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EXACT SEQUENCES OF TORSION FREE SHEAVES ON A NODAL CURVE

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Abstract: Here we use a paper of U. Bhosle to study the existence of short exact sequences of stable torsion free sheaves on a projective curve with only ordinary nodes as singularities.

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Let Y be an integral projective curve we only ordinary nodes as singularities. Set $g := p_a(Y)$. We always assume $g \geq 3$. Here we use [1] to study exact sequences of stable torsion free sheaves on Y. For any $P \in Y$ let m_P denote the maximal ideal of the local ring $\mathcal{O}_{Y,P}$. Fix a rank r torsion free sheaf F on Y. For any $P \in Y$ let F_P (resp. $F|\{P\}$) denote the germ of F at P. Since P is an ordinary node of Y Y, then the classification of all torsion free modules on an A_n-singularity (see [2]) gives the existence of a unique integer b such that $0 \leq b \leq r$ and $F_P \cong \mathcal{O}_{Y,P}^{\oplus(r-b)} \oplus m_P^{\oplus b}$. We will say that F as local type b at P. Notice that F has local type b at P (for a fixed rank r) if and only if $\dim(F|\{P\}) = r + b$. If every singular point of Y is either an ordinary node or an ordinary cusp and F_P has local type b_P at $P \in \mathrm{Sing}(Y)$, then we will say that the ordered set of integers $\{b_P\}_{P\in\mathrm{Sing}(Y)}$ is the type of Y.

We prove the following result.

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Theorem 1. *Assume that every singular point of Y is an ordinary node. Fix integers r, s, a, b, a_P, $P \in \mathrm{Sing}(Y)$, such that $r > 0$, $s > 0$, $0 \le a_P \le \min\{r, s\}$ for all $P \in \mathrm{Sing}(Y)$, $x := \sum_{P \in \mathrm{Sing}}(Y) a_P \le g - 2$ and $s(a + x) < r(b - x)$. Then there exists an exact sequence of stable torsion free sheaves on Y:*

$$0 \to A \to E \to B \to 0 \tag{1}$$

in which $\deg(A) = a$, $\mathrm{rank}(A) = r$, $\deg(B) = b$, $\mathrm{rank}(B) = s$, A and B have type $\{a_P\}_{P \in \mathrm{Sing}(Y)}$, and E is locally free.

Remark 1. Fix integers a, b, r, s such that $r > 0$, $b > 0$ and $a/r < b/s$. By [1], Part (1) of Proposition 3.7, there is an exact sequence of stable vector bundles on Y:

$$0 \to F \to E \to G \to 0 \tag{2}$$

such that $\mathrm{rank}(F) = r$, $\mathrm{rank}(G) = s$, $\deg(F) = a$ and $\deg(G) = 0$. When Y is smooth, this important result was proved in [3].

Remark 2. For all choices of integers $r, b_P, P \in \mathrm{Sing}(Y)$ such that $0 \le b_P \le r$ for all $P \in \mathrm{Sing}(Y)$ there are many torsion free sheaves G with rank r and local type $\{b_P\}_{P \in \mathrm{Sing}(Y)}$. Fix integers $r, b_P, a_P, P \in \mathrm{Sing}(Y)$ such that $0 \le b_P \le r$, $0 \le a_P \le r$ for all $P \in \mathrm{Sing}(Y)$ and $a_Q > 0$ for at least one $Q \in \mathrm{Sing}(Y)$. Fix any torsion free sheaf G with rank r and local type $\{b_P\}_{P \in \mathrm{Sing}(Y)}$. Set $A := \{P \in \mathrm{Sing}(Y) : a_P > 0\}$. Let T be the skyscraper sheaf of Y such that each connected component of T is supported by some $P \in A$, that component is killed by m_P and it has dimension a_P. Let $u : G \to T$ be a general surjection. Since A is finite and each connected component of T is supported by some $P \in A$, we see that the restriction map $\rho : H^0(Y, \mathrm{Hom}(G, T)) \to H^0(A, G|A)$ is bijective. Notice that $h^0(A, G|A) = \sum_{P \in A} a_P(r_P + b_P)$. Take a general $u \in H^0(Y, \mathrm{Hom}(G, T))$ and set $E := \mathrm{Ker}(u)$. Hence E is a rank r torsion free sheaf and $\deg(E) = \deg(G) - \sum_{P \in A} a_P$. The surjectivity of ρ gives that E has local type b_P at each $P \in \mathrm{Sing}(Y) \backslash A$ and local type $\max\{b_P - a_P, a_P - b_P\}$ at each $P \in A$.

Proposition 1. *Fix integers r, d, a_P, $P \in \mathrm{Sing}(Y)$, such that $0 \le a_P \le r$ for all P, and $0 < x := \sum_{P \in \mathrm{Sing}(Y)} a_P \le g - 2$. Let G be a general stable vector bundle on Y with rank r and degree d. Set $A := \{P \in \mathrm{Sing}(Y) : a_P > 0\}$. Let T be the skyscraper sheaf of Y such that each connected component of T is supported by some $P \in A$, that component is killed by m_P and it has dimension a_P. Let $u : G \to T$ be any surjection. Set $E := \mathrm{Ker}(u)$. Then E is stable and it has type $\{a_P\}_{P \in \mathrm{Sing}(Y)}$.*

Proof. It is easy to check that E has type $\{a_P\}_{P\in \mathrm{Sing}(Y)}$ and degree $d-x$ even if u is an arbitrary surjection. Assume that E is not stable and let F be a proper subsheaf of E with maximal slope. Set $k := \mathrm{rank}(F)$ and $c := \deg(F)$. Since E is assumed to be unstable, we have $c/k \ge (d-x)/r$. Let U be the saturation of F in G. We have $\mathrm{rank}(U) = k$ and $c \le \deg(U) \le c + x$. By [1], Th. 2, we have $(d - \deg(U))/(r-k) \ge \deg(U)/k + g - 1$. Hence $k(d-c) \ge (r-k)c + k(r-k)(g-1)$, i.e. $kd - cr \ge k(r-k)(g-1)$. Since $rc \ge kd - kx$, we get $x \ge (r-k)(g-1)$, contradiction. □

Proposition 2. *Fix integers r, d, a_P, $P \in \mathrm{Sing}(Y)$, such that $0 \le a_P \le r$ for all P, and $0 \le x := \sum_{P\in \mathrm{Sing}(Y)} a_P \le g-2$. Let G be a general stable vector bundle on Y with rank r and degree d. Set $A := \{P \in \mathrm{Sing}(Y) : a_P > 0\}$. Let T be the skyscraper sheaf of Y such that each connected component of T is supported by some $P \in A$, that component is killed by m_P and it has dimension a_P. The set* Ext *of all extensions of T by E is parametrized by a vector space of dimension rx. Take any $\beta \in$* Ext *such that the associated middle term sheaf E is torsion free. We may take as β a general element of* Ext*. Then E is stable and it has type $\{a_P\}_{P\in \mathrm{Sing}(Y)}$.*

Proof. It is easy to check that E has type $\{a_P\}_{P\in \mathrm{Sing}(Y)}$ and degree $d+x$. Assume that E is not stable and let F be a proper subsheaf of E with maximal slope. Set $k := \mathrm{rank}(F)$ and $c := \deg(F)$. Since E is assumed to be unstable, we have $cr \ge kd + kx$. See G as a subsheaf of E and set $F' := F \cap G$. Hence $\mathrm{rank}(F') = k$ and $c - x \deg(F') \le c$. By [1], Th. 2, we have $(d - \deg(F'))/(r-k) \ge \deg(F') + g - 1$ and hence $(d - c + x)/(r-k) \ge (c-x)/k + g - 1$, i.e. $kd + xr \ge rc + k(r-k)(g-1)$. Since $cr \ge kd + kx$, we get $kx \ge k(r-k)(g-1)$, contradiction. □

Proof of Theorem 1. The case $x = 0$ and $b/s \le a/r + g - 1$ is just [1]. Hence we may assume $x > 0$. Since $(a+x)/r < (b-x)/r$, the case $x = 0$ just done gives the existence of an exact sequence of stable vector bundles on Y:

$$0 \to E_1 \to E_2 \to E_3 \to 0 \tag{3}$$

such that $\deg(E_1) = a + x$, $\mathrm{rank}(E_1) = r$, $\deg(E_3) = b - x$, and $\mathrm{rank}(E_3) = s$. Furthermore, we may assume that E_1 and E_3 are general in their moduli space. Let T be the skyscraper sheaf on Y associated to the datum $\{a_P\}_{P\in \mathrm{Sing}(Y)}$ as in the statement of Propositions 1 and 2. Take a general surjection $u : E_1 \to T$ and set $F_1 := \mathrm{Ker}(u)$. F_1 is stable and has type $\{a_P\}_{P\in \mathrm{Sing}(Y)}$ (Proposition 1). Let F_3 be a torsion free sheaf obtained as an extension of T by F_3. F_3 is stable and has type $\{a_P\}_{P\in \mathrm{Sing}(Y)}$ (Proposition 2). Let F_2 be the general sheaf obtained as an extension of F_3 by F_2. The proof of [3], Lemmas 1.9 and 1.10, shows that E_2 is a specialization of a family of extensions of F_3 by F_1. Since local freeness and stability are open conditions, F_2 is stable and locally free. Set $A := F_1$, $E := F_2$ and $B := F_3$. □

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$.

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