International Journal of Pure and Applied Mathematics
Volume 38 No. 2 2007, 199-207

OSCILLATION CRITERIA FOR SECOND-ORDER NONLINEAR DELAYED DIFFERENCE EQUATIONS WITH PULSES

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Abstract: The present paper is devoted to the investigation of the oscillation of a class of second-order nonlinear impulsive delay difference equations. Some interesting results are obtained, and some examples which illustrate that impulsive perturbations play a very important role in giving rise to oscillations of equations are also included.

AMS Subject Classification: 34K11
Key Words: oscillation, impulsive, difference equation, delay

1. Introduction

In the recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of ordinary differential equations and impulsive delay differential equations, numerous papers have been published and good results were obtained (see [9], [7], [2], [6], [8] and the references therein). But fewer paper are on impulsive difference equations [5], [4], [10].

In this paper, we consider the following impulsive delay difference equation

$$\begin{cases} \triangle(\varphi(x(n-1))(\triangle_\alpha x(n-1))^\sigma) + f(n, x(n-l)) = g(x(n)), \\ \qquad\qquad\qquad\qquad \alpha \geq 1, n \neq n_k, k \in N, \\ (\triangle_\alpha x(n_k))^\sigma = b_k(\triangle_\alpha x(n_k-1))^\sigma, \\ x(n_k) = b'_k x(n_k - 1), \end{cases} \tag{1}$$

where $\triangle x(n) = x(n+1) - x(n)$, $\triangle_\alpha x(n) = x(n+1) - \alpha x(n)$, σ is a quotient of

Received: April 10, 2007
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odd positive integer, $l \in N$, N is the natural number set, $0 \le n_0 < n_1 < \cdots < n_k, < \cdots$, and $\lim_{k\to\infty} n_k = \infty$, $n_k - n_{k-1} > l$.

Throughout this paper, we assume that the following conditions hold:

(C1) $uf(n,u) > 0(u \neq 0)$ and there exists a nonnegative sequence $\{p_n\}$ and a function u such that $\frac{f(n,u)}{u^\sigma} \ge p_n$;

(C2) $\{b_k\}$, $\{b'_k\}$ are positive sequences;

(C3) $\varphi(\mu)$ is nonincreasing continuous, $\mu\varphi(\mu) > 0$ $(\mu \neq 0)$ and $\varphi(\lambda\mu) \le \lambda\mu \; for \; \lambda \in R^+$; sequence;

(C4) $vg(v) \le 0(v \neq 0)$.

For convince, we let

$$N[n_1, n_2] = \{n|n \in N, n_1 \le n \le n_2\}, \quad N[n_1, n_2) = \{n|n \in N, n_1 \le n < n_2\},$$

$$N[n_1, \infty) = \{n|n \in N, n_1 \le n < \infty\}$$

and

$$S(n) = \varphi(x(n-1))(\triangle_\alpha x(n-1))^\sigma$$

Solution of (1) is a real valued sequence $\{x(n)\}$ defined on $N[n_0 - l, \infty)$ which satisfies (1) for $n \ge n_0$. It is obvious that (1) has a unique solution $\{x(n)\}_{n_0-l}^\infty$, under the initial conditions

$$x(i) = y_i, \; i = n_0 - l, \cdots, n_0, \tag{2}$$

in which y_i $(i = n_0 - l, \cdots, n_0)$ are given real constants.

The solution of (1) is said to be nonoscillatory if this solution is eventually positive or eventually negative. Otherwise, this solution is said to be oscillatory.

2. Some Lemmas

Lemma 1. *Assume that*

$$\begin{cases} \triangle m(n) \le l_n m(n) + q_n, & n \neq n_k, k \in N, \\ m(n_k + 1) \le b_k m(n_k) + e_k, \end{cases}$$

where $\{l_n\}$ *and* $\{q_n\}$ *are two real valued sequences and* $l_n > -1$, e_k, b_k *are constants and* $b_k \ge 0$. *Then*

$$
\begin{aligned}
m(n) \le m(n_0) \prod_{n_0<n_k<n} b_k \prod_{n_0<i<n, i\neq n_k, k\in N} (1+l_i) \\
+ \sum_{n_0<n_k<n} e_k \prod_{n_k<n_j<n} b_j \prod_{n_k<i<n, i\neq n_j, j\in N} (1+l_i) \\
+ \sum_{i=n_0, i\neq n_k}^{n-1} \prod_{i<n_k<n} b_k \prod_{i<s<n, s\neq n_k} (1+l_s) q_i, \quad n \ge n_0. \quad (3)
\end{aligned}
$$

Proof. This lemma is a discrete version of Theorem 1.4.1 in [3] and Lemma 1 in [1]. The proof can be followed from mathematical induction and direct analysis.

If $n \in N[n_0, n_1]$, $m(n) \le m(n_0) \prod_{n_0<i<n}(1+l_i) + \sum_{i=n_0}^{n-1} \prod_{i<s<n}(1+l_s)q_i$, obviously, for $n \in N[n_0, n_1]$, (3) holds. We might assume for $n \in N[n_0, n_p]$, (3) also holds, thus, for $n \in N(n_p, n_{p+1}]$, we get

$$
\begin{aligned}
m(n) &\le m(n_p+1) \prod_{n_p<i<n} (1+l_i) + \sum_{i=n_p}^{n-1} \prod_{i<s<n} (1+l_s) q_i \\
&\le (b_p m(n_p) + e_p) \prod_{n_p<i<n} (1+l_i) + \sum_{i=n_p}^{n-1} \prod_{i<s<n} (1+l_s) q_i \,.
\end{aligned}
$$

From the induction hypothesis, the above inequality turns into

$$
\begin{aligned}
m(n) \le \{ b_p [m(n_0) \prod_{n_0<n_k<n_p} b_k \prod_{n_0<i<n_p, i\neq n_k, k\in N} (1+l_i) \\
+ \sum_{n_0<n_k<n_p} e_k \prod_{n_k<n_j<n_p} b_j \prod_{n_k<i<n_p, i\neq n_j, j\in N} (1+l_i) \\
+ \sum_{i=n_0, i\neq n_k}^{n_p-1} \prod_{i<n_k<n_p} b_k \prod_{i<s<n_p, s\neq n_k} (1+l_s) q_i] + e_p \} \prod_{n_p<i<n} (1+l_i) \\
+ \sum_{i=n_p}^{n-1} \prod_{i<s<n} (1+l_s) q_i,
\end{aligned}
$$

which on simplification gives the estimate (3) for $n \in N[n_0, n_{p+1}]$. The proof is complete. □

Lemma 2. *Let $x(n)$ be a solution of* (1). *Suppose that there exist some $N^* \ge n_0$ such that $x(n) > 0$ for $n \ge N^*$, and the following conditions hold:*

(h1) $(C1) - (C4)$;

(h2) *For all sufficiently large $n_j \geq n_1$ the following inequality holds*

$$\lim_{n\to\infty} \sum_{m=0}^{n-n_j} \alpha^{n-n_j-m} \prod_{n_j \leq n_k \leq n_j+m} (b_k b'_k)^{\frac{1}{\sigma}} = +\infty\,.$$

Then

$$\triangle_\alpha x(n_k - 1) \geq 0, \triangle_\alpha x(n) \geq 0, \quad n \in N[n_k, n_{k+1})(n_k - l > N^*)\,.$$

Proof. Firstly, we show that

$$\triangle_\alpha x(n_k - 1) \geq 0$$

for any $n_k \geq N^*$. Otherwise, there exists some j such that

$$\triangle_\alpha x(n_j - 1) < 0$$

for $n_j - 1 > N^*$, from (1), $(C2)$ and $(C3)$, we get

$$\begin{aligned}\varphi(x(n_j))(\triangle_\alpha x(n_j))^\sigma &= \varphi(b'_j(x(n_j - 1)))b_j(\triangle_\alpha x(n_j - 1))^\sigma) \\ &\leq b_j b'_j \varphi((x(n_j - 1)))(\triangle_\alpha x(n_j - 1))^\sigma < 0\,.\end{aligned}$$

Let

$$\varphi((x(n_j - 1)))(\triangle_\alpha x(n_j - 1))^\sigma = -\beta^\sigma \quad (\beta > 0).$$

By (1), for $n \in N(n_{j+i-1}, n_{j+i})$, $i = 1, 2, \cdots$, we have

$$\begin{aligned}\triangle S(n) &= \triangle(\varphi(x(n-1))(\triangle_\alpha x(n-1))^\sigma) \\ &= -f(n, x(n-l)) + g(x(n)) \leq -p_n(x(n-l))^\sigma\,.\end{aligned}$$

Hence, $S(n)$ is monotonically decreasing in $N(n_{j+i-1}, n_{j+i})$. So,

$$\varphi(x(n_{j+1} - 1))(\triangle_\alpha x(n_{j+1} - 1))^\sigma \leq \varphi(x(n_j))(\triangle_\alpha x(n_j))^\sigma \leq -b_j b'_j \beta^\sigma < 0$$

and

$$\begin{aligned}\varphi(x(n_{j+2} - 1))(\triangle_\alpha x(n_{j+2} - 1))^\sigma &\leq \varphi(x(n_{j+1}))(\triangle_\alpha x(n_{j+1}))^\sigma \\ &\leq b_{j+1} b'_{j+1} \varphi(x(n_{j+1} - 1))(\triangle_\alpha x(n_{j+1} - 1))^\sigma \leq -b_{j+1} b'_{j+1} b_j b'_j \beta^\sigma < 0.\end{aligned}$$

By induction, we obtain

$$\varphi(x(n))(\triangle_\alpha x(n))^\sigma \le -\beta^\sigma \prod_{n_j\le n_k\le n} b_k b'_k < 0\,.$$

Due to $x(n)>0$ for $n\ge N^*$, it follows from $(C3)$ that there exists a positive constant $c(<x(n))$ such that $\varphi(x(n))<\varphi(c)$. Thus

$$\triangle_\alpha x(n) \le -\frac{\beta}{(\varphi(c))^{\frac{1}{\sigma}}} \prod_{n_j\le n_k\le n} (b_k b'_k)^{\frac{1}{\sigma}}\,.$$

Now we go on the following calculation

$$x(n_j+1)-\alpha x(n_j) = x(n_j+1)-\alpha b'_j x(n_j-1) \le -\frac{\beta}{(\varphi(c))^{\frac{1}{\sigma}}}(b_j b'_j)^{\frac{1}{\sigma}},$$

$$x(n_j+2)-\alpha^2 x(n_j) \le -\frac{\alpha\beta}{(\varphi(c))^{\frac{1}{\sigma}}}(b_j b'_j)^{\frac{1}{\sigma}} - \frac{\beta}{(\varphi(c))^{\frac{1}{\sigma}}} \prod_{n_j\le n_k\le n_j+1}(b_k b'_k)^{\frac{1}{\sigma}},$$

$$\begin{aligned} x(n_j+3)-\alpha^3 x(n_j) \le& -\frac{\alpha^2\beta}{(\varphi(c))^{\frac{1}{\sigma}}}(b_j b'_j)^{\frac{1}{\sigma}} - \frac{\alpha\beta}{(\varphi(c))^{\frac{1}{\sigma}}} \prod_{n_j\le n_k\le n_j+1}(b_k b'_k)^{\frac{1}{\sigma}} \\ & - \frac{\beta}{(\varphi(c))^{\frac{1}{\sigma}}} \prod_{n_j\le n_k\le n_j+2}(b_k b'_k)^{\frac{1}{\sigma}}\,. \end{aligned}$$

By induction, we obtain

$$\begin{aligned} &x(n+1) \\ &\le \alpha^{n-n_j+1} b'_j x(n_j-1) - \frac{\alpha^{n-n_j}\beta}{(\varphi(c))^{\frac{1}{\sigma}}}(b_j b'_j)^{\frac{1}{\sigma}} - \frac{\alpha^{n-n_j-1}\beta}{(\varphi(c))^{\frac{1}{\sigma}}} \prod_{n_j\le n_k\le n_j+1}(b_k b'_k)^{\frac{1}{\sigma}} \\ &\quad - \frac{\alpha^{n-n_j-2}\beta}{(\varphi(c))^{\frac{1}{\sigma}}} \prod_{n_j\le n_k\le n_j+2}(b_k b'_k)^{\frac{1}{\sigma}} - \cdots - \frac{\alpha\beta}{(\varphi(c))^{\frac{1}{\sigma}}} \prod_{n_j\le n_k\le n-1}(b_k b'_k)^{\frac{1}{\sigma}} \\ &\quad - \frac{\beta}{(\varphi(c))^{\frac{1}{\sigma}}} \prod_{n_j\le n_k\le n}(b_k b'_k)^{\frac{1}{\sigma}}, \qquad (4) \end{aligned}$$

in view of $x(n)>0$, by $(h2)$, we can find that the right side of (4) converges to $-\infty$, however, the left side of (4) is eventually positive, which is a contradiction. Therefore

$$\triangle_\alpha x(n_k-1)\ge 0, \quad n_k-1\ge N^*\,.$$

By $(C2)$, for $\forall n_k \geq N^* + 1, \varphi(x(n_k))(\triangle_\alpha x(n_k))^\sigma = b_k(\varphi(x(n_k - 1))(\triangle_\alpha x(n_k - 1))^\sigma) \geq 0$. Because $S(n)$ is monotonically decreasing in $N(n_{j+i-1}, n_{j+i}]$, therefore, we get $S(n) > 0$ for $n \in N(n_{j+i-1}, n_{j+i}]$, which implies

$$\triangle_\alpha x(n) \geq 0 \,.$$

The proof is complete. □

Remark 3. Suppose that $x(n)$ is eventually negative, if (h1) and (h2) hold true, then we get $\triangle_\alpha x(n_k - 1) \leq 0, \triangle_\alpha x(n) \leq 0,\ n \in N[n_k, n_{k+1})(n_k - l > N^*)$.

3. Oscillation Criteria

Theorem 4. *Suppose that condition (h1) and (h2) hold, and for all sufficiently large n_j,*

$$\sum_{i=n_j+1, i\neq n_k}^{n} p_i \prod_{n_j < n_k \leq i} \frac{\alpha^\sigma}{b_k b'_k} \longrightarrow +\infty (n \to \infty) \tag{5}$$

holds. Then every solution of (1) *is oscillatory.*

Proof. If (1) has a nonoscillatory solution $x(n)$, without loss of generality, we can assume that $x(n) > 0 (n > n_0)$. From Lemma 2, get $\triangle_\alpha x(n) \geq 0,\ n \in N[n_k, n_{k+1}](n_1 > n_0 + l),\ k = 1, 2, \cdots$.

Let

$$w(n) = \frac{\varphi(x(n-1))(\triangle_\alpha x(n-1))^\sigma}{(x(n-l))^\sigma} \,.$$

Then

$$w(n_k) \geq 0 (k = 1, 2, \cdots), \quad w(n) \geq 0 (n \geq n_0) \,.$$

Using (1) and (C1), we get

$$\begin{aligned}
\triangle w(n) &= \frac{\varphi(x(n))(\triangle_\alpha x(n))^\sigma}{(x(n+1-l))^\sigma} - \frac{\varphi(x(n-1))(\triangle_\alpha x(n-1))^\sigma}{(x(n-l))^\sigma} \\
&= \frac{\triangle(\varphi(x(n-1))(\triangle_\alpha x(n-1))^\sigma)}{(x(n-l))^\sigma} - \frac{\varphi(x(n))(\triangle_\alpha x(n))^\sigma}{(x(n-l))^\sigma} + \frac{\varphi(x(n))(\triangle_\alpha x(n))^\sigma}{(x(n+1-l))^\sigma} \\
&= \frac{-f(n, x(n-l)) + g(x(n))}{(x(n-l))^\sigma} - \frac{\varphi(x(n))(\triangle_\alpha x(n))^\sigma \triangle((x(n-l))^\sigma)}{(x(n-l))^\sigma (x(n+1-l))^\sigma} \leq -p_n,
\end{aligned}$$

$$w(n_k+1) = \frac{\varphi(x(n_k))(\triangle_\alpha x(n_k))^\sigma}{(x(n_k+1-l))^\sigma} \le \frac{b_k b'_k \varphi(x(n_k-1))(\triangle_\alpha x(n_k-1))^\sigma}{(x(n_k+1-l))^\sigma} \le \frac{b_k b'_k}{\alpha^\sigma} w(n_k)\,.$$

It follows from the above inequalities that $w(n)$ satisfies the following difference inequalities

$$\begin{cases} \triangle w(n) \le -p_n, \quad n \ne n_k, k \in N, \\ w(n_k+1) \le \frac{b_k b'_k}{\alpha^\sigma} w(n_k). \end{cases}$$

Applying Lemma 1, we have

$$w(n+1) \le w(n_j) \prod_{n_j<n_k\le n} \frac{b_k b'_k}{\alpha^\sigma} - \sum_{i=n_j+1, i\ne n_k}^{n} p_i \prod_{i<n_k\le n} \frac{b_k b'_k}{\alpha^\sigma}$$

$$= \prod_{n_j<n_k\le n} \frac{b_k b'_k}{\alpha^\sigma} \left(w(n_j) - \sum_{i=n_j+1, i\ne n_k}^{n} p_i \prod_{n_j<n_k\le i} \frac{\alpha^\sigma}{b_k b'_k}\right), \quad n \ge n_j. \tag{6}$$

By (4), (6) and $w(n) > 0$, we can draw a contradiction as $n \to \infty$. Hence, every solution of (1) is oscillatory. The proof is complete. □

Corollary 5. *Assume that (h1) and (h2) hold and there exists a positive integer k_0 such that $\alpha \ge (b_k b'_k)^{\frac{1}{\sigma}}$ for $k \ge k_0$. If*

$$\sum_{n\ne n_k, k\in N} p_n = +\infty, \tag{7}$$

then every solution of (1) *is oscillatory.*

Proof. Without loss of generality, let $k_0 = 1$. It follows from $\alpha \ge (b_k b'_k)^{\frac{1}{\sigma}}$ that

$$\sum_{i=n_j+1, i\ne n_k, k\in N}^{n} p_i \prod_{n_j<n_k\le i} \frac{\alpha^\sigma}{b_k b'_k} \ge \sum_{i=n_j+1, i\ne n_k, k\in N}^{n} p_i\,. \tag{8}$$

Let $n \to \infty$, applying (7) and (8), we get (4). According to Theorem 4 we get every solution of (1) is oscillatory. □

Corollary 6. *Assume that* $(h1)$ *and* $(h2)$ *hold and there exists a positive integer k_0 such that $\alpha \ge \frac{n_{k+1}}{n_k}(b_k b'_k)^{\frac{1}{\sigma}}$ for $k \ge k_0$. If*

$$\sum_{n\ne n_k, k\in N} n^\sigma p_n = +\infty, \tag{9}$$

then every solution of (1) *is oscillatory.*

Proof. Without loss of generality, let $k_0 = 1$. It follows from $\alpha \geq \frac{n_{k+1}}{n_k}(b_k b'_k)^{\frac{1}{\sigma}}$ that

$$\begin{aligned}
\sum_{i=n_0, i\neq n_k, k\in N}^{n} p_i \prod_{n_0\leq n_k\leq i} \frac{\alpha^\sigma}{b_k b'_k} &= \sum_{i=n_o}^{n_1-1} p_i + \frac{\alpha^\sigma}{b_1 b'_1}\sum_{i=n_1+1}^{n_2-1} p_i + \cdots \\
&\quad + \frac{\alpha^{j\sigma}}{b_1 b'_1\ b_2 b'_2\ b_3 b'_3 \cdots b_j b'_j} \sum_{i=n_j+1, i\neq n_k, k\in N[j+1,\infty)}^{n} p_i \\
&\geq \frac{\alpha^\sigma}{b_1 b'_1}\sum_{s=n_1+1}^{n_2-1} p_s + \cdots + \frac{\alpha^{j\sigma}}{b_1 b'_1\ b_2 b'_2\ b_3 b'_3 \cdots b_j b'_j} \sum_{s=n_j+1, i\neq n_k, k\in N[j+1,\infty)}^{n} p_s \\
&\geq \frac{1}{n_1^\sigma}[\sum_{s=n_1+1}^{n_2-1} (n_2)^\sigma p_s + \cdots + \sum_{s=n_j+1}^{n} (n_{j+1})^\sigma p_s] \\
&= \frac{1}{n_1^\sigma} \sum_{s=n_1+1, s\neq n_k, k\in N}^{n} s^\sigma p_s, \qquad n \in N(n_j, n_{j+1}).
\end{aligned}$$

Let $n \to \infty$, applying (9), we get (4). According to Theorem 4 we get every solution of (1) is oscillatory. □

4. An Example

Consider impulsive delay difference equation

$$\begin{cases} \triangle(\frac{x(n-1)}{\ln(n-1)}\triangle_2 x(n-1)) + \frac{(\ln\frac{n^2}{n^2-1} + \ln\frac{n}{n-1} - \frac{1}{n^2})}{\ln(n-l)} x(n-l) \\ \qquad\qquad = -\frac{1}{n^2}\operatorname{sgn}(x(n)), \quad n \neq 2k, k \in N, \\ \triangle_2 x(2k) = \frac{k}{k+1}\triangle_2 x(2k-1)), \\ x(2k) = \frac{k+1}{k} x(2k-1), \end{cases} \tag{10}$$

in which $\sigma = 1$, $\varphi(x) = \frac{x}{\ln(n-1)}$, $\alpha = 2$, $p_n = \frac{(\ln\frac{n^2}{n^2-1} + \ln\frac{n}{n-1} - \frac{1}{n^2})}{\ln(n-l)}$, $b_k = \frac{k}{k+1}$, $b'_k = \frac{k+1}{k}$. Applying Corollary 5, we get every solution of (10) is oscillatory. But the delay difference equation

$$\triangle(\frac{x(n-1)}{\ln(n-1)}\triangle_2 x(n-1)) + \frac{(\ln\frac{n^2}{n^2-1} + \ln\frac{n}{n-1} - \frac{1}{n^2})}{\ln(n-l)} x(n-l) = -\frac{1}{n^2}\operatorname{sgn}(x(n))$$

has a nonoscillatory solution $x(n) = \ln n$.

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