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ON ℓ-KÖTHE SPACES

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Abstract: Let ℓ be a Banach sequence space with a monotone norm $\|\cdot\|_\ell$, in which the canonical system (e_i) is a normalized symmetric basis. We consider this monotone norm and considered ℓ-Köthe spaces which is generalization of usual Köthe spaces. In this study, we generalized some results in [6] and understood that they are valid also for ℓ-Köthe spaces.

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1. Preliminaries

Let ℓ be a Banach sequence space in which $\{e_i = (\delta_{i,j})_{j\in\mathbb{N}} : i \in \mathbb{N}\}$ forms an unconditional basis. The norm $\|\cdot\|_\ell$ is called *monotone* (see [1]) if $\|x\|_\ell \le \|y\|_\ell$ whenever $x = (\xi_i)$, $y = (\eta_i)$, $|\xi_i| \le |\eta_i|$, $i \in \mathbb{N}$. By Λ we denote the set of all such spaces ℓ with monotone norm.

A matrix $A := (a_{i,n})_{i,n\in\mathbb{N}}$ of real numbers is called a *Köthe matrix* if $0 \le a_{i,n} \le a_{i,n+1}$ for each $i, n \in \mathbb{N}$; and for each $i \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that $a_{i,n} > 0$.

Definition 1. Let $\ell \in \Lambda$. The *ℓ-Köthe space* $K^\ell(A)$, defined by the Köthe matrix $A = (a_{i,n})_{i,n\in\mathbb{N}}$, is a Fréchet space of number sequences $\xi = (\xi_i)$ such that $(\xi_i a_{i,n}) \in \ell$, for each n, with the topology generated by the system of seminorms $\{|(\xi_i)|_n := \|(\xi_i a_{i,n})\| : n \in \mathbb{N}\}$.

Note that $|(e_i)|_n = \|(e_i a_{i,n})\| = a_{i,n}$. Hereafter the notation $e = (e_i)_{i\in\mathbb{N}}$, $e_i := (\delta_{i,k})_{k\in\mathbb{N}}$, will be always used for the canonical basis of $K^\ell(A)$ regardless of a matrix A.

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When ℓ is an l_p, we obtain the usual Köthe space

$$K^{l_p}(A) = \{(\xi) = (\xi_i) : |(\xi_i)|_n = \left(\sum_{i=1}^{\infty} |\xi_i|^p (a_{i,n})^p\right)^{\frac{1}{p}} < +\infty, \forall n \in \mathbb{N}\}.$$

In some sources, usual Köthe spaces are also denoted by $\lambda^p(A)$.

Due to [2], it is known that every Fréchet space with an absolute basis is isomorphic to some ℓ_1-Köthe space.

Set $\omega_+ := \{a = (a_i)_{i\in\mathbb{N}} : a_i \geq 1, \forall i\}$. For $a \in \mathcal{P}$ we consider the weighted ℓ-space as $\ell(a) := \{x = (\xi_i) : \|x\|_{\ell(a)} := \|(\xi_i\, a_i)\|_\ell < \infty\}$.

For a given sequences $a = (a_i)_{i\in\mathbb{N}} \in \omega_+$ and $\lambda_n \to \alpha$, $-\infty < \alpha \leq \infty$, we call the ℓ-Köthe space $E^\ell_\alpha(a) := K^\ell(\exp(\lambda_n a_i))$, *$\ell$-power series space of finite* (respectively, *infinite*) *type* if $\alpha < \infty$ (respectively, $\alpha = \infty$).

For any set S, we denote by $|S|$ the number of elements in S if it is finite and the symbol ∞ if S is infinite.

Let $X = K^\ell(A)$ and $\tilde{X} = K^\ell(\tilde{A})$ be ℓ-Köthe spaces with the canonical bases (e_i). We say that X is quasidiagonally isomorphic to $\tilde{X}$ and write $X \overset{qd}{\simeq} \tilde{X}$ if there exists $T : X \to \tilde{X}$ such that

$$Te_i := t_i e_{\varphi(i)}, \quad i \in \mathbb{N}, \tag{1}$$

is an isomorphism, where t_i is a sequence of numbers and $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection. Also, we denote by $X \overset{qd}{\hookrightarrow} \tilde{X}$, a quasidiagonal isomorphic imbedding, for which φ in (1) is an injection.

2. Main Results

The following statement is proved in [6] (see also [4]), for Köthe spaces. We generalized to ℓ-Köthe spaces.

Theorem 2. *Let X and $\tilde{X}$ be ℓ-Köthe spaces with $X \overset{qd}{\hookrightarrow} \tilde{X}$ and $\tilde{X} \overset{qd}{\hookrightarrow} X$ then $X \overset{qd}{\simeq} \tilde{X}$.*

Proof. Let $(e_i)_{i\in\mathbb{N}}$ and $(e_j)_{j\in\mathbb{N}}$ be bases of X and $\tilde{X}$, respectively. Let the quasidiagonal embedding $X \overset{qd}{\hookrightarrow} \tilde{X}$ and $\tilde{X} \overset{qd}{\hookrightarrow} X$ be defined respectively by (r_i), $\varphi : \mathbb{N} \to \mathbb{N}$, and (t_j), $\psi : \mathbb{N} \to \mathbb{N}$. By Cantor-Bernstein Theorem, there exist complementary subsets $I_1, I_2 \subset \mathbb{N}$ and $J_1, J_2 \subset \mathbb{N}$ such that $\varphi(I_1) = J_1$ and $\psi(J_2) = I_2$. Since, any part of an unconditional basis is a basis in its closed linear span, and any permutation of an unconditional basis is also basis, then $(e_{\varphi(i)})_{i\in I_1} \cup (e_{\psi^{-1}(i)})_{i\in I_2}$ is a basis in $\tilde{X}$. We define the quasidiagonal isomorphism T between X and $\tilde{X}$ as

$$Te_i = \begin{cases} r_i e_{\varphi(i)} & \text{if} \quad i \in I_1, \\ t^{-1}_{\psi^{-1}(i)} e_{\psi^{-1}(i)} & \text{if} \quad i \in I_2. \end{cases} \qquad \square$$

Let $a = (a_n)$ where $a_n \geq 1$. In [3], [4] Mitiagin investigated isomorphism of some non-Montel power series spaces by using the following counting function:

$$\mu_a(t,\tau) := |\{n \in \mathbb{N} : \tau < a_n \leq t\}|, \qquad 0 < \tau < t < \infty \,.$$

We use the notation $\mu_a \approx \mu_{\tilde{a}}$ if both $\mu_a(t,\tau) \leq \mu_{\tilde{a}}(\Delta t, \frac{\tau}{\Delta})$ and $\mu_{\tilde{a}}(t,\tau) \leq \mu_a(\Delta t, \frac{\tau}{\Delta})$ hold for some constant $\Delta > 0$.

The following two propositions are proved in the survey [6].

Proposition 3. *Let the number sequences $a = (a_i), b = (b_j)$ be such that $a_i \geq 1, b_j \geq 1$, $\lim_{i\to\infty} a_i = \infty, \lim_{j\to\infty} b_j = \infty$, and satisfy the following condition:*

$$\mu_a(t,\tau) \leq \mu_b(\Delta t, \frac{\tau}{\Delta}) \,, \quad 1 \leq \tau \leq t < \infty \,, \tag{2}$$

with some constant $\Delta > 1$. Then there exists an injection $\sigma : \mathbb{N} \to \mathbb{N}$ such that the inequalities

$$\frac{1}{\Delta} a_i \leq b_{\sigma(i)} \leq \Delta a_i \,, \quad \forall i \in \mathbb{N} \,, \tag{3}$$

hold.

Lemma 4. *Let for arbitrary sequences $a = (a_i), b = (b_j)$, $a_i \geq 1, b_j \geq 1$, the condition (2) hold. Then there exists an injection $\varphi : \mathbb{N} \to \mathbb{N}$ such that the inequalities*

$$\frac{1}{\Delta^2} a_i \leq b_{\varphi(i)} \leq \Delta^2 a_i \,, \quad i \in \mathbb{N}, \tag{4}$$

hold.

Theorem 5. *If $a = (a_k)$ and $b = (b_k)$ are sequences of positive numbers satisfying (2), then $E^\ell_\nu(a)$ can be quasidiagonally imbedded into $E^\ell_\nu(b)$, where $\nu = 0$ or ∞.*

Proof. Because of the similarity, we restrict ourselves to the case $\nu = \infty$. By Lemma 4, there exists an injection $\sigma : \mathbb{N} \to \mathbb{N}$ satisfying

$$\frac{1}{\Delta^2} a_i < b_{\sigma(i)} \leq \Delta^2 a_i, \quad \forall i. \tag{5}$$

Define an operator $T : E^\ell_\infty(a) \to E^\ell_\infty(b)$ by $e_i \mapsto e_{\sigma(i)}$.

By (5), we obtain that $\exp\big(p\frac{1}{\Delta^2}a_i\big) \le \exp\big(pb_{\sigma(i)}\big) \le \exp\big(\lambda\Delta^2 a_i\big)$. By monotonicity of the norm, we get that

$$\|\big(\exp\big(p\frac{1}{\Delta^2}a_i\big)\big)\xi_i\| \le \|\big(\exp\big(pb_{\sigma(i)}\big)\big)x_{\sigma(i)}\| \le \|\big(\exp\big(\lambda\Delta^2 a_i\big)\big)\xi_i\|\,.$$

From which continuity of T and $T^{-1}|_{R(T)}$ follows. □

Theorem 6. *If a sequence $a = (a_n)$ of positive numbers is bounded, then $E^{\ell}_{\nu}(a) \overset{qd}{\cong} \ell$, where $\nu = 0$ or ∞.*

Proof. It is sufficient to show that identity operator from ℓ to $E^{\ell}_0(a)$ is quasidiagonal isomorphism. Let I be identity operator from $\ell = \{x = (\xi_n) : \|(\xi_n)\|_{\ell} < \infty\}$ to $E^{\ell}_0(a) = \{x = (\xi_n) : \|(\xi_n \exp(\frac{-a_n}{p}))\|_{\ell} < \infty\}$.

Let $x = (\xi_n) \in E^{\ell}_0(a)$. Since $a = (a_n)$ is bounded, then there exists $C_1, C_2 > 0$ such that $C_1 \le (\exp(\frac{-a_n}{p})) \le C_2$. By monotonicity of the norm, we get $C_1\|(\xi_n)\|_{\ell} \le \|(\xi_n \exp(\frac{-a_n}{p}))\|_{\ell} \le C_2\|(\xi_n)\|_{\ell}$ from which continuity of I and I^{-1} follow. I is quasidiagonal, because $\sigma(i) = i$ and $t_i = 1$ for each $i \in I$ in (1). □

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