{R}_f} \bigcup_{i \in J} \Omega_{i,\varepsilon} = \bigcup_{\varepsilon \in \mathcal{R}_f} M_\varepsilon = M$, where $\bigcup_{i \in J} \Omega_{i,\varepsilon} = M_\varepsilon$, is called an atlas. The coordinates of an object $P \in \Omega$ related to the local chart $(\Omega, \varphi, T \circ \varphi)$ are the coordinates of the object $\varphi(P)$ in $\bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\}$, and of the object $T \circ \varphi(P)$ in $\bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\varepsilon}^- \times \{\varepsilon\}$.

Definition 24. An atlas of class \mathcal{C}^1 on a fractal manifold M is an atlas for which all families of changes of charts are \mathcal{C}^1 . That is to say, if we have an atlas of class \mathcal{C}^1 on M_ε for all $\varepsilon \in \mathcal{R}_f$, then we have a family of atlases of class \mathcal{C}^1 , which gives us an atlas of class \mathcal{C}^1 on M .

We consider the following *equivalence relation* between atlases of class \mathcal{C}^1 on a fractal manifold M : two atlases $(\Omega_i, \varphi_i, (T \circ \varphi)_i)_{i \in I}$, and $(\Omega_j, \varphi_j, (T \circ \varphi)_j)_{j \in J}$ of class \mathcal{C}^1 are said to be equivalent if their union is an atlas of class \mathcal{C}^1 on M . That is to say that if we have an equivalence relation between atlases of class \mathcal{C}^1 on M_ε for all $\varepsilon \in \mathcal{R}_f$, then we have a family of equivalence relations,

which gives us an equivalence relation between atlases of class \mathcal{C}^1 on the fractal manifold M .

Definition 25. A fractal manifold M of class \mathcal{C}^1 is a fractal manifold together with an equivalence class of \mathcal{C}^1 atlases.

Remark. 1) The element $x(\varepsilon) \in M_\varepsilon$ has 6 local coordinates in a rectangular coordinate system (considered as a subspace of (\mathbb{R}^6, d_h)). In a local chart we have: $\varphi_\varepsilon(x(\varepsilon)) = \left((x_1(\varepsilon), y_1(\varepsilon)), (x_2(\varepsilon), y_2(\varepsilon)), (x_3(\varepsilon), y_3(\varepsilon)) \right)$, element of $\prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\}$, and

$$T_\varepsilon \circ \varphi_\varepsilon(x(\varepsilon)) = \left((x_1(\varepsilon) + \varepsilon, y_1(\varepsilon)), (x_2(\varepsilon) + \varepsilon, y_2(\varepsilon)), (x_3(\varepsilon) + \varepsilon, y_3(\varepsilon)) \right),$$

element of $\prod_{i=1}^3 \Gamma_{i\varepsilon}^- \times \{\varepsilon\}$, where $y_i(\varepsilon) = f_i(x_i(\varepsilon) + \frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ for $i = 1, 2, 3$.

2) The sets $(\prod_{i=1}^3 \Gamma_{i\varepsilon}^\sigma)_{\varepsilon \in \mathcal{R}_f}$, for $\sigma = \pm$, are disjoint for the Hausdorff distance,

but not necessarily for another metric. Even when the sets $\prod_{i=1}^3 \Gamma_{i\varepsilon}^\sigma \times \{\varepsilon\}$, for $\sigma = \pm$, $\varepsilon \in \mathcal{R}_f$, are always disjoint.

3.3. Example of Fractal Manifold

An example of fractal manifold is given by the following lemma and we will see later on why this name.

Lemma 26. Let f_1, f_2 , and f_3 be nowhere differentiable functions. Let $f_i(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, and $f_i(x + \frac{3\varepsilon}{4}, \frac{\varepsilon}{4})$ be mean functions of the functions f_i , for $i=1,2,3$. If we associate the graph $\Gamma_{i\varepsilon}^+$ to the function $f_i(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ (respectively, $\Gamma_{i\frac{\varepsilon}{2}}^+$ to the function $f_i(x + \frac{3\varepsilon}{4}, \frac{\varepsilon}{4})$) for $i=1,2,3$. Then $\bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\}$ is a \mathcal{C}^1 scale manifold homeomorphic to $\bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\frac{\varepsilon}{2}}^+ \times \{\varepsilon\}$, and we have the following diagram:

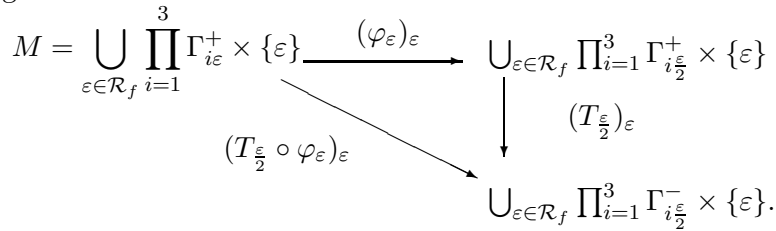


Figure 6: Diagram of the scale manifold $\bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\}$

Proof. If for $i = 1, 2, 3$, $\Gamma_{i\varepsilon^+}$ represents the graph of the forward mean function $f_i(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ of the nowhere differentiable function f_i , at ε resolution, and $\Gamma_{\frac{i\varepsilon}{2}^+}$ represents the graph of the forward mean function $f_i(x + \frac{3\varepsilon}{4}, \frac{\varepsilon}{4})$ of f_i at $\frac{\varepsilon}{2}$ resolution, then one can define

$$\begin{aligned} \varphi_\varepsilon : \Gamma_{i\varepsilon^+} &\longrightarrow \Gamma_{\frac{i\varepsilon}{2}^+} \\ (x, f_i(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2})) &\longmapsto (x + \frac{\varepsilon}{4}, f_i(x + \frac{3\varepsilon}{4}, \frac{\varepsilon}{4})). \end{aligned}$$

It is not difficult to see that:

1) φ_ε is continuous: each coordinate function is continuous, as composite function of continuous functions.

2) φ_ε^{-1} exists: $\varphi_\varepsilon^{-1}(x, y_i(x, \varepsilon)) = (x - \frac{\varepsilon}{4}, y_i(x - \frac{\varepsilon}{4}, 2\varepsilon))$

3) φ_ε^{-1} is continuous for the same reason as φ_ε .

Then φ_ε is an homeomorphism from $\Gamma_{i\varepsilon^+}$ onto $\Gamma_{\frac{i\varepsilon}{2}^+}$, for $i = 1, 2, 3$. We can generalize this result to the product of three graphs, and we obtain an homeomorphism of the form $\varphi_\varepsilon : \prod_{i=1}^3 \Gamma_{i\varepsilon^+} \times \{\varepsilon\} \longrightarrow \prod_{i=1}^3 \Gamma_{\frac{i\varepsilon}{2}^+} \times \{\varepsilon\}$, and we have the following diagram:

$$\begin{array}{ccc} M_\varepsilon = \prod_{i=1}^3 \Gamma_{i\varepsilon^+} \times \{\varepsilon\} & \xrightarrow{\varphi_\varepsilon} & \prod_{i=1}^3 \Gamma_{\frac{i\varepsilon}{2}^+} \times \{\varepsilon\} \\ & \searrow T_{\frac{\varepsilon}{2}} \circ \varphi_\varepsilon & \downarrow T_{\frac{\varepsilon}{2}} \\ & & \prod_{i=1}^3 \Gamma_{\frac{i\varepsilon}{2}^-} \times \{\varepsilon\}. \end{array}$$

Figure 7: Diagram of $\frac{\varepsilon}{2}$ -manifold

Internal structure can be found on $\bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\varepsilon^+} \times \{\varepsilon\}$.

Indeed, $\forall P \in \bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\varepsilon^+} \times \{\varepsilon\}$, there exists $\varepsilon' \in \mathcal{R}_f$ such that $P = x(\varepsilon') = (x_1, y(x_1, \varepsilon'), x_2, y(x_2, \varepsilon'), x_3, y(x_3, \varepsilon'))$, where $y(x_i, \varepsilon') = f_i(x_i + \frac{\varepsilon'}{2}, \frac{\varepsilon'}{2})$ for $i = 1, 2, 3$, and, where the internal structure is given by the \mathcal{C}^0 parametric path:

$$\begin{aligned} x : \mathcal{R}_f &\longrightarrow \bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\varepsilon^+} \times \{\varepsilon\} \\ \varepsilon &\longmapsto x(\varepsilon) = (x_1, y(x_1, \varepsilon), x_2, y(x_2, \varepsilon), x_3, y(x_3, \varepsilon)) \in \prod_{i=1}^3 \Gamma_{i\varepsilon^+} \times \{\varepsilon\}, \end{aligned} \quad (18)$$

and we have $\forall \varepsilon \in \mathcal{R}_f$, $\text{Range}(x) \cap \prod_{i=1}^3 \Gamma_{i\varepsilon^+} \times \{\varepsilon\} = \{x(\varepsilon)\}$. The x_i are constant

and the $y(x_i, \varepsilon)$ are of class \mathcal{C}^1 . Using Definition 22, we obtain a fractal manifold

$$M = \bigcup_{\varepsilon \in \mathcal{R}_f} M_\varepsilon = \bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\}. \tag{19}$$

□

3.4. Elements of the Fractal Manifold

By Definitions 17, the map $x : \mathcal{R}_f \rightarrow M$ describes the evolution of one representative element $x(\varepsilon)$ of x . This map is a continuous path of ε , and objects of M are not “points” but ranges of differentiable paths parameterized by $\varepsilon \in \mathcal{R}_f$. An element $x(\varepsilon)$ of M_ε is represented in local coordinates by two points, then an object P of M is represented in local coordinates by $\text{Range}(x^+) \cup \text{Range}(x^-)$ where the paths x^+ and x^- are given by:

$$\begin{aligned} x^+ : \mathcal{R}_f &\longrightarrow \bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\} \\ \varepsilon &\longmapsto \varphi_\varepsilon(x(\varepsilon)) \end{aligned}, \tag{20}$$

and

$$\begin{aligned} x^- : \mathcal{R}_f &\longrightarrow \bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\varepsilon}^- \times \{\varepsilon\} \\ \varepsilon &\longmapsto T_\varepsilon \circ \varphi_\varepsilon(x(\varepsilon)) \end{aligned}. \tag{21}$$

Then locally, an object of the fractal manifold M is not a “point” but a disjoint union of sets of points (double copy), it can be seen as a double string.

$$\bigvee \tag{22}$$

Remark. We have $d(x^+(\varepsilon), x^-(\varepsilon)) = \varepsilon\sqrt{3}$ for all $\varepsilon \in \mathcal{R}_f$ then the ranges of the two paths become closer as ε tends to 0, but $\varepsilon \neq 0$. If an object of fractal manifold is always locally represented by a double string it will never deserve the name fractal. In the following section we will prove the appearance of substructure into it.

4. Process Mean of the Mean

Actually, the principle of approximation of the nowhere differentiable function by an infinity of forward and backward mean functions, can be made for differentiable functions too. That means that we can define our nowhere differentiable function by a family of differentiable functions, and we can also define the

family of differentiable functions by another family of differentiable functions. The procedure can be repeated indefinitely. Visibly, repeating the procedure seems to be useless. In this part we will explore the hidden information we get by repeating the procedure indefinitely. But first of all, we have to introduce “points” in our new manifold.

4.1. Points and Elements: Toward a New Geometry

All the classical physical spaces are manifolds, sets of points with structures. In physics, a point is a pyridoxal object with zero extension, it is a limit entity that no one have ever seen. The points are without depth and their accumulation gives a space or a continuous manifold. The point is the origin of many difficulties in physics: divergence in classical and quantum physics, singularity in cosmology and general relativity, divergence in quantum field theory. . . , etc. The Heisenberg’s uncertainly principle prohibits a perfect localization, which means that the point is inaccessible. Many physicists tried to introduce a new space without points, with elementary cell “Atom Space”, with the hope to make disappear the divergences and to clarify naturally the impossible localization in quantum mechanics. In mathematics, point is assumed to be dimensionless, in axiomatic geometry, usually a completely undefined⁵ (primitive) element, although there are axiomatizations of geometry in which those properties of *point* that are desired, are given by postulates⁶. Any physical representation of a point must be of some size, if we draw a dot (point) and we magnify it, we will find a small surface. At this level, if we erase the small surface and we put another dot instead, and we magnify it again, we will obtain another small surface etc. We can repeat this procedure indefinitely, and if we want to introduce a notion of point, we have to take into account all the points of view given by magnification. An intuitive new postulate is as follows: *A point is that which has no parts for a given scale.*

Lemma 27. *Let $g_1, g_2,$ and g_3 be differentiable functions, and let $g_i(x + \frac{\delta}{2}, \frac{\delta}{2})$, be the mean functions of the functions g_i , for $i = 1, 2, 3$. If we associate the graph Γ_{i0} to the functions $g_i(x)$ (respectively, $\Gamma_{i\epsilon}^\sigma$ to the functions $g_i(x + \sigma \frac{\epsilon}{2}, \frac{\epsilon}{2})$, $\sigma = \mp$, for $i = 1, 2, 3$). The product $\prod_{i=1}^3 \Gamma_{i0}$ is a fractal manifold of class \mathcal{C}^1 homeomorphic to $\bigcup_{\epsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\delta}^+ \times \{\delta\}$, and we have the following diagram:*

⁵In Euclid’s geometry: a point is that which has no parts [11, 6].

⁶In a geometry in which line is a primitive element, points may be defined as classes of lines that conform to certain requirements (postulates). In n-dimensional metric analytic point geometry, a point may be defined as an ordered n-tuple of numbers, etc.

$$\begin{array}{ccc}
 M = \prod_{i=1}^3 \Gamma_{i0} & \xrightarrow{(\varphi_\delta)_\delta} & \bigcup_{\delta \in \mathcal{R}_g} \prod_{i=1}^3 \Gamma_{i\delta}^+ \times \{\delta\} \\
 & \searrow (T_\delta \circ \varphi_\delta)_\delta & \downarrow (T_\delta)_\delta \\
 & & \bigcup_{\delta \in \mathcal{R}_g} \prod_{i=1}^3 \Gamma_{i\delta}^- \times \{\delta\}.
 \end{array}$$

Proof. Let g be a differentiable function, and let $\delta \in \mathcal{R}_g$. If we denote Γ_0 the graph of the function g and Γ_δ^+ the graph of the forward mean function $g(x + \frac{\delta}{2}, \frac{\delta}{2})$ of g at δ resolution. We consider $\varphi_\delta : \Gamma_0 \rightarrow \Gamma_\delta^+$ defined by $(x, g(x, 0)) \mapsto (x + \frac{\delta}{2}, g(x + \frac{\delta}{2}, \frac{\delta}{2}))$, where $g(x + \frac{\delta}{2}, \frac{\delta}{2}) = \frac{1}{\delta} \int_x^{x+\delta} g(t) dt$, and $g(x, 0) = g(x)$. Then we have:

- 1) φ_δ is continuous: each coordinate function is continuous, as composite function of continuous functions.
- 2) φ_δ^{-1} exists: $\varphi_\delta^{-1}(x, y(x, \delta)) = (x - \frac{\delta}{2}, y(x - \frac{\delta}{2}))$
- 3) φ_δ^{-1} is continuous for the same reason as φ_δ .

Then φ_δ is an homeomorphism from Γ_0 onto $\Gamma_\delta^+ \times \{\delta\}$. For three dimensions we have to consider $\Gamma_{i\delta}^+$ the graph associated to the forward mean functions of the differentiable functions g_i , and we denote Γ_{i0} the graph associated to the functions g_i , for $i = 1, 2, 3$. By the same, we obtain an homeomorphism $\varphi_\delta : \prod_{i=1}^3 \Gamma_{i0} \rightarrow \prod_{i=1}^3 \Gamma_{i\delta}^+ \times \{\delta\}$, and then $\prod_{i=1}^3 \Gamma_{i0}$ is a δ -manifold, and we have the diagram given by Figure 8.

$$\begin{array}{ccc}
 M_\delta = \prod_{i=1}^3 \Gamma_{i0} & \xrightarrow{\varphi_\delta} & \prod_{i=1}^3 \Gamma_{i\delta}^+ \times \{\delta\} \\
 & \searrow T_\delta \circ \varphi_\delta & \downarrow T_\delta \\
 & & \prod_{i=1}^3 \Gamma_{i\delta}^- \times \{\delta\}.
 \end{array}$$

Figure 8: Diagram of δ -manifold

An internal structure can be found on $\bigcup_{\delta \in \mathcal{R}_g} \prod_{i=1}^3 \Gamma_{i0} = \prod_{i=1}^3 \Gamma_{i0}$. Indeed, for

all $P \in \bigcup_{\varepsilon \in \mathcal{R}_g} \prod_{i=1}^3 \Gamma_{i0}$, we have $P = (x_1, y(x_1), x_2, y(x_2), x_3, y(x_3))$ and then for $y(x_i) = y_i$ constant for $i = 1, 2, 3$, $P = (x_1, y_1, x_2, y_2, x_3, y_3)$ (which means that

for every scale we consider the same point), and where a C^0 parametric path x is given by

$$\begin{aligned} x : \mathcal{R}_g &\longrightarrow \cup_{\delta \in \mathcal{R}_g} \prod_{i=1}^3 \Gamma_{i0} \\ \delta &\longmapsto x(\delta) = (x_1, y_1, x_2, y_2, x_3, y_3) \in \prod_{i=1}^3 \Gamma_{i0} \end{aligned} \tag{23}$$

and we have $\forall \delta \in \mathcal{R}_g, \text{Range}(x) \cap \prod_{i=1}^3 \Gamma_{i0} = \{P\}$. Using the definition 22,

for $M_\delta = \prod_{i=1}^3 \Gamma_{i0}$, we obtain a fractal manifold $M = \bigcup_{\delta \in \mathcal{R}_g} M_\delta = \bigcup_{\delta \in \mathcal{R}_g} \prod_{i=1}^3 \Gamma_{i0} =$

$\prod_{i=1}^3 \Gamma_{i0}$, homeomorphic to $\bigcup_{\delta \in \mathcal{R}_g} \prod_{i=1}^3 \Gamma_{i\delta}^+ \times \{\delta\}$, and then we have the result. Since $\prod_{i=1}^3 \Gamma_{i0}$ is homeomorphic to $\prod_{i=1}^3 \Gamma_{i0} \times \{\varepsilon\}$ for $\varepsilon \in \mathcal{R}_g$, then $\prod_{i=1}^3 \Gamma_{i0} \times \{\varepsilon\}$ is homeomorphic to $\bigcup_{\delta \in \mathcal{R}_g} \prod_{i=1}^3 \Gamma_{i\delta}^+ \times \{\delta\} \times \{\varepsilon\}$. \square

Corollary 28. For $\varepsilon \in \mathcal{R}_g$, the manifold given by $\prod_{i=1}^3 \Gamma_{i0} \times \{\varepsilon\}$ is a fractal manifold of class C^1 homeomorphic to $\bigcup_{\delta \in \mathcal{R}_g} \prod_{i=1}^3 \Gamma_{i\delta}^+ \times \{\delta\} \times \{\varepsilon\}$, and we have the following diagram:

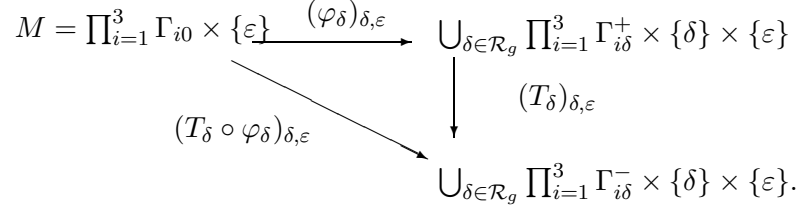


Figure 9: Diagram of δ -manifold at a given ε

We can generalize the last result to a three dimensional differentiable manifold M_0 , indeed, if any three dimensional differentiable manifold M_0 is homeomorphic to $\prod_{i=1}^3 \Gamma_{i0}$ (Lemma 27) then M_0 is a fractal manifold.

Corollary 29. Let $g_1, g_2,$ and g_3 be differentiable functions, and let $g_i(x + \frac{\delta}{2}, \frac{\delta}{2})$, be the mean functions of the functions g_i , for $i = 1, 2, 3$. We associate the graph Γ_{i0} to the functions $g_i(x)$. If M_0 is a three dimensional fractal manifold homeomorphic to the product $\prod_{i=1}^3 \Gamma_{i0}$ then M_0 is a fractal manifold.

If we consider an object P of the fractal manifold $\prod_{i=1}^3 \Gamma_{i0} \times \{\varepsilon\}$, it is

represented in local coordinates, in $\prod_{i=1}^3 \Gamma_{i\delta}^\sigma \times \{\delta\} \times \{\varepsilon\}$, by two points denoted by $x_\varepsilon^+(\delta)$, $x_\varepsilon^-(\delta)$ for all $\delta \neq 0$, and by one point for $\delta = 0$ given by $x_\varepsilon^+(0) = x_\varepsilon^-(0)$ (using a classical notation of point). At this level, an object P of $\prod_{i=1}^3 \Gamma_{i0} \times \{\varepsilon\}$ is represented in local coordinates, in $\bigcup_{\delta \in \mathcal{R}_g} \prod_{i=1}^3 \Gamma_{i\delta}^+ \times \{\delta\} \times \{\varepsilon\}$, by $\text{Range}(x_\varepsilon^+) \cup \text{Range}(x_\varepsilon^-)$, where the two paths x_ε^+ and x_ε^- are given by:

$$x_\varepsilon^+ : \mathcal{R}_g \longrightarrow \bigcup_{\delta \in \mathcal{R}_g} \prod_{i=1}^3 \Gamma_{i\delta}^+ \times \{\delta\}, \quad \delta \longmapsto \varphi_\delta(x_\varepsilon(\delta)), \tag{24}$$

and

$$x_\varepsilon^- : \mathcal{R}_g \longrightarrow \bigcup_{\delta \in \mathcal{R}_g} \prod_{i=1}^3 \Gamma_{i\delta}^- \times \{\delta\}, \quad \delta \longmapsto T_\delta \circ \varphi_\delta(x_\varepsilon(\delta)), \tag{25}$$

with $x_\varepsilon^+(0) = x_\varepsilon^-(0)$, then an object of the fractal manifold $M = \prod_{i=1}^3 \Gamma_{i0}$ at the resolution ε looks like

$$\bigvee \tag{26}$$

and we have $d(x_\varepsilon^+(\delta), x_\varepsilon^-(\delta)) = \sqrt{3}\delta$ for all $\delta \in \mathcal{R}_g$, then the two paths become closer when δ tends to 0, and for $\delta = 0$ the two strings intersect at one point. An element $x(\varepsilon)$ of M_ε introduced in Definition 22 is then represented in local coordinates by two elements, one from $\prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\}$ which has the form given by (26), and one from $\prod_{i=1}^3 \Gamma_{i\varepsilon}^- \times \{\varepsilon\}$ which has also the form given by (26)⁷, and, where the distance in a Hausdorff metric space between the two elements is $\sqrt{3}\varepsilon$. Then an object P of M is represented in local coordinates by $\text{Range}(x^+) \cup \text{Range}(x^-)$, where the two paths x^+ and x^- are given by the formula (20) and (21) respectively, such that $x^+(\varepsilon)$ and $x^-(\varepsilon)$ are two copies given by the form (26):

$$\bigvee \bigvee . \tag{27}$$

At the end an object of the fractal manifold $M = \bigcup_{\varepsilon \in \mathcal{R}_f} M_\varepsilon$ is represented by the union of the range of x^+ and x^- given by the formula (20) and (21) respectively (Figure 10a).

Remark. If we use definition 22, an element of the ε -manifold M_ε corresponds to two points (with a classical notion of point). If we take into account the corollary 28, an element of the ε -manifold M_ε becomes a double set of points of the form (27). The procedure given by corollary 28 is a transformation of one string of length L to one surface (see Figure 10b), then we have appearance of new structure.

⁷We apply corollary 28 for the graphs $\Gamma_{i0} = \Gamma_{i\varepsilon}^+$ of $g_i = f_i(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ for $i=1,2,3$.

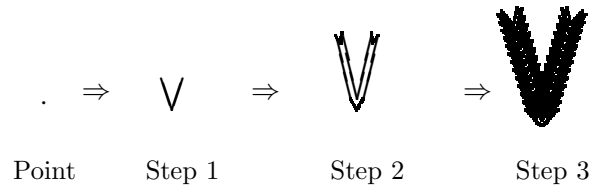


Figure 10a: One illustration of classical point in fractal manifold after 3 steps

4.2. Substructure

The fractal manifold deserve this name because of the appearance of new structure at every step. Indeed

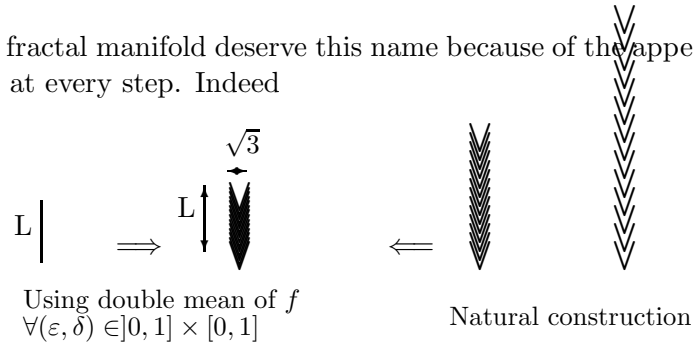


Figure 10b: One illustration of one string in fractal manifold M as represented in $\bigcup_{\epsilon \in \mathcal{R}_f} \bigcup_{\delta \in \mathcal{R}_g} \prod_{i=1}^3 \Gamma_{i\delta}^\sigma \times \{\delta\} \times \{\epsilon\}$.

4.2.1. Step 1: Mean of Nowhere Differentiable Functions

For the nowhere differentiable function f_i , $i = 1, 2, 3$, introduced before, Definition 22 gives us the following diagram, where $\varphi_1 = (\varphi_\epsilon)_\epsilon$, $T_1 = (T_\epsilon)_\epsilon$, and $T_1 \circ \varphi_1 = (T_\epsilon \circ \varphi_\epsilon)_\epsilon$.

$$\begin{array}{ccc}
 M = \bigcup_{\varepsilon \in \mathcal{Q}_f} M_\varepsilon & \xrightarrow{\varphi_1} & \bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\} \\
 & \searrow T_1 \circ \varphi_1 & \downarrow T_1 \\
 & & \bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\varepsilon}^- \times \{\varepsilon\},
 \end{array}$$

We can obtain the same diagram if we use an unknown differentiable function. The diagram denoted before for the fractal manifold represents actually a family of diagram of ε -manifold for all $\varepsilon \in \mathcal{R}_f$.

4.2.2. Step 2: Mean of the Mean

Using the process mean of the mean to approximate the mean functions $f_i(x, \varepsilon)$ of f_i by another family of mean functions $g_i(x, \delta_1)$ (we just replace $\prod_{i=1}^3 \Gamma_{i0}$ by $\prod_{i=1}^3 \Gamma_{i\varepsilon}^{\sigma_1}$ in Corollary 28) for $\sigma_1 = \pm$, we obtain a second diagram:

$$\begin{array}{ccc}
 M_\varepsilon = \prod_{i=1}^3 \Gamma_{i\varepsilon}^{\sigma_1} \times \{\varepsilon\} & \xrightarrow{(\varphi_{\delta_1})_{\delta_1, \varepsilon}^{\sigma_1}} & \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{\sigma_1, +} \times \{\delta_1\} \times \{\varepsilon\} \\
 & \searrow (T_{\delta_1} \circ \varphi_{\delta_1})_{\delta_1, \varepsilon}^{\sigma_1} & \downarrow (T_{\delta_1})_{\delta_1, \varepsilon}^{\sigma_1} \\
 & & \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{\sigma_1, -} \times \{\delta_1\} \times \{\varepsilon\},
 \end{array}$$

where for $\sigma_1 = \pm$, $(\varphi_{\delta_1})_{\delta_1, \varepsilon}^{\sigma_1}$ represents two families $(\varphi_{\delta_1})_{\delta_1 \in \mathcal{R}_{\delta_1}}^{\sigma_1}$ at the resolution ε (respectively for $(T_{\delta_1})_{\delta_1, \varepsilon}^{\sigma_1}$ and $(T_{\delta_1} \circ \varphi_{\delta_1})_{\delta_1, \varepsilon}^{\sigma_1}$), with $\mathcal{R}_{\delta_1} = [0, 1]$. The union over $\varepsilon \in \mathcal{R}_f$ gives us a new diagram :

$$\begin{array}{ccc}
 \bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\varepsilon}^{\sigma_1} \times \{\varepsilon\} & \xrightarrow{\varphi_2^{\sigma_1}} & \bigcup_{\varepsilon \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{\sigma_1, +} \times \{\delta_1\} \times \{\varepsilon\} \\
 & \searrow (T_2 \circ \varphi_2)^{\sigma_1} & \downarrow T_2^{\sigma_1} \\
 & & \bigcup_{\varepsilon \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{\sigma_1, -} \times \{\delta_1\} \times \{\varepsilon\},
 \end{array}$$

where $\varphi_2^{\sigma_1} = ((\varphi_{\delta_1})_{\delta_1, \varepsilon}^{\sigma_1})_\varepsilon$, $(T_2 \circ \varphi_2)^{\sigma_1} = ((T_{\delta_1} \circ \varphi_{\delta_1})_{\delta_1, \varepsilon}^{\sigma_1})_\varepsilon$, and $T_2^{\sigma_1} = ((T_{\delta_1})_{\delta_1, \varepsilon}^{\sigma_1})_\varepsilon$,

$\sigma_1 = \pm$. Using the diagram of Step 1, we obtain two new diagrams for the fractal manifold M (at this step we observe an appearance of a new structure [13, 15, 21]):

$$\begin{array}{ccc}
 M & \xrightarrow{\varphi_2^+ \circ \varphi_1} & \bigcup_{\varepsilon \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{+,+} \times \{\delta_1\} \times \{\varepsilon\} \\
 & \searrow (T_2 \circ \varphi_2)^+ \circ \varphi_1 & \downarrow T_2^+ \\
 & & \bigcup_{\varepsilon \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{+,-} \times \{\delta_1\} \times \{\varepsilon\},
 \end{array}$$

and we have

$$\begin{array}{ccc}
 M & \xrightarrow{\varphi_2^- \circ T_1 \circ \varphi_1} & \bigcup_{\varepsilon \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{-,+} \times \{\delta_1\} \times \{\varepsilon\} \\
 & \searrow (T_2 \circ \varphi_2)^- \circ T_1 \circ \varphi_1 & \downarrow T_2^- \\
 & & \bigcup_{\varepsilon \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{-,-} \times \{\delta_1\} \times \{\varepsilon\}.
 \end{array}$$

4.2.3. Step n : Mean of the Mean n Times

By induction over $n \geq 1$ and using Step 2, we will find $2^{(n-1)}$ diagrams of the form given by Figure 11, where φ_k is a family of homeomorphisms for $2^{n-1} \leq k \leq 2^n - 1$, T_k is a family of translations for $2^{n-1} \leq k \leq 2^n - 1$, where $\sigma_j = \pm$ for $j = 1, \dots, n - 2$. We know that φ_k, T_k are families which depend on $\varepsilon, \delta_1, \dots, \delta_{n-1}$ for all $2^{n-1} \leq k \leq 2^n - 1$, we denoted them with only one index k just for simplicity. The number n represents the steps of magnification, and k represents the number of diagrams at a given step.

changing at every step. Following the given tree we can see that:

In Step 1, the local chart is given by a triplet $(\Omega, \varphi_1, T_1 \circ \varphi_1)$.

In Step 2, the local chart is given by the quintuplet $(\Omega, \psi_1, \psi_2, \psi_3, \psi_4)$ given by $(\Omega, \varphi_2 \circ \varphi_1, T_2 \circ \varphi_2 \circ \varphi_1, \varphi_3 \circ T_1 \circ \varphi_1, T_3 \circ \varphi_3 \circ T_1 \circ \varphi_1), \dots$, etc. That is why we call it a fractal manifold.

In the diagram given by Figure 12 the following notation are used for simplicity

i) $N_{\delta_0}^{\sigma_1} = \prod_{i=1}^3 \Gamma_{i\delta_0}^{\sigma_1} \times \{\delta_0\}$, $\sigma_1 = \pm$, and the integrals used in this sets are given by $y_i^+ = \frac{1}{\delta_0} \int_{x_i(\delta_0)}^{x_i(\delta_0)+\delta_0} f(t)dt$, and $y_i^- = \frac{1}{\delta_0} \int_{x_i(\delta_0)-\delta_0}^{x_i(\delta_0)} f(t)dt$.

ii) $N_{\delta_0\delta_1}^{\sigma_1\sigma_2} = \prod_{i=1}^3 \Gamma_{i\delta_1}^{\sigma_1\sigma_2} \times \{\delta_1\} \times \{\delta_0\}$, $(\sigma_1, \sigma_2) = (\pm, \pm)$, and the integrals used in this sets are given by

$$\begin{aligned} y_{i1} &= \frac{1}{\delta_0\delta_1} \int_{x_{i1}(\delta_0, \delta_1)}^{x_{i1}(\delta_0, \delta_1)+\delta_1} \int_{t_1}^{t_1+\delta_0} f(t_0)dt_0dt_1, & \text{for } (\sigma_1, \sigma_2) &= (+, +), \\ y_{i2} &= \frac{1}{\delta_0\delta_1} \int_{x_{i2}(\delta_0, \delta_1)}^{x_{i2}(\delta_0, \delta_1)+\delta_1} \int_{t_1-\delta_0}^{t_1} f(t_0)dt_0dt_1, & \text{for } (\sigma_1, \sigma_2) &= (+, -), \\ y_{i3} &= \frac{1}{\delta_0\delta_1} \int_{x_{i3}(\delta_0, \delta_1)-\delta_1}^{x_{i3}(\delta_0, \delta_1)} \int_{t_1}^{t_1+\delta_0} f(t_0)dt_0dt_1, & \text{for } (\sigma_1, \sigma_2) &= (-, +), \end{aligned}$$

$$y_{i4} = \frac{1}{\delta_0\delta_1} \int_{x_{i4}(\delta_0, \delta_1)-\delta_1}^{x_{i4}(\delta_0, \delta_1)} \int_{t_1-\delta_0}^{t_1} f(t_0)dt_0dt_1, \quad \text{for } (\sigma_1, \sigma_2) = (-, -),$$

etc.

4.3. Relationship with Classical Point

We start this part by an example of fractal manifold.

Example. Let $\Gamma_{\frac{\varepsilon}{2}}$ be the graph of the mean function $f(x, \frac{\varepsilon}{2})$ of a differentiable function f at $\frac{\varepsilon}{2}$ resolution, and Γ_{ε}^+ be the graph of the right mean function $f(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ of f at ε resolution. We define

$$\begin{aligned} \varphi_{\varepsilon} : \Gamma_{\frac{\varepsilon}{2}} &\longrightarrow \varphi_{\varepsilon}(\Gamma_{\frac{\varepsilon}{2}}) & \text{and} & & T_{\frac{-\varepsilon}{2}} : \varphi_{\varepsilon}(\Gamma_{\frac{\varepsilon}{2}}) &\longrightarrow \Gamma_{\varepsilon}^+ \\ (x, f(x, \frac{\varepsilon}{2})) &\longmapsto (x + \frac{\varepsilon}{2}, f(x + \frac{\varepsilon}{2}, \frac{\varepsilon}{2})) & & & (x, y) &\longmapsto (x - \frac{\varepsilon}{2}, y). \end{aligned} \quad \text{We}$$

see that:

1) φ_{ε} and $T_{\frac{-\varepsilon}{2}}$ are continuous: each coordinate function is continuous, as

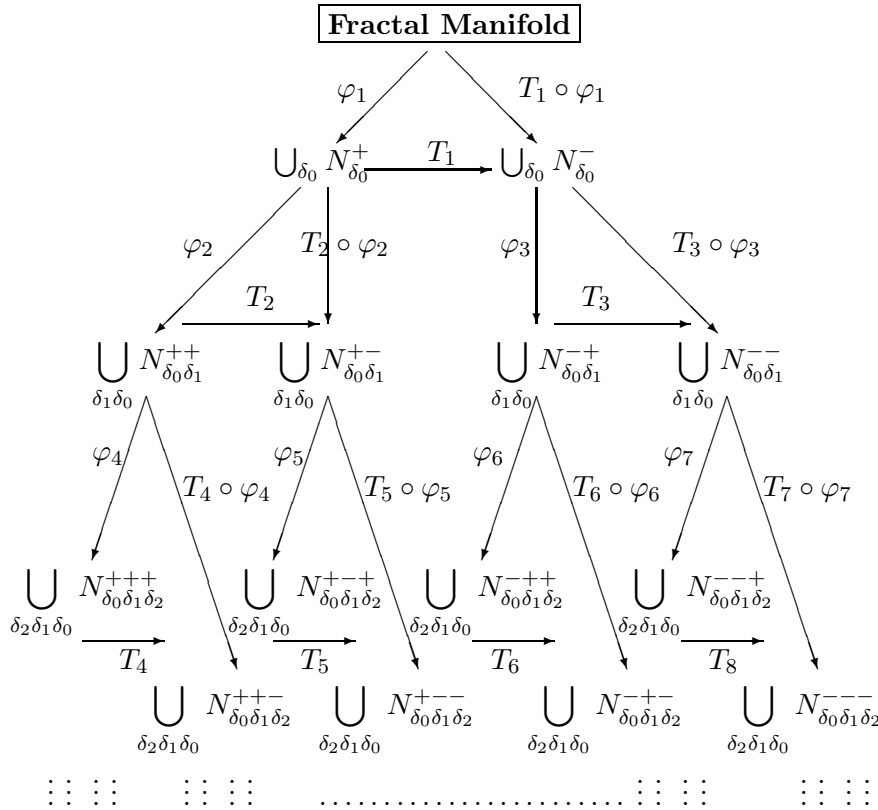


Figure 12: A fractal diagram of a fractal manifold

composite function of continuous functions.

2) φ_ε^{-1} , and $T_{\frac{-\varepsilon}{2}}^{-1}$ exist, with: $\varphi_\varepsilon^{-1}(x, y(x, \varepsilon)) = (x - \frac{\varepsilon}{2}, y(x - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}))$

and $T_{\frac{-\varepsilon}{2}}^{-1}(x, y) = (x + \frac{\varepsilon}{2}, y)$

3) φ_ε^{-1} and $T_{\frac{-\varepsilon}{2}}^{-1}$ are continuous for the same reason as φ_ε and $T_{\frac{-\varepsilon}{2}}$. Then $T_{\frac{-\varepsilon}{2}} \circ \varphi_\varepsilon$ is an homeomorphism from $\Gamma_{\frac{\varepsilon}{2}}$ to Γ_ε^+ . It is not difficult to generalize the homeomorphism for the product of three graphs. Conclusion, we obtain an homeomorphism $\phi_\varepsilon : \prod_{i=1}^3 \Gamma_{\frac{i\varepsilon}{2}} \times \{\varepsilon\} \rightarrow \prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\}$, and then we obtain the following diagram given in Figure 13.

$$\begin{array}{ccc}
M_\varepsilon = \prod_{i=1}^3 \Gamma_{\frac{i\varepsilon}{2}}^+ \times \{\varepsilon\} & \xrightarrow{\phi_\varepsilon} & \prod_{i=1}^3 \Gamma_{i\varepsilon}^+ \times \{\varepsilon\} \\
& \searrow T_\varepsilon \circ \phi_\varepsilon & \downarrow T_\varepsilon \\
& & \prod_{i=1}^3 \Gamma_{i\varepsilon}^- \times \{\varepsilon\}.
\end{array}$$

Figure 13: Diagram of $\frac{\varepsilon}{2}$ -manifold

An internal structure can be found on $\bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{\frac{i\varepsilon}{2}}^+ \times \{\varepsilon\}$.

Indeed, $\forall P \in \bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{\frac{i\varepsilon}{2}}^+ \times \{\varepsilon\}$, there exists $\varepsilon' \in \mathcal{R}_f$ such that

$$\begin{aligned}
P &= x(\varepsilon') = \left(x_1, y(x_1, \varepsilon'), x_2, y(x_2, \varepsilon'), x_3, y(x_3, \varepsilon') \right) \text{ with } y(x_i, \varepsilon') = f_i(x_i, \frac{\varepsilon'}{2}) \\
&\text{for } i = 1, 2, 3, \text{ and, where a } \mathcal{C}^1 \text{ parametric path } x \text{ is given by} \\
x : \mathcal{R}_f &\longrightarrow \bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{\frac{i\varepsilon}{2}}^+ \times \{\varepsilon\} \\
\varepsilon &\longmapsto x(\varepsilon) = \left(x_1, y(x_1, \varepsilon), x_2, y(x_2, \varepsilon), x_3, y(x_3, \varepsilon) \right) \in \prod_{i=1}^3 \Gamma_{\frac{i\varepsilon}{2}}^+ \times \{\varepsilon\}, \tag{30}
\end{aligned}$$

and we have $\forall \varepsilon \in \mathcal{R}_f$, $\text{Range}(x) \cap \prod_{i=1}^3 \Gamma_{\frac{i\varepsilon}{2}}^+ \times \{\varepsilon\} = \{x(\varepsilon)\}$, the x_i are constant and the $y(x_i, \varepsilon)$ are of class \mathcal{C}^1 for $i = 1, 2, 3$. Using Definition 22, we obtain then a fractal manifold

$$M = \bigcup_{\varepsilon \in \mathcal{R}_f} M_\varepsilon = \bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{\frac{i\varepsilon}{2}}^+ \times \{\varepsilon\}, \tag{31}$$

and we have the following diagram:

$$\begin{array}{ccc}
 M = \bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{\frac{i\varepsilon}{2}} \times \{\varepsilon\} & \xrightarrow{(\phi_\varepsilon)_\varepsilon} & \bigcup_{\varepsilon \in \mathcal{R}_f} (\prod_{i=1}^3 \Gamma_{i\varepsilon}^+) \times \{\varepsilon\} \\
 & \searrow (T_\varepsilon \circ \phi_\varepsilon)_\varepsilon & \downarrow (T_\varepsilon)_\varepsilon \\
 & & \bigcup_{\varepsilon \in \mathcal{R}_f} (\prod_{i=1}^3 \Gamma_{i\varepsilon}^-) \times \{\varepsilon\}.
 \end{array}$$

Figure 14: Example of fractal manifold $\bigcup_{\varepsilon \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{\frac{i\varepsilon}{2}} \times \{\varepsilon\}$

On one hand, we have $\prod_{i=1}^3 \Gamma_{\frac{i\varepsilon}{2}}^+ \times \{\varepsilon\}$ homeomorphic to \mathbb{R}^3 , then the set M given by the formula (31) is homeomorphic to \mathbb{R}^3 , and the elements of this set are points (classical definition). On the other hand, the set M given by the formula (31) is a fractal manifold, where the elements are objects given by (26). We observe two kinds of objects obtained by the process mean of mean: one concerning nowhere differentiable function, and another one concerning differentiable function (see illustration Figure 15). As we can see, the mechanism is the same, the only difference is the appearance of discontinuity.

If we consider only the mean function (1) of f (and not the right- left mean function), the same construction leads us to obtain a differentiable family of manifold given by $M = \bigcup_{\varepsilon \in \mathcal{R}_f} M_\varepsilon$, but this family $(M_\varepsilon)_{\varepsilon \in \mathbb{R}}$ does not give us an appearance of new structure: indeed, $\forall \varepsilon \in \mathcal{R}_f$, we obtain the same local object, and then we have the same object (only one string) for differentiable and nowhere differentiable functions.

Following Remark (4.2 Substructure), the dimension of a fractal manifold is variable. It varies between 5 and infinity. More details will be given in [5] about a non linear analysis on fractal manifold, and details will be given about the nature of the expansion into this kind of manifold in [2]. An application in modern cosmology will also be given in [3] using fractal manifold.

We constructed a geometric space which is neither a continuum nor a discrete space, but a mixture of both. Locally this space is a disjoint union over $\varepsilon \in]0, \alpha]$ of a family of two ordinary continuums. The space acquires a variable geometry, namely, it becomes explicitly dependent on the resolution. However,

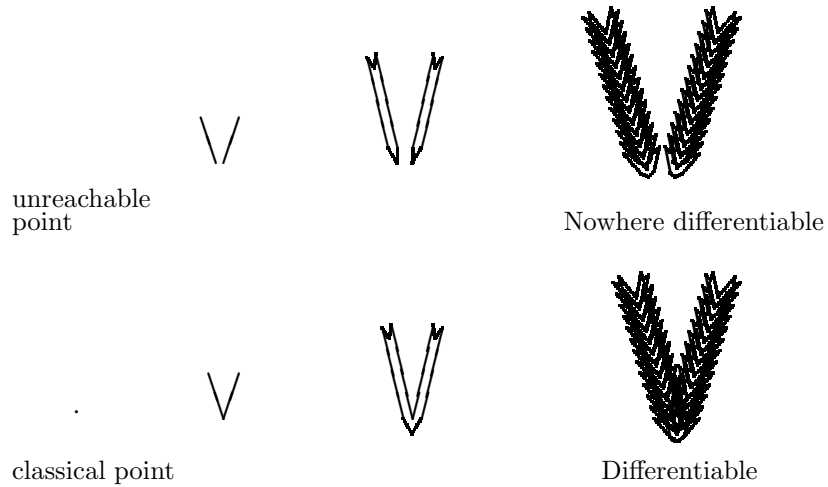


Figure 15: One illustration of the process mean of the mean

following this framework, the geometry of space is assumed to be characterized not only by curvature, but also by appearance of new structure at every step. The classical differentiable geometry does not give us useful information of this kind of double space, however one should construct a non linear analysis over this kind of manifolds.

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