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FIXED POINT THEOREMS IN METRIC SPACES

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Abstract: In this paper several fixed point theorems for five kinds of mappings in complete metric spaces and metric spaces are proved.

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1. Introduction

Throughout this paper N denotes the set of all positive integers. Let T be a mapping from a metric space (X, d) into itself, $r \in [0, 1)$ be a constant and

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$x \in X$. The set $O_T(x) = \{T^n x : n \geq 0\}$ is called the *orbit* of T at x. A mapping T is said to be *orbitally continuous* on X if $\lim_{k\to\infty} T^{n_k}x = u$ implies that $\lim_{k\to\infty} TT^{n_k}x = Tu$ for any $x \in X$.

Bianchini [1] proved a fixed point theorem for the following contractive mapping:

$$d(Tx, Ty) \leq r \max\{d(x, Tx), d(y, Ty)\} \tag{1.1}$$

for all $x, y \in X$. Rhoades [2] studied the existence of fixed points for various contractive mappings in complete metric spaces and Banach spaces.

The purpose of this paper is to establish the existence of fixed points for the following mappings:

$$d(Tx, Ty) + d(Ty, Tz) \leq r[d(x, y) + d(y, z)] \tag{1.2}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$;

$$d(Tx, Ty)d(Ty, Tz) \leq rd(x, y)d(y, z) \tag{1.3}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$;

$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) \leq r[d(x, y) + d(y, z) + d(z, x)] \tag{1.4}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$,

$$\begin{aligned} &\max\{d(Tx, Ty), d(Ty, Tz)\} \\ &\leq r \max\{d(x, y), d(y, z), d(x, Tx), d(y, Ty), d(z, Tz), \\ &\qquad \min\{d(x, Ty), d(y, Tx)\}, \min\{d(y, Tz), d(z, Ty)\}\}, \end{aligned} \tag{1.5}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$;

$$\begin{aligned} &\max\{d(Tx, Ty), d(Ty, Tz), d(Tz, Tx)\} \\ &\leq r \max\{d(x, y), d(y, z), d(z, x), d(x, Tx), d(y, Ty), \\ &\qquad d(z, Tz), d(x, Ty), d(y, Tx), d(y, Tz), d(z, Ty)\}, \end{aligned} \tag{1.6}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$;

$$d(Tx, Ty) + d(Ty, Tz) < d(x, y) + d(y, z) \tag{1.7}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$,

$$d(Tx, Ty)d(Ty, Tz) < d(x, y)d(y, z) \tag{1.8}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$,

$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) < d(x, y) + d(y, z) + d(z, x) \tag{1.9}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$.

Remark 1.1. It is easy to show that (1.1) $\Rightarrow$ (1.5), (1.1) $\Rightarrow$ (1.6), (1.2) $\Rightarrow$ (1.7), (1.3) $\Rightarrow$ (1.8), (1.4) $\Rightarrow$ (1.9). But the reversions do not hold. Example 2.1 in this paper illustrates that (1.5) $\Rightarrow$ (1.1) and (1.6) $\Rightarrow$ (1.1) are not available. Now, we give an example to show that (1.7) $\Rightarrow$ (1.2), (1.8) $\Rightarrow$ (1.3), (1.9) $\Rightarrow$ (1.4) are not available.

Example 1.1. Let $X = [0, \infty)$ with the usual metric, $T : X \to X$ be a mapping defined by $Tx = \frac{x^2}{x+1}$ for all $x \in [0, \infty)$. For all $x, y, z \in X$ with $x \neq y \neq z \neq x$, we have

$$d(Tx, Ty) + d(Ty, Tz) = \frac{xy + x + y}{xy + x + y + 1}|x - y| + \frac{yz + y + z}{yz + y + z + 1}|y - z| < |x - y| + |y - z| = d(x, y) + d(y, z),$$

$$d(Tx, Ty)d(Ty, Tz) = \frac{xy + x + y}{xy + x + y + 1}|x - y|\frac{yz + y + z}{yz + y + z + 1}|y - z| < |x - y||y - z| = d(x, y)d(y, z),$$

and

$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) = \frac{xy + x + y}{xy + x + y + 1}|x - y| + \frac{yz + y + z}{yz + y + z + 1}|y - z| + \frac{zx + z + x}{zx + z + x + 1}|z-x| < |x-y|+|y-z|+|z-x| = d(x, y)+d(y, z)+d(z, x),$$

which means that T satisfies (1.7)-(1.9).

For any $r \in [0, 1)$, there exists some $n \in N$ such that

$$\frac{n^2 + 3n + 1}{n^2 + 3n + 2} + \frac{n^2 + 5n + 5}{n^2 + 5n + 6} > 2r, \quad \frac{n^2 + 3n + 1}{n^2 + 3n + 2} \cdot \frac{n^2 + 5n + 5}{n^2 + 5n + 6} > r,$$

$$\frac{n^2 + 3n + 1}{n^2 + 3n + 2} + \frac{n^2 + 5n + 5}{n^2 + 5n + 6} + \frac{2n^2 + 8n + 4}{n^2 + 4n + 3} > 4r.$$

Choose $x = n, y = n + 1, z = n + 2$. From the above inequalities, we get that

$$d(Tx, Ty) + d(Ty, Tz) = \frac{n^2 + 3n + 1}{n^2 + 3n + 2} + \frac{n^2 + 5n + 5}{n^2 + 5n + 6} > 2r = r[d(x, y) + d(y, z)],$$

$$d(Tx, Ty)d(Ty, Tz) = \frac{n^2 + 3n + 1}{n^2 + 3n + 2} \cdot \frac{n^2 + 5n + 5}{n^2 + 5n + 6} > r = rd(x, y)d(y, z),$$

and

$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) = \frac{n^2 + 3n + 1}{n^2 + 3n + 2} + \frac{n^2 + 5n + 5}{n^2 + 5n + 6} + \frac{2n^2 + 8n + 4}{n^2 + 4n + 3} > 4r = r[d(x, y) + d(y, z) + d(z, x)],$$

which imply that T does not satisfy (1.2)-(1.4).

2. Main Results

Theorem 2.1. *Let T be a mapping from a complete metric space (X, d) into itself and satisfy* (1.5). *Then:*

(a) *T has no k-periodic points in X for $k \geq 3$;*

(b) *T has a fixed point in X if T has an orbit without 2-periodic points;*

(c) *T has at most two fixed points in X.*

Proof. Suppose that there exists some $a_0 \in X$ is a k-periodic point of T for $k \geq 3$. Set $a_n = T^n a_0$, $d_n = d(a_n, a_{n+1})$ for all $0 \leq n \leq k$. According to the definition of k-periodic point and (1.5), we get that

$$\begin{aligned}\max\{d_0, d_1\} &= \max\{d_k, d_{k+1}\} = \max\{d(Ta_{k-1}, Ta_k), d(Ta_k, Ta_{k+1})\} \\ &\leq r \max\{d(a_{k-1}, a_k), d(a_k, a_{k+1}), d(a_{k-1}, Ta_{k-1}), d(a_k, Ta_k), \\ &\quad d(a_{k+1}, Ta_{k+1}), \min\{d(a_{k-1}, Ta_k), 0\}, \min\{d(a_k, Ta_{k+1}), 0\}\} \\ &= r \max\{d_{k-1}, d_k, d_{k+1}\} = rd_{k-1} < d_0,\end{aligned}$$

which is a contradiction. Then we exclude the presence of k-periodic point for $k \geq 3$ in X.

Assume that there exists a point $x_0 \in X$ such that T has no 2-periodic points in $O_T(x_0)$. Let $x_n = T^n x_0$ and $d_n = d(x_n, x_{n+1})$ for any $n \geq 0$. We consider the following cases:

Case 1. There exists some $n \geq 0$ with $x_n = x_{n+1}$. Then x_n is a fixed point of T.

Case 2. For any $n \geq 0$, $x_n \neq x_{n+1}$. From (a), we easily conclude that $x_m \neq x_n$ for all $m > n \geq 0$. In view of (1.5) we have

$$\begin{aligned}\max\{d_n, d_{n+1}\} &= \max\{d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1})\} \\ &\leq r \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\ &\quad d(x_{n+1}, Tx_{n+1}), \min\{d(x_{n-1}, Tx_n), 0\}, \min\{d(x_n, Tx_{n+1}), 0\}\} \\ &\leq r \max\{d_{n-1}, d_n, d_{n+1}\} = rd_{n-1} \leq r^n d_0. \quad (2.1)\end{aligned}$$

We now assert that $\{x_n\}_{n \geq 0}$ is a Cauchy sequence. For each $n, p \in N$, using triangular inequality and (2.1), we get that

$$d(x_n, x_{n+p}) \leq \sum_{i=n}^{n+p-1} d_i \leq d_0 \sum_{i=n}^{n+p-1} r^i \leq \frac{r^n}{1-r} d_0 \to 0 \quad \text{as } n \to \infty.$$

Since (X, d) is a complete metric space, there exists a point $a \in X$ such that $\lim_{n\to\infty} x_n = a$. It is easy to see that there exists an integer $k \in N$ with $x_n \neq a$

for all $n \geq k$. From (1.5) we infer that

$$\begin{aligned} &\max\{d(x_{n+1}, Ta), d(Ta, x_{n+2})\} \\ &\leq r\max\{d(x_n, a), d(a, x_{n+1}), d(x_n, x_{n+1}), d(a, Ta), d(x_{n+1}, x_{n+2}), \\ &\qquad \min\{d(x_n, Ta), d(a, x_{n+1})\}, \min\{d(a, x_{n+2}), d(x_{n+1}, Ta)\}\}. \end{aligned} \tag{2.2}$$

Letting $n \to \infty$ in (2.2), we get that

$$d(a, Ta) \leq rd(a, Ta). \tag{2.3}$$

Note that $r \in [0, 1)$. Therefore (2.3) implies that $a = Ta$.

Finally we prove that T has at most two distinct fixed points in X. Otherwise, T has three different fixed points a, b, c in X. In the light of (1.5), we have

$$\begin{aligned} &\max\{d(a, b), d(b, c)\} = \max\{d(Ta, Tb), d(Tb, Tc)\} \leq r\max\{d(a, b), d(b, c), \\ &d(a, Ta), d(b, Tb), d(c, Tc), \min\{d(a, Tb), d(b, Ta)\}, \min\{d(b, Tc), d(c, Tb)\}\} \\ &\leq r\max\{d(a, b), d(b, c)\} < \max\{d(a, b), d(b, c)\}, \end{aligned}$$

which is impossible. This completes the proof. □

Theorem 2.2. *Let T be a mapping from a complete metric space (X, d) into itself and satisfy (1.6). Then the conclusions of Theorem 2.1 hold.*

Proof. We first exclude the presence of k-periodic point for $k \geq 3$ in X. Suppose that $a_0 \in X$ is a k-periodic point of T for $k \geq 3$. Set $a_n = T^n a_0$ for $0 \leq n \leq k$. By the definition of k-periodic point for $k \geq 3$ and (1.6), we have

$$\begin{aligned} &\max\{d(a_0, a_1), d(a_1, a_2), d(a_2, a_0)\} \\ &= \max\{d(a_k, a_{k+1}), d(a_{k+1}, a_{k+2}), d(a_{k+2}, a_k)\} \\ &\leq r\max\{d(a_{k-1}, a_k), d(a_k, a_{k+1}), d(a_{k+1}, a_{k-1}), d(a_{k-1}, a_k), \\ &d(a_k, a_{k+1}), d(a_{k+1}, a_{k+2}), d(a_{k-1}, a_{k+1}), d(a_k, a_k), d(a_k, a_{k+2}), d(a_{k+1}, a_{k+1})\} \\ &= r\max\{d(a_{k-1}, a_k), d(a_k, a_{k+1}), d(a_{k+1}, a_{k-1}), d(a_k, a_{k+2}), d(a_{k+1}, a_{k+2})\} \\ &\leq r\max\{d(a_{k-1}, a_k), d(a_k, a_{k+1}), d(a_{k+1}, a_{k-1})\} \leq \\ &r^k\max\{d(a_0, a_1), d(a_1, a_2), d(a_2, a_0)\} < \max\{d(a_0, a_1), d(a_1, a_2), d(a_2, a_0)\}, \end{aligned}$$

which is a contradiction.

Suppose that there exists $x_0 \in X$ such that T has no 2-periodic points in $O_T(x_0)$. Let $x_n = T^n x_0$ for any $n \geq 0$. We consider the following cases:

Case 1. There exists some $n \geq 0$ with $x_n = x_{n+1}$. Then x_n is a fixed point of T.

Case 2. For any $n \geq 0$, $x_n \neq x_{n+1}$. From (a), we easily conclude that $x_m \neq x_n$ for all $m > n \geq 0$. Using (1.6), we have

$$\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_{n+2}, x_n)\}$$

$$\begin{aligned}&\le r\max\{d(x_{n-1},x_n),d(x_n,x_{n+1}),d(x_{n+1},x_{n-1}),d(x_{n-1},x_n),d(x_n,x_{n+1}),\\&\quad d(x_{n+1},x_{n+2}),d(x_{n-1},x_{n+1}),d(x_n,x_n),d(x_n,x_{n+2}),d(x_{n+1},x_{n+1})\}\\&= r\max\{d(x_{n-1},x_n),d(x_n,x_{n+1}),d(x_{n+1},x_{n-1}),d(x_{n+1},x_{n+2}),d(x_n,x_{n+2})\}\\&\le r\max\{d(x_{n-1},x_n),d(x_n,x_{n+1}),d(x_{n+1},x_{n-1})\}\\&\le r^n\max\{d(x_0,x_1),d(x_1,x_2),d(x_2,x_0)\}. \qquad (2.4)\end{aligned}$$

For each $n \in N$ and $p \in N$, using triangular inequality and (2.4), we have

$$\begin{aligned}d(x_n,x_{n+p}) &\le \sum_{i=n}^{n+p-1} d(x_i,x_{i+1}) \le \max\{d(x_0,x_1),d(x_1,x_2),d(x_2,x_0)\}\\&\quad\times\sum_{i=n}^{n+p-1} r^i \le \frac{r^n}{1-r}\max\{d(x_0,x_1),d(x_1,x_2),d(x_2,x_0)\}\to 0\end{aligned}$$

as $n \to \infty$. That is, $\{x_n\}_{n\ge 0}$ is a Cauchy sequence. It follows from the completeness of (X,d) that $\lim_{n\to\infty} x_n = a$ for some $a \in X$. Clearly there exists an integer $k \in N$ with $x_n \ne a$ for all $n \ge k$. From (1.6) we conclude that

$$\begin{aligned}&\max\{d(x_{n+1},Ta),d(Ta,x_{n+2}),d(x_{n+2},x_{n+1})\}\\&\quad\le r\max\{d(x_n,a),d(a,x_{n+1}),d(x_{n+1},x_n),d(x_n,x_{n+1}),d(a,Ta),\\&\qquad d(x_{n+1},x_{n+2}),d(x_n,Ta),d(a,x_{n+1}),d(a,x_{n+2}),d(x_{n+1},Ta)\}, \qquad (2.5)\end{aligned}$$

letting $n \to \infty$ in (2.5), we obtain that

$$d(a,Ta) \le r d(a,Ta),$$

which implies that $a = Ta$.

At last, we conclude that T has at most two distinct fixed points in X. Otherwise, suppose that T has three distinct fixed points a,b,c in X. By (1.6) we have

$$\begin{aligned}&\max\{d(a,b),d(b,c),d(c,a)\} = \max\{d(Ta,Tb),d(Tb,Tc),d(Tc,Ta)\}\\&\quad\le r\max\{d(a,b),d(b,c),d(c,a),d(a,Ta),d(b,Tb),d(c,Tc),\\&\qquad d(a,Tb),d(b,Ta),d(b,Tc),d(c,Tb)\}\\&\quad\le r\max\{d(a,b),d(b,c),d(c,a)\} < \max\{d(a,b),d(b,c),d(c,a)\},\end{aligned}$$

which is a contradiction. This completes the proof. □

Example 2.1. Let $X = \{1,2,3\}$, $T : X \to X$ be a mapping defined by $T1 = 1$, $T2 = 2$, $T3 = 2$, $d : X \times X \to [0,\infty)$ be a function defined by $d(1,2) = 3$, $d(1,3) = 4$, $d(2,3) = 5$, $d(x,x) = 0$ and $d(x,y) = d(y,x)$ for all $x,y \in X$. Put $r = 4/5$. It is easy to show that the conditions of Theorems 2.1 and 2.2 are satisfied, and T has two fixed points 1 and 2 in X. But T does not satisfy (1.1) since

$$d(T1, T2) = 3 > 0 = r \max\{d(1, T1), d(2, T2)\}$$

for any $r \in [0, 1)$.

Theorem 2.3. *Let (X, d) be a metric space and let $T : X \to X$ be orbitally continuous and satisfy (1.7). Then:*

(a) *T has a fixed point in X provided that there exist $x_0, w \in X$ such that w is a cluster point of $O_T(x_0)$ and w is not a 2-periodic point of T;*

(b) *If T has 2-periodic points in X, then they are exactly two;*

(c) *T has at most two distinct fixed points in X.*

Proof. Suppose that x_0 and w are in X such that w is a cluster point of $O_T(x_0)$ and w is not a 2-periodic point of T. Set $x_n = T^n x_0$ for all $n \geq 0$. Since w is a cluster point of $O_T(x_0)$, it is easy to show that $x_n \neq x_{n+p}$ for all $n \geq 0$ and $p \in N$. Let $d_n = d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$ for all $n \geq 0$. By (1.7), we get that

$$d_n = d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) < d(x_{n-1}, x_n) + d(x_n, x_{n+1}) = d_{n-1},$$

which means that $\{d_n\}_{n \geq 0}$ is a decreasing sequence. Hence $\lim_{n\to\infty} d_n = r$ for some $r \in [0, \infty)$. Since w is a cluster point of $O_T(x_0)$, there exists a subsequence $\{x_{n_i}\}_{i \geq 0}$ of $\{x_n\}_{n \geq 0}$ with $\lim_{i\to\infty} x_{n_i} = w$. By the orbital continuity of T, we infer that

$$\begin{aligned} r &= \lim_{i\to\infty} d_{n_i} = \lim_{i\to\infty} [d(x_{n_i}, x_{n_i+1}) + d(x_{n_i+1}, x_{n_i+2})] \\ &= d(w, Tw) + d(Tw, T^2 w), \\ r &= \lim_{i\to\infty} d_{n_i+1} = \lim_{i\to\infty} [d(x_{n_i+1}, x_{n_i+2}) + d(x_{n_i+2}, x_{n_i+3})] \\ &= d(Tw, T^2 w) + d(T^2 w, T^3 w). \end{aligned}$$

Note that $T^2 w \neq w$. We claim that w or Tw is a fixed point of T. Otherwise $w \neq Tw$ and $Tw \neq T^2 w$. According to (1.7), we have

$$r = d(Tw, T^2 w) + d(T^2 w, T^3 w) < d(w, Tw) + d(Tw, T^2 w) = r,$$

which is a contradiction. Therefore $w = Tw$ or $Tw = T^2 w$.

Suppose that $b \in X$ is a 2-periodic point of T. Then Tb is also a 2-periodic point of T. We show that b and Tb are the only two 2-periodic points of T. Suppose that there exists a point $c \in X$ which is different from b and Tb. By (1.7) we obtain that

$$\begin{aligned} d(Tb, Tc) + d(Tc, b) &= d(Tb, Tc) + d(Tc, T^2 b) < d(b, c) + d(c, Tb) \\ &= d(T^2 b, T^2 c) + d(T^2 c, Tb) < d(Tb, Tc) + d(Tc, b), \end{aligned}$$

which is impossible.

We finally assert that T has at most two distinct fixed points. Otherwise

T has three different fixed points $a, b, c \in X$. In the light of (1.7), we conclude that

$$d(a,b) + d(b,c) = d(Ta,Tb) + d(Tb,Tc) < d(a,b) + d(b,c),$$

which is a contradiction. This completes the proof. □

The proofs of the following theorems go in a similar fashion as that of Theorem 2.3, so we omit the proofs.

Theorem 2.4. *Let* (X,d) *be a metric space and let* $T : X \to X$ *be orbitally continuous and satisfy* (1.8). *Then the conclusions of Theorem 2.3 hold.*

Theorem 2.5. *Let* (X,d) *be a metric space and let* $T : X \to X$ *be orbitally continuous and satisfy* (1.9). *Then the conclusions of Theorem 2.3 hold.*

As consequence of Theorems 2.3-2.5, we have:

Corollary 2.1. *Let* (X,d) *be a metric space and let* $T : X \to X$ *be orbitally continuous and satisfy* (1.2). *Then the conclusions of Theorem 2.3 hold.*

Corollary 2.2. *Let* (X,d) *be a metric space and let* $T : X \to X$ *be orbitally continuous and satisfy* (1.3). *Then the conclusions of Theorem 2.3 hold.*

Corollary 2.3. *Let* (X,d) *be a metric space and let* $T : X \to X$ *be orbitally continuous and satisfy* (1.4). *Then the conclusions of Theorem 2.3 hold.*

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