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THE BEHAVIOUR OF SOLUTIONS OF NONHOMOGENEOUS THIRD ORDER DIFFERENTIAL EQUATIONS

Pakize Temtek
Department of Mathematics
Erciyes University
Kayseri, 38039, TURKEY
e-mail: temtek@erciyes.edu.tr

Abstract: In this paper, we consider the equation

$$y^{'''} + q(t)(y')^{\gamma} + p(t)h(y) = f(t)\,,$$

where p , q and f are real valued continuous functions on $[\,0,\infty)$ such that $p(t) \leq 0$, $q(t) \leq 0$,$f(t) \geq 0$,$\gamma > 0$ is ration of odd integers and h is continuous in $(-\infty,\infty)$ such that $h(y)y > 0$ for $y \neq 0$. We obtain sufficient conditions for solutions of the considered equation to be nonoscillatory.

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1. Introduction

This paper concerns with qualitative behaviour of nonoscillatory solutions of the third-order nonhomogeneous equation

$$y^{'''} + q(t)(y')^{\gamma} + p(t)h(y) = f(t)\,, \tag{1}$$

where p, q and f are real valued continuous functions on $[0,\infty)$ such that $p(t) \leq 0$, $q(t) \leq 0, f(t) \geq 0$ and $\gamma > 0$ is ration of odd integers and h is continuous of $y \in (-\infty,\infty)$ satisfying $h(y)y > 0$ for $y \neq 0$. We restrict our considerations to those real solutions of (1) which exist on the half line $[T,\infty)$, where $T \geq 0$ depends on the particular solution and are nontrivial in any neighbourhood of infinity. We classify solutions of (1) as follows: a solution $y(t)$ is said to be nonoscillatory if there exists a $t_1 \geq T$ such that $y(t) \neq 0$ for

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$t \geq t_1$; y(t) is said to be oscillatory if for any $t_1 \geq T$ there exist t_2 and t_3 satisfying $t_1 < t_2 < t_3$ such that $y(t_2) > 0$ and $y(t_3) < 0$; and it is said to be a z-type solution if it has arbitrarily large zeros but is ultimately non-negative or non-positive.

The oscillatory properties of solutions of

$$y''' + P(t)y'' + Q(t)y' + R(t)y = 0$$

were first studied by Birkhoff [2]. More recent work on this equation can be found in Hanan [4], Lazer [6], Heidel [5], Barret [1], and Erbe [3]. For more information, the reader is referred to the references in Erbe [3], Heidel [5] and Lazer [6]. Heidel [5] investigated the qualitative behaviour of nonoscillatory solutions of nonlinear homogeneous third order differential equation

$$y''' + q(t)y' + p(t)y^{\beta} = 0.$$

He considered two cases: (i) $p(t) \leq 0$, $q(t) \leq 0$; and (ii) $p(t) \geq 0$, $q(t) \geq 0$. N. Parhi [8] was concerned with the equation

$$(r(t)y'')' + q(t)(y')^{\gamma} + p(t)y^{\beta} = f(t)$$

by using the result of Heidel and [7].

2.

We consider equation (1). Let $y_1(t)$, $y_2(t)$ and $y_3(t)$ are solutions of (1) on $[t_0, \infty)$, $t_0 \geq 0$, respectively with initial conditions $y_1(t_0) = 0$, $y_1'(t_0) = 1$, $y_1''(t_0) = 0$; $y_2(t_0) = 1$, $y_2'(t_0) = 1$, $y_2''(t_0) = 0$ and $y_3(t_0) = 0$, $y_3'(t_0) = 0$, $y_3''(t_0) = 1$.

Theorem 1. *Let $q(t)$ be once continuously differentiable such that $q'(t) \geq 0$ and $\gamma = 1$ in (1). If $q(t_0) = 0$, $y_1(t)$ cannot meet $y_2(t)$ in $[t_0, \infty)$. If $q(t_0) = 0$, $y_1(t)$ and $y_2(t)$ cannot meet in the strip $[t_0, t_0 + \sqrt{\frac{2}{-q(t_0)}})$.*

Proof. Integrating $y_1''' + q(t)y_1'(t) + p(t)h(y_1(t)) = f(t)$ from t_0 to t, we get

$$y_1''(t) = -q(t)y_1(t) + \int_{t_0}^{t} q'(s)y_1(s)ds - \int_{t_0}^{t} p(s)h(y_1(s))ds + \int_{t_0}^{t} f(s)ds.$$

Further integration from t_0 to t yields

$$y_1(t) = (t - t_0) - \int_{t_0}^{t}(\int_{t_0}^{\theta} q(s)y_1(s)ds)d\theta + \int_{t_0}^{t}\int_{t_0}^{u}\int_{t_0}^{\theta} q'(s)y_1(s)dsd\theta du$$

$$- \int_{t_0}^{t}\int_{t_0}^{u}\int_{t_0}^{\theta} p(s)h(y_1(s))dsd\theta du + \int_{t_0}^{t}\int_{t_0}^{u}\int_{t_0}^{\theta} f(s)dsd\theta du.$$

Integrating $y_2^{'''}(t) + q(t)y_2^{'}(t) + p(t)h(y_2(t)) = f(t)$ from t_0 to t, we get

$$\begin{aligned} y_2(t) = 1 + (t - t_0) - \int_{t_0}^{t}\int_{t_0}^{\theta} q(s)y_2(s)dsd\theta + q(t_0)\frac{(t-t_0)^2}{2} \\ + \int_{t_0}^{t}\int_{t_0}^{u}\int_{t_0}^{\theta} q^{'}(s)y_2(s)dsd\theta du \\ - \int_{t_0}^{t}\int_{t_0}^{u}\int_{t_0}^{\theta} p(s)h(y_2(s))dsd\theta du + \int_{t_0}^{t}\int_{t_0}^{u}\int_{t_0}^{\theta} f(s)dsd\theta du. \end{aligned}$$

Suppose that $y_1(t)$ meets $y_2(t)$ first at $t = t_1 > t_0$. So $y_1(t) < y_2(t)$ for $t \in [t_0, t_1)$ and $y_1(t_1) = y_2(t_1)$. Consequently, we have

$$y_2(t_1) \geq 1 + q(t_0)\frac{(t_1 - t_0)^2}{2} + y_1(t_1).$$

If $q(t_0) = 0$, the above inequality leads to a contradiction $1 \leq 0$.If $q(t_0) = 0$, then $t_1 \geq t_0 + \sqrt{\frac{2}{-q(t_0)}}$.

Hence the theorem. □

Theorem 2. *Suppose that conditions of Theorem 1 are satisfied.Then $y_1(t)$ and $y_3(t)$ cannot meet in the strip $(t_0, t_0 + 2)$ and $y_2(t)$ and $y_3(t)$ cannot meet in the strip*

$$[t_0, t_0 + \frac{1 + \sqrt{3 - 2q(t_0)}}{1 - q(t_0)}).$$

The proof of this theorem is similar to that of Theorem 1.

In the following, we state some lemmas which will be used in the sequel.

Lemma 3. *Let $p \equiv 0$ in (1). If $y(t)$ is a solution of (1) on $[T, \infty), T \geq 0$, then there exists a $c \geq T$ such that $y^{'}(t) > 0$ or ≤ 0 for $t \geq c$.*

This lemma has been already proved in [8] by Parhi.

Lemma 4. *If $y(t)$ is a solution of (1) on $[T, \infty), T \geq 0$, such that it is ultimately positive, then there exists a $c \geq T$ such that $y^{'}(t) > 0$ or ≤ 0 for $t \geq c$.*

Proof. If possible, let $y^{'}(t)$ be oscillatory. Let a and b $(T \leq a < b)$ be consecutive zeros of $y^{'}(t)$ such that $y^{'}(t) > 0$ for $t \in (a, b), y^{''}(a) \geq 0$ and $y^{''}(b) \leq 0$. Integrating (1) from a to b, we get

$$0 > r(b)y^{''}(b) - r(a)y^{''}(a) = \int_{a}^{b} [-q(t)(y^{'}(t))^{\gamma} + f(t)]dt > 0,$$

a contradiction. Similarly, it can be shown that $y^{'}(t)$ cannot be of nonnegative z-type.

Hence the lemma. □

The following lemma is due to Heidel [5].

Lemma 5. *Let* $g : [t_0, \infty) \to R$, $t_0 \geq 0$. *Suppose that* $g(t) > 0$ *and that* $g^{'}(t)$, $g^{''}(t)$ *exist for* $t \geq t_0$. *Suppose also that if* $g^{'}(t) \geq 0$ *ultimately, then*

$$\lim_{t\to\infty} g(t) = A < \infty.$$

Then

$$\liminf_{t\to\infty} |t^{\alpha} g^{''}(t) - \alpha t^{\alpha-1} g^{'}(t)| = 0$$

for any α *such that* $0 \leq \alpha \leq 2$.

Theorem 6. *Consider (1) with* $\gamma = 1$. *For some fixed* α *with* $0 \leq \alpha \leq 2$, *suppose that* $-\infty < -M < t^2 q(t)$, $\int_0^{\infty} t^{\alpha} p(t) h(y(t)) dt = -\infty$ *and* $\int_0^{\infty} t^{\alpha} f(t) dt < \infty$. *If* $y(t)$ *is a nonoscillatory solution of (1) such that* $y(t)y^{'}(t) \leq 0$ *ultimately then*

$$\lim_{t\to\infty} y(t) = 0.$$

Proof. If $y(t)$ is ultimately positive, we proceed as in Theorem 2.3 of Heidel [5] to obtain

$$\lim_{t\to\infty} y(t) = 0.$$

Suppose that $y(t)$ is ultimately negative. So there exists a $t_0 > 0$ that $y(t) < 0$ and $y^{'}(t) \geq 0$ for $t \geq t_0$. Let

$$\lim_{t\to\infty} y(t) = -A, A > 0.$$

Multiplying (1) through by t^{α}, $t \geq t_0$, and integrating the result identity from t_0 to t, we get

$$[s^{\alpha} y^{''}(s)]_{t_0}^{t} - \alpha [s^{\alpha-1} y^{'}(s)]_{t_0}^{t} + \alpha(\alpha-1) \int_{t_0}^{t} s^{\alpha-2} y^{'}(s) ds + \int_{t_0}^{t} s^{\alpha} q(s) y^{'}(s) ds + \int_{t_0}^{t} s^{\alpha} p(s) h(y(s)) ds = \int_{t_0}^{t} s^{\alpha} f(s) ds,$$

that is,

$$t^{\alpha} y^{''}(t) - \alpha t^{\alpha-1} y^{'}(t) \leq t_0^{\alpha} y^{''}(t_0) - \alpha t_0^{\alpha-1} y^{'}(t_0) + M \int_{t_0}^{t} y^{'}(s) ds$$

$$-\alpha(\alpha-1)\int_{t_0}^{t} s^{\alpha-2}y^{'}(s)ds-\int_{t_0}^{t} s^{\alpha}p(s)h(y(s))ds+\int_{t_0}^{t} s^{\alpha}f(s)ds. \quad (2)$$

Clearly,

$$\lim_{t\to\infty}\int_{t_0}^{t} y^{'}(s)ds=-A-y(t_0)\,,$$

$$0\le\lim_{t\to\infty}\int_{t_0}^{t} s^{\alpha-2}y^{'}(s)ds\le t_0^{\alpha-2}(-A-y(t_0))$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t} s^{\alpha}p(s)h(y(s))ds=\infty.$$

Hence, taking $\liminf$ in (2) as $t\to\infty$, we get

$$\liminf_{t\to\infty}(t^{\alpha}y^{''}(t)-\alpha t^{\alpha-1}y^{'}(t))=-\infty.$$

But, by Lemma 5, $\liminf_{t\to\infty}(t^{\alpha}y^{''}(t)-\alpha t^{\alpha-1}y^{'}(t))=0$. This contradiction proves the theorem. □

Remark. Lazer [6] and Heidel [5] gave sufficient conditions so that nonoscillatory solutions $y(t)$ of homogeneous third order equations satisfy the property $y(t)y^{'}(t)\ge 0$ ultimately. In the following we give sufficient conditions so that $y(t)y^{'}(t)\le 0$ ultimately.

Theorem 7. *Consider (1) with $f(t)\equiv 0$, $h(y)$ is nondecreasing and γ as the ratio of odd integers. Suppose that $\int_0^{\infty} p(t)dt=-\infty$. If $y(t)$ is a bounded nonoscillatory solution of (1), then $y(t)y^{'}(t)\le 0$ ultimately.*

Proof. Without any loss of generality we can assume $y(t)$ to be ultimately positive. By Lemma 4, there exists a $t_0>0$ such that $y(t)>0$ and $y^{'}(t)>0$ or ≤ 0 for $t\ge t_0$. Suppose that $y^{'}(t)>0$ for $t\ge t_0$. Since $y^{'''}\ge 0$ and $y(t)$ is bounded. $y^{''}(t)$ is monotonic increasing and $y^{''}(t)\le 0$. Integrating (1) from t_0 to t, we get

$$y^{''}(t)\ \ \ge\ \ y^{''}(t_0)\ -\ \int_{t_0}^{t} p(s)h(y(s))ds\ \ \ge\ \ y^{''}(t_0)\ -\ h(y(t_0))\int_{t_0}^{t} p(s)ds.$$

So $\lim_{t\to\infty} y^{''}(t)=\infty$, which contradicts the fact that

$$\lim_{t\to\infty} y^{''}(t)\le 0.$$

Hence $y(t)y^{'}(t) \leq 0$ for $t \geq t_0$. □

Corollary. *In (1), let $f(t) \equiv 0$ and $\gamma = 1$. Suppose that $\int_0^{\infty} p(t)dt = -\infty$ and $-\infty < -M < t^{\alpha}q(t)$ for some fixed α with $0 \leq \alpha \leq 2$. If $y(t)$ is a bounded nonoscillatory solution of (1), then*

$$\lim_{t\to\infty} y(t) = 0.$$

This follows from Theorem 6 and 7.

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