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ON THE MONOTONICITY OF THE TRINOMIAL ARCS $C(p,k,r,n)$ OUTSIDE THE UNIT DISK, WHEN $\alpha > 1$

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Abstract: In this work, we deal with the family $C(p,k,r,n)$ of trinomial arcs defined as the set of roots of trinomial equation $z^n = \alpha z^k + (1-\alpha)$, where $z = \rho\, e^{i\theta}$, n and k are two integers such that $1 \leq k \leq n-1$, α is a real number greater than 1 and p and r are nonzero integers satisfying some conditions. These curves $C(p,k,r,n)$ are continuous arcs expressed in polar coordinates (ρ,θ) by a function $\rho(\theta)$, where $\rho \geq 1$ and θ is a feasible angle, an angle verifying some conditions. In this paper, the question is to prove that ρ changes monotonically with respect to θ and that $\rho(\theta)$ is an increasing function, for each trinomial arc $C(p,k,r,n)$.

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1. Introduction

Consider the trinomial equation

$$z^n = \alpha\, z^k + (1-\alpha), \tag{1}$$

where z is a complex number, n is an integer greater than 1, $k = 1, 2, \ldots, n-1$

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and α is a real number such that $\alpha > 1$.

Noting that the first complete discussion of the behaviour of the roots of trinomial equation was fulfilled by Fell [5]. She has established a large description of the trajectories of these roots, called *trinomial arcs*, which are continuous arcs, corresponding to a number α which is whether between 0 and 1, or between 1 and $+\infty$, or also between $-\infty$ and 0. Each of these arcs can be expressed in polar coordinates (ρ, θ) by a function $\rho(\theta)$, where θ satisfy some conditions. She has studied equally the monotonicity of the function $\alpha(\theta)$ and gave one bound for the modulus of roots. However, she could not establish the monotonicity of ρ as a function of θ.

The descriptive results of Fell [5] gave us the information about the form and the localization of the trinomial arcs. However, these types of arcs are not well-defined, in order to be studied. In this paper, we will restrict our attention to a family of trinomial arcs, solutions of equation (1) with $\alpha > 1$, outside the unit disk $D = \{z\ ;\ |z| \leq 1\}$. First, we formulate and define this family of trinomial arcs, denoted by $C(p, k, r, n)$, where p, k, r and n satisfy some conditions. Afterwards, we prove that $\rho(\theta)$ is a derivable function for these arcs $C(p, k, r, n)$. In order to solve the problem of monotonicity of $\rho(\theta)$, an intermediate result is showed. Finally, we prove that $\rho(\theta)$ is an increasing function, for each trinomial arc $C(p, k, r, n)$.

2. Study of the Trinomial Equation

In the trinomial equation (1), fix n and k. For $z = \rho\, e^{i\theta}$ in (1), one gets

$$\rho^n\, e^{in\theta} = \alpha\, \rho^k\, e^{ik\theta} + (1-\alpha).$$

So $\rho^n \cos n\theta + i\,\rho^n \sin n\theta = \alpha\,\rho^k \cos k\theta + i\,\alpha\,\rho^k \sin k\theta + (1-\alpha)$. Separating real and imaginary parts of both sides of this equality, one has

$$\rho^n \sin n\theta = \alpha\rho^k \sin k\theta, \quad \rho^n \cos n\theta = \alpha\rho^k \cos k\theta + (1-\alpha)\,.$$

So, when $\theta \neq l\pi/n$, where $l \in \mathbb{N}$, we obtain

$$\rho^{n-k} = \alpha\ \sin k\theta\ /\ \sin n\theta\,. \tag{2}$$

On the other hand, divide the equation (1) by z^n and consider the imaginary part. When $\theta \neq l\pi/(n-k)$, where $l \in \mathbb{N}$, we get

$$\rho^k = \frac{(\alpha-1)}{\alpha}\,\frac{\sin n\theta}{\sin(n-k)\,\theta}\,. \tag{3}$$

At last, we have the following α-free equation for the trajectories of roots of (1):

$$\rho^{n-k}\ \sin n\theta - \rho^n\ \sin(n-k)\theta = \sin k\theta\,. \tag{4}$$

Otherwise, in [5], Fell has studied the trinomial equation

$$\lambda\, z^n \;+\; (1-\lambda)\; z^k \;-\; 1 \;=\; 0\,, \tag{5}$$

where z is a complex number, n is an integer larger than one, $k = 1, 2, \ldots,$ $n-1$ and λ is a real number.

In order to pass from equation (1) to equation (5), we can set $\alpha = 1 - 1/\lambda$.

Indeed, substituting into equation (5) the expression given for z^n by equation (1) yields $\left(z^k - 1\right)\; [1 \;-\; \lambda\,(1-\alpha)] \;=\; 0$. Then, $z^k = 1$ or $\lambda\,(1-\alpha) = 1$. As z is a complex number, so $\alpha = 1 - 1/\lambda$.

From this equality, we easily conclude that the case $1 \le \alpha < +\infty$ of equation (1) corresponds to the case $-\infty < \lambda < 0$ of equation (5).

According to Fell [5] and taking into account this correspondence between α and λ, we are interested in those angles θ for which

$$\operatorname{sgn}(\sin n\theta) = \operatorname{sgn}(\sin k\theta) = \operatorname{sgn}(\sin(n-k)\theta)\,. \tag{6}$$

Definition 1. An angle θ which fulfills (6) will be called (n, k) feasible angle for the equation (1) with $\alpha > 1$.

Remark 2. The upper and lower half-planes are symmetrical. So, we will restrict our study to the upper half-plane.

3. Description and Definition of Trinomial Arcs $C(p, k, r, n)$

For $\alpha = 1$, the equation (1) becomes $z^k\,\left[z^{n-k} - 1\right] = 0$. Then, the n roots of equation (1) are exactly the $(n-k)$-th roots of unity, which are simple roots and 0, a root of multiplicity k. According to Fell [5], when α moves from 1 to $+\infty$, the n trajectories of the n roots are continuous arcs. In fact, outside the unit disk, there exist $(n-k)$ trajectories of these arcs, those of the $(n-k)$-th roots of unity. In [5], these $(n-k)$ trajectories are described and located as following:

Let

$$C = \{\ n\text{-th roots of unity}\ \}\,, \quad D \;=\; \{\ (n-k)\text{-th roots of unity}\ \}\,,$$

and

$$E \;= \{\ k\text{-th roots of } -1\ \}\,.$$

Let $\gamma \in C$ and δ be the unique nearest neighbor of γ in $D \cup E$. Then, we distinguish the following cases:

1. $\delta \in D$, $\delta \notin E$. As α moves from 1 to $+\infty$, there is a trajectory starting at δ such that ρ moves from 1 to $+\infty$. When ρ goes to infinity, the curve is asymptotic to the line $\theta = \arg(\delta)$.

2. $\delta \in D \cap E$. As α moves from 1 to $+\infty$, ρ changes from 1 to $+\infty$. This

trajectory is the line of equation $\theta = \arg(\delta)$. Thus, $\delta \in D \cap E$ corresponds to one particular case of trinomial arcs.

3. $\theta = 0$. As α moves from 1 to $+\infty$, ρ changes from 1 to $+\infty$. There exists always a root at $z = 1$.

In [5], Fell has showed that, in the case $\delta \in D$, $\delta \notin E$ and $\alpha > 1$, the feasible angles θ belong to separate intervals I_j of length less than or equal to π/n and bounded on one side by β_j and on the other side by β'_j, such that the number $e^{i\beta_j}$ is an $(n-k)$-th root of unity and the number $e^{i\beta'_j}$ is an n-th root of -1. Thus, the trajectories of roots of (1) corresponding to this case move into these intervals I_j outside the unit disk.

In this paper, by leaning on this result, we will consider and restrict our attention to the trinomial arcs, such that the feasible angles θ belong to an interval of the form $[2\pi p/(n-k),\ (2r+1)\pi/n]$, where p and r are nonzero integers satisfying some conditions. Indeed, $2\pi p/(n-k) = \beta_j$ verify that the number $e^{i\beta_j}$ is an $(n-k)$-th root of unity and $(2r+1)\pi/n = \beta'_j$ verify that the number $e^{i\beta'_j}$ is an n-th root of -1. These arcs will be denoted by $C(p,k,r,n)$. Notice that the two cases $n = 2$ and $n = 3$ are particular cases, because the trajectories, solutions of (1), where $\alpha > 1$, for these cases are linear. Thus, we define the family of trinomial arcs $C(p,k,r,n)$ as follows.

Definition 3. If n is an integer greater than or equal to 4, so $C(p,k,r,n)$ is the set of roots of equation (1) such that $\alpha > 1$ and the feasible angles belong to the interval $[2\pi p/(n-k),\ (2r+1)\pi/n]$, where p and r are nonzero integers verifying $r \geq p$ and k is an integer such that $(r-p)n/r < k < [2(r-p)+1]n/(2r+1)$.

This type of curves $C(p,k,r,n)$ (see the picture below) exists in view of the following lemma.

Lemma 4. *If n is an integer greater than or equal to 4, k is an integer such that $1 \leq k \leq n-1$ and $\alpha > 1$, then in the trinomial equation (1) with $(r-p)n/r < k < [2(r-p)+1]n/(2r+1)$, where p and r are nonzero integers verifying $r \geq p$, any angle of the interval $[2\pi p/(n-k),\ (2r+1)\pi/n]$ is feasible.*

Proof. We assume that k is an integer verifying $(r-p)n/r < k < [2(r-p)+1]n/(2r+1)$. In fact, because $0 < k < n$, the nonzero integers p and r satisfy the condition $1 - p/r \geq 0$, i.e. $r \geq p$. Let θ be an angle such that $2\pi p/(n-k) < \theta < (2r+1)\pi/n$. So:

— $2\pi pn/(n-k) < n\theta < (2r+1)\pi$. As $(r-p)n/r < k$, then $2\pi r < 2\pi pn/(n-k)$. Then $\sin n\theta > 0$.

— $2\pi p < (n-k)\theta < (2r+1)\pi(1-k/n)$. Since $r \geq p$, we have $2(r-p)n/(2r+1) < (r-p)n/r < k$. It follows that $(2r+1)\pi(1-k/n) < (2p+1)\pi$ and that

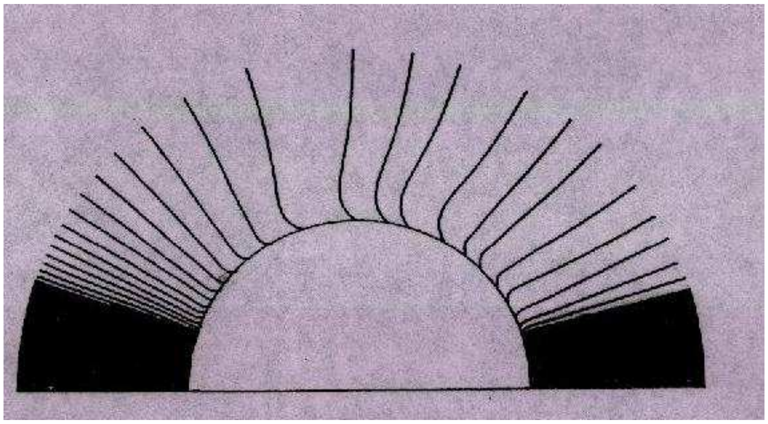

Figure 1: Trinomial arcs $C(p,k,r,n)$

$\sin(n-k)\theta > 0$.

— $2\pi pk/(n-k) < k\theta < (2r+1)\pi k/n$. Because $(r-p)n/r < k < [2(r-p)+1]n/(2r+1)$, so $2(r-p)\pi < 2\pi pk/(n-k)$ and $(2r+1)\pi k/n < [2(r-p)+1]\,\pi$. Then $\sin k\theta > 0$.

Therefore, the conditions (6) are fulfilled. □

4. Derivability of the Function $\rho\,(\theta)$ for the Arcs $C(p,k,r,n)$

In this section, we will show that the derivative $d\rho/d\theta$ exists and it is well-defined.

Lemma 5. *For each trinomial arc $C(p,k,r,n)$, the function $\rho(\theta)$ is derivable.*

Proof. Let $C(p,k,r,n)$ be a trinomial arc. From the equality (3), we get $\rho^k(\theta) = (1-1/\alpha)\,\sin n\theta/\sin(n-k)\theta$. By Lemma 4, the feasible angles θ satisfy $\sin n\theta > 0$ and $\sin(n-k)\theta > 0$. If we set $f : \theta \longmapsto (1-1/\alpha)\,\sin n\theta/\sin(n-k)\theta$ and because $\alpha > 1$, the denominator of $f\,(\theta)$ is never zero. The function f is so well-defined. In addition, f is derivable and positive. So, the function $\rho(\theta) = [f\,(\theta)]^{1/k} = [(1-1/\alpha)\,\sin n\theta/\sin(n-k)\theta]^{1/k}$ is derivable. Thus, its derivative $d\rho/d\theta$ exists and it is well-defined. □

5. Monotonicity of the Function $\rho(\theta)$ for the Arcs $C(p,k,r,n)$

Now, our main interest is to prove that $d\rho/d\theta \neq 0$, i.e. that the function $\rho(\theta)$ is monotonic, for each trinomial arc $C(p,k,r,n)$. For that, in equality (4), differentiating both sides with respect to θ, we obtain

$$\left[(n-k)\,\rho^{n-k-1}\,\sin n\theta - n\rho^{n-1}\,\sin(n-k)\theta\right]\,d\rho/d\theta$$
$$= k\,\cos k\theta \,+\, (n-k)\,\rho^n\,\cos(n-k)\theta \,-\, n\,\rho^{n-k}\,\cos n\theta.$$

Supposing that $d\rho/d\theta = 0$, we will consider ρ^{n-k} and ρ^n as solutions of the system:

$$k\cos k\theta + (n-k)\rho^n\cos(n-k)\theta - n\rho^{n-k}\cos n\theta = 0\,,$$
$$\rho^{n-k}\sin n\theta - \rho^n\sin(n-k)\theta - \sin k\theta = 0\,.$$

This last system is equivalent to the following system:

$$R\,(\theta)\cdot\rho^{n-k} = N_1\,(\theta)\ , \qquad R\,(\theta)\cdot\rho^n = N_2\,(\theta)\ , \tag{7}$$

where

$$R(\theta) = (n-k)\sin k\theta - k\cos n\theta\sin(n-k)\theta\,,$$
$$N_1(\theta) = (n-k)\sin n\theta - n\sin(n-k)\theta\cos k\theta\,,$$
$$N_2(\theta) = (n-k)\sin n\theta\cos k\theta - n\sin(n-k)\theta\,.$$

The difference of the two equations of (7) leads to the relation:

$$R(\theta)[\rho^n - \rho^{n-k}] = U(\theta)[1-\cos k\theta]\,, \tag{8}$$

with

$$U(\theta) = -[n\sin(n-k)\theta + (n-k)\sin n\theta].$$

In what follows, the question is to contradict the hypothesis $d\rho/d\theta = 0$ for the type of arcs $C(p,k,r,n)$. For that, we will need the next lemma.

Lemma 6. *For any integer k such that$(r-p)n/r < k < [2(r-p)+1]n/(2r+1)$, we have $R(\theta) = (n-k)\sin k\theta - k\cos n\theta\sin(n-k)\theta > 0$, for any angle θ in the interval $]2\pi p/(n-k),(2r+1)\pi/n[$.*

Proof Let be $\theta \in\,]2\pi p/(n-k),(2r+1)\pi/n[$. Remarking that $R(\theta) = n\sin k\theta - k\sin n\theta\cos(n-k)\theta$, we obtain that $R(\theta) \geq V(\theta)$, where $V(\theta) = n\sin k\theta - k\sin n\theta$. The function $V(\theta)$ is derivable, of derivative $V'(\theta) = nk[\cos k\theta - \cos n\theta]$. Then, the zeros of this derivative verify the equation $\cos n\theta = \cos k\theta$, of which the solutions are of the form $\theta = 2l\pi/(n-k)$ or of the form $\theta = 2l\pi/(n+k)$, where $l \in \mathbb{N}$.

On the one hand, $2l\pi/(n-k) \in\,]2\pi p/(n-k),(2r+1)\pi/n[$ if and only if $p < l < (2r+1)(1/2-k/2n)$. Since $(r-p)n/r < k$, then $(2r+1)(1/2-k/2n) < p(2r+1)/2r$ and since $r \geq p$, so $p(2r+1)/2r \leq (p+1/2)$. It follows that the integer l satisfy $p < l < (p+1/2)$, which is not possible.

On the other hand, $2l\pi/(n+k) \in]2\pi p/(n-k), (2r+1)\pi/n[$ if and only if $2p[1/2 + k/(n-k)] < l < (2r+1)(1/2 + k/2n)$. Because $(r-p)n/r < k$, so $(2r-p) < 2p[1/2 + k/(n-k)]$ and as $k < [2(r-p)+1]n/(2r+1)$, then $(2r+1)(1/2 + k/2n) < (2r-p+1)$. So $(2r-p) < l < (2r-p+1)$, which is impossible.

Therefore, $V'(\theta) \neq 0$, namely $V(\theta)$ is monotonic on the interval of feasible angles $]2\pi p/(n-k), (2r+1)\pi/n[$. At the bounds of this interval, we have $V(2\pi p/(n-k)) > 0$ and $V([2r+1]\pi/n) > 0$. Then, we conclude that $V(\theta) > 0$, and consequently $R(\theta) > 0$ for any angle θ in

$$]2\pi p/(n-k), (2r+1)\pi/n[\,. \qquad \square$$

Lemma 6 allows us to state the next main result for the trinomial arcs $C(p,k,r,n)$.

Theorem 7. *For any feasible angle θ in $]2\pi p/(n-k), (2r+1)\pi/n[$, where n is an integer greater than or equal to 4, p and r are nonzero integers such that $r \geq p$ and for any integer k such that $(r-p)n/r < k < [2(r-p)+1]n/(2r+1)$, we have $d\rho\,/\,d\theta \neq 0$.*

Proof. Let be θ a feasible angle in $]2\pi p/(n-k), (2r+1)\pi/n[$. From Lemma 4 stems that $U(\theta) < 0$ and from Lemma 6 stems that $R(\theta) > 0$ for any θ in the interval $]2\pi p/(n-k), (2r+1)\pi/n[$. Therefore, the equality (8) implies that $\rho^n - \rho^{n-k} < 0$, which is not possible because $\rho > 1$. Thus, we have proved that for each trinomial arc $C(p,k,r,n)$, we have $d\rho/d\theta \neq 0$, namely $\rho(\theta)$ is a monotonic function, for any feasible angle θ in the interval $]2\pi p/(n-k), (2r+1)\pi/n[$. $\square$

Remark 8. The particular case $k = (r-p)n/r$, where $r \neq p$ corresponds to a linear trinomial arc outside the unit disk. We can prove that as follows, in this case, we have $\theta = 2\pi p/(n-k) = 2\pi r/n$. Using the notations of Fell [5], we can tell that $2\pi r/n = \arg(\gamma)$, such that $\gamma \in C = \{n\text{-th } \textit{roots of unity}\}$ and that $2\pi p/(n-k) = \arg(\delta)$, such that $\delta \in D = \{(n-k)\text{-th } \textit{roots of unity}\}$. According to Fell [5], as $\arg(\delta) = \arg(\gamma)$, this particular case corresponds to a half-line outside the unit disk

Remark 9. The particular case $k = [2(r-p)+1]n/(2r+1)$ corresponds to a linear trinomial arc outside the unit disk. Indeed, when $k = [2(r-p)+1]n/(2r+1)$, we have $\theta = 2\pi p/(n-k) = (2r+1)\pi/n$. Then, the trinomial arc $C(p,k,r,n)$ is such that the bounds of the interval of feasible angles are identical. So, this particular case corresponds to a half-line outside the unit disk

Lastly, Theorem 7 allows us to prove the following main result.

Theorem 10. *For the trinomial arcs $C(p,k,r,n)$, $\rho(\theta)$ is an increasing* *function on the interval* $[2\pi p/(n-k),(2r+1)\pi/n]$.

Proof. Let $C(p,k,r,n)$ be a trinomial arc. By first, we estimate $\rho\,(\theta)$ at the bound $2\pi p/(n-k)$ of the interval of feasible angles. Then, we put $\theta = 2\pi p/(n-k)$ in the equation $\rho^{n-k}\sin n\theta - \rho^n \sin(n-k)\theta = \sin k\theta$ given by (4). So, we obtain the equality $\left[\rho^{n-k}-1\right]\sin(2\pi pn/(n-k)) = 0$. But $\sin(2\pi pn/(n-k)) \neq 0$, because $2\pi r < (2\pi pn/(n-k)) < (2r+1)\,\pi$, so, we deduce that $\rho\,[2\pi p/(n-k)] = 1$.

On the other hand, $\rho\,(\theta)$ changes between 1 and $+\infty$ for any angle θ included between $2\pi p/(n-k)$ and $(2r+1)\pi/n$. Then, from Theorem 7, we have that $\rho(\theta)$ is an increasing function on the interval $[2\pi p/(n-k),(2r+1)\pi/n]$, for each trinomial arc $C(p,k,r,n)$. □

References

[1] M. Biernacki, Sur un nouveau théorème d'algèbre, *C.R. Académie des Sciences,* ***177*** *(1923), 1193-1194.*

[2] D.Z. Djokovic, Cyclic polygons, roots of polynomials with decreasing non-negative coefficients, and eigenvalues of stochastic matrices, *Linear Algebra Appl.*, **142** (1990), 173-193.

[3] S. Dubuc, A. Malik, Convex hull of powers of a complex number, trinomial equations and the Farey sequence, *Num. Algorithms*, **2** (1992), 1-32.

[4] S. Dubuc, M. Zaoui, Sur la quasi-convexité des arcs trinomiaux, *Rend. Circ. Mat. Palermo, Série 2*, **XLV** (1996), 493-514.

[5] H. Fell, The geometry of zeros of trinomial equations, *Rend. Circ. Mat. Palermo, Série 2*, **29** (1980), 303-336.

[6] F. Pécastaings, J. Sevin, *Chemins vers la Géométrie,* Edition Vuibert, Paris (1981).