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LINEAR SPACES OF MATRICES, SYMMETRIC MATRICES OR HERMITIAN MATRICES WITH A FIXED RANK OVER A FINITE FIELD

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Abstract: Here we raise several questions concerning linear spaces of matrices with fixed rank over $\mathbb{F}_q$.

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Here we raise the following questions.

Question 1. Fix integers $m \geq n \geq k > 0$, $e \geq 1$, and a prime power q. What is the maximal integer $a(q,m,n,k)$, (resp. $b(q,m,n,k)$) such that there is an $a(q,m,n,k)$-dimensional (resp. $d(q,m,n,k)$-dimensional $\mathbb{F}_q$-vector space of $n \times m$ matrices over $\mathbb{F}_q$ such that each non-zero element of it has rank k (resp. $\leq k$)? What is the maximal integer $a(q,m,n,k,e)$, (resp. $b(q,m,n,k)$) such that there is an $a(q,m,n,k,e)$-dimensional (resp. $b(q,m,n,k,e)$-dimensional) $\mathbb{F}_{q^e}$-vector space of $n \times m$ matrices with a basis defined over $\mathbb{F}_q$ such that each non-zero element of it has rank k (resp. $\leq k$)? Say something about the linear subspaces with maximal dimension.

Question 2. Fix integers $n \geq k > 0$, $e \geq 1$, and a prime power q. What is the maximal integer $c(q,n,k)$, (resp. $d(q,n,k)$) such that there is a $c(q,n,k)$-dimensional (resp. $d(c,q,n,k)$-dimensional) $\mathbb{F}_q$-vector space of symmetric $n \times n$

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matrices over $\mathbb{F}_q$ such that each non-zero element of it has rank k (resp. $\le k$)? What is the maximal integer $c(q,n,k,e)$, (resp. $d(q,m,n,k)$) such that there is an $a(q,m,m,k,e)$-dimensional $\mathbb{F}_{q^e}$-vector space of $n \times m$ matrices with a basis defined over $\mathbb{F}_q$ such that each non-zero element of it has rank k (resp. $\le k$)? Say something about the linear subspaces with maximal dimension.

Question 3. Fix integers $n \ge k > 0$ and a prime power q. What is the maximal integer $e(q,n,k)$ (resp. $f(q,n,k)$) such that there is an $e(q,n,k)$-dimensional (resp. $f(q,n,k)$-dimensional) $\mathbb{F}_q$-linear spaces of Hermitian matrices over $\mathbb{F}_{q^2}$ such that each non-zero element of it has rank k (resp. $\le k$)?

Question 4. Fix integers $m \ge n \ge k\ t > 0$ and a prime power q such that $0 < t < a(q,m,n,k)$. Show the existence of a t-dimensional $\mathbb{F}_q$-vector space of $n \times m$ matrices over $\mathbb{F}_q$ such that each non-zero element of it has rank k, but that it is not contained in a bigger linear space with the same property. Hopefully, prove the same guess for the other integers introduced in Questions 1, 2 and 3.

For backgroud on Hermitian matrices over a finite field, see [6], Chapter 23; however, here we use the notation $\mathbb{F}_q$ (resp. $\mathbb{F}_{q^2}$) instead of $\mathbb{F}_{\sqrt{q}}$) resp. $\mathbb{F}_q$. For similar problems for the real, complex and quaternionic division rings, see [1], [2], [3], [4], [5] and [7].

Question 5. Fix as the base field F either the real or the complex or the quaternionic division ring. Take the set-up of Questions 1 or 2 for $e = 1$. Is there a maximal dimension linear subspace defined over the rational field?

Remark 1. Fix as the base field F either the real or the complex or the quaternionic division ring. Take the case $m = n$ and $k = n - 1$. The existence parts of the proofs in [1], [2] and [3] use an induction on n in which all the 3 division rings play simultaneously. In this way we get at least are not terms of n an explicit upper bound for the degree of the extension L of $\mathbb{Q}$ on which a maximal dimension linear subspace may be defined over L.

Remark 2. Take the case $k = n$. Let $A, A_1, \dots, A_t$ be $n \times m$ matrices over $\mathbb{F}_q$. $\mathrm{rank}(A) < n$ if and only if $m - n + 1$ $n \times n$ submatrices of A have zero-determinant. Hence the set $B(A_1, \dots, A_t =$ of all $(x_1, \dots, x_t) \in {\mathbb{F}_{q^e}}^{\oplus t}$ such nthat $\mathrm{rank}(x_1A_1 + \cdots + x_tA_t) < n$ is defined by $m - n + 1$ homogeneous degree n equations with coefficients in $\mathbb{F}_q$. By Chevalley-Warning Theorem for systems of homogeneous equations we get $a(q,m,n,n) \le n(m-n+1)$. The same theorem gives an upper bound for the integer $a(q,m,n,k)$ for any $k < n$. However, these upper bounds do not use the structure of the determinantal equations and hence they probably are very bad.

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