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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR A QUASILINEAR ELLIPTIC PROBLEM ON TIME SCALES

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Abstract: We study a boundary-value quasilinear elliptic problem on a generic time scale. Making use of the fixed-point index theory, sufficient conditions are given to obtain existence, multiplicity, and infinite solvability of positive solutions.

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1. Introduction

We are interested in the study of the following quasilinear elliptic problem:

$$\begin{aligned} -\Big(\phi_p(u^{\triangle}(t))\Big)^{\nabla} &= f(u(t)) + h(t)\,, \quad \forall t \in (0,T)_{\mathbb{T}} = \mathbb{T}\,, \\ u^{\triangle}(0) &= 0\,, \\ u(T) - u(\eta) &= 0\,, \quad 0 < \eta < T\,, \end{aligned} \tag{1}$$

where $\phi_p(\cdot)$ is the $p-$ Laplacian operator defined by $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$ with q the Holder conjugate of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, and $\mathbb{T}$ is a time-scale. We assume the following hypotheses:

(H1) The function $f : (0,T) \to \mathbb{R}^{+*}$ is a continuous function;

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(H2) The function $h : T \to \mathbb{R}^{+*}$ is left dense continuous (i.e., $h \in \mathbb{C}_{ld}(\mathbb{T}, \mathbb{R}^{+*})$). Here $\mathbb{C}_{ld}(\mathbb{T}, \mathbb{R}^{+*})$ denotes the set of all left dense continuous functions from $\mathbb{T} \to \mathbb{R}^{+}$; and $h \in L^{\infty}(0, T)$.

Results on existence of radially infinity many solutions for (1) are proved using: (i) variational methods [5, 10], where solutions are obtained as critical points of some energy functional on a Sobolev space, by imposing appropriate conditions on f; (ii) methods based on phase-plane analysis and the shooting method [11]; (iii) by adapting the technique of time maps [12]. For $p = 2$, $h = 0$, $\mathbb{T} = \mathbb{R}$, problem (1) becomes a boundary-value problem of differential equations. Our results extend and include results of the earlier works to the case of a generic time-scale $\mathbb{T}$, $p \neq 2$ and, where h is not identically zero. In the case of $h = 0$, $p = 2$, many existence results of dynamic equations on time scales are available, using different fixed point theorems, see [6, 17]. We remark that there are not many results concerning the p-Laplacian problems on time scales [18]. In this paper we prove existence of solutions by constructing an operator whose fixed points are solutions of (1). Our main ingredient is the following well-known fixed-point theorem of index theory.

Theorem 1. (see [13, 14]) *Suppose E is a real Banach space, and $K \subset E$ is a cone in X. Let $\Omega_r = \{u \in K, \|u\| < r\}$, and $F : \Omega_r \to K$ be a completely continuous operator satisfying $Fx \neq x$, for all $x \in \partial\Omega_r$. The following holds:*

(i) if $\|Fx\| \leq \|x\|$, $\forall x \in \partial\Omega_r$, then $i(F, \Omega_r, K) = 1$;

(ii) if $\|Fx\| \geq \|x\|$, $\forall x \in \partial\Omega_r$, then $i(F, \Omega_r, K) = 0$,

where i is the index of F.

The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing. The theory of time scales has been created in order to unify continuous and discrete analysis, allowing a simultaneous treatment of differential and difference equations, and to extend those theories to so-called delta/nabla-dynamic equations. A vast literature has already emerged in this field: see e.g. [2, 4, 7]. For an introduction to time scales with applications, we refer the reader to [8, 9].

The outline of the paper is as follows. In Section 2 we give some preliminary results with respect to the calculus on time scales. Section 3 is devoted to the existence of positive solutions using fixed-point index theory. The remaining sections deal with multiplicity and infinite solvability solutions for (1).

2. Preliminary Results on Time Scales

We begin by recalling some basic concepts of time scales. Then, we prove some preliminary results that will be needed in the sequel.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The operators σ and ρ from $\mathbb{T}$ to $\mathbb{T}$ are defined in [15, 16]:

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \in \mathbb{T}, \quad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\} \in \mathbb{T}.$$

They are called the forward jump operator and the backward jump operator, respectively.

The point $t \in \mathbb{T}$ is said to be left-dense, left-scattered, right-dense, or right-scattered, if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise we set $\mathbb{T}_k = \mathbb{T}$. Similarly, if $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise we set $\mathbb{T}^k = \mathbb{T}$.

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$ (assume t is not left-scattered if $t = \sup \mathbb{T}$). We define $f^{\triangle}(t)$ to be the number (provided it exists) such that given any $\epsilon > 0$ there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq |\sigma(t) - s|, \quad \text{for all } s \in U.$$

We call $f^{\triangle}(t)$ the delta derivative of f at t. We remark that $f^{\triangle}$ is the usual derivative f' if $\mathbb{T} = \mathbb{R}$ and the usual forward difference Δf (defined by $\triangle f(t) = f(t+1) - f(t)$) if $\mathbb{T} = \mathbb{Z}$.

Similarly, for $t \in \mathbb{T}$ (assume t is not right-scattered if $t = \inf \mathbb{T}$), the nabla derivative of f at the point t is defined in [7] to be the number $f^{\nabla}(t)$ (provided it exists) with the property that for each $\epsilon > 0$ there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq |\rho(t) - s|, \quad \text{for all } s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$. If $\mathbb{T} = Z$, then $f^{\nabla}(t) = f(t) - f(t-1)$ is the backward difference operator.

We say that a function f is left-dense continuous (ld-continuous for short), provided f is continuous at each left-dense point in $\mathbb{T}$ and its right-sided limit exists at each right-dense point in $\mathbb{T}$. It is well-known that if f is ld-continuous and if $F^{\nabla}(t) = f(t)$, then one can define the nabla integral by

$$\int_a^b f(t) \nabla t = F(b) - F(a).$$

If $F^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Function F is said to be an antiderivative of f. For more details on time scales, the reader can consult [1, 2, 3, 4, 8, 9] and references therein.

In the rest of the paper, $\mathbb{T}$ is a closed subset of $\mathbb{R}$ with $0 \in \mathbb{T}_k$, $T \in \mathbb{T}^k$. We denote $E = \mathbb{C}_{ld}([0,T],\mathbb{R})$, which is a Banach space with the maximum norm $\|u\| = \max_{[0,T]} |u(t)|$.

Lemma 2. *Suppose that conditions $(H1)$ and $(H2)$ hold. Then, $u(t)$ is a solution of the boundary-value problem (1) if and only if $u(t) \in E$ is a solution of the following equation:*

$$u(t) = \phi_q\left(\int_\eta^T (f(u(r) + h(r))\,\nabla r\right) + \int_0^t \phi_q\left(\int_s^T (f(u(r) + h(r))\,\nabla r\right) \triangle s. \quad (2)$$

Proof. By integrating the equation (1) on (s,T), we have

$$\phi_p(u^{\triangle}(T)) = \phi_p(u^{\triangle}(s)) - \int_s^T (f(u(r) + h(r))\,\nabla r.$$

Then,

$$\phi_p(u^{\triangle}(s)) = \phi_p(u^{\triangle}(T)) + \int_s^T (f(u(r) + h(r))\,\nabla r.$$

Using the boundary conditions, we have

$$\phi_p(u^{\triangle}(s)) = \int_s^T (f(u(r) + h(r))\,\nabla r.$$

Thus,

$$u^{\triangle}(s) = \phi_q\left(\int_s^T (f(u(r) + h(r))\,\nabla r\right).$$

Integrating the last equation on $(0,t)$, we have

$$\begin{aligned} u(t) &= u(0) + \int_0^t \phi_q\left(\int_s^T (f(u(r) + h(r))\,\nabla r\right) \triangle s \\ &= u^{\triangle}(\eta) + \int_0^t \phi_q\left(\int_s^T (f(u(r) + h(r))\,\nabla r\right) \triangle s \\ &= \phi_q\left(\int_\eta^T (f(u(r) + h(r))\,\nabla r\right) + \int_0^t \phi_q\left(\int_s^T (f(u(r) + h(r))\,\nabla r\right) \triangle s. \end{aligned}$$

Inversely, if we suppose that (2) holds, it is easy to get the first equation of (1) by derivation, and to see that u satisfies the boundary value conditions in (1). Furthermore, u is obviously positive since ϕ_q is non decreasing function and f and h are also positives functions. The proof of Lemma 2 is now complete. $\square$

On the other hand, we have $-(\phi_p(u^\triangle))^\nabla = f(u(t)) + h(t)$. Then, since $f, h \geq 0$, we have $(\phi_p(u^\triangle))^\nabla \leq 0$ and $(\phi_p(u^\triangle(t_2))) \leq (\phi_p(u^\triangle(t_1)))$ for any t_1, $t_2 \in [0,T]$ with $t_1 \leq t_2$. It follows that $u^\triangle(t_2) \leq u^\triangle(t_1)$ for $t_1 \leq t_2$. Hence, $u^\triangle(t)$ is a decreasing function on $[0,T]$. This means that the graph of $u(t)$ is concave on $[0,T]$.

Let $K \subset E$ be the cone defined by

$$K = \{u \in E : u(t) \geq 0, u(t) \text{ is a concave function}, t \in [0,1]\},$$

and $F : K \to E$ the operator

$$Fu(t) = \phi_q\left(\int_\eta^T (f(u(r)) + h(r))\,\nabla r\right) + \int_0^t \phi_q\left(\int_s^T (f(u(r) + h(r))\,\nabla r\right) \triangle s.$$

It is easy to see that (1) has a solution $u = u(t)$ if and only if u is a fixed point of the operator F. One can also verify that $F(K) \subset K$ and $F : K \to K$ is completely continuous.

3. Existence of Positive Solutions

We define two open subsets Ω_1 and Ω_2 of E:

$$\Omega_1 = \{u \in K : \|u\| < a\}, \quad \Omega_2 = \{u \in K : \|u\| < b\}.$$

Without loss of generality, we suppose that $b < a$. For convenience, we introduce the following notation:

$$A = \frac{a - \alpha\|h\|_\infty^{1/p-1}}{\alpha a}, \text{ where } \alpha = \phi_q(2^{p-2})\phi_q(T)(T+1),$$
$$B = \phi_p(T - \eta).$$

Theorem 3. *Besides* $(H1)$ *and* $(H2)$, *suppose that* f *also satisfies:*
(i) $\max_{0\leq u\leq a} f(u) \leq \phi_p(aA)$;
(ii) $\min_{0\leq u\leq b} f(u) \geq \phi_p(bB)$.
Then, (1) *has a positive solution.*

Proof. If $u \in \partial\Omega_1$, we have:

$$
\begin{aligned}
\|F(u)\| &\le \phi_q\left(\int_\eta^T ((aA)^{p-1} + \|h\|_\infty)\nabla r\right) \\
&\quad + \int_0^T \phi_q\left(\int_s^T ((aA)^{p-1} + \|h\|_\infty)\nabla r\right)\triangle s \\
&\le \phi_q\left(((aA)^{p-1} + \|h\|_\infty)(T-\eta)\right) + \int_0^T \phi_q\left(((aA)^{p-1} + \|h\|_\infty)(T-s)\right)\triangle s \\
&\le \phi_q\left((aA)^{p-1} + \|h\|_\infty\right)\phi_q(T) + \phi_q\left((aA)^{p-1} + \|h\|_\infty\right)\int_0^T \phi_q(T-s)\triangle s \\
&\le \phi_q\left((aA)^{p-1} + (\|h\|_\infty^{1/p-1})^{p-1}\right)\phi_q(T) \\
&\quad + \phi_q\left((aA)^{p-1} + (\|h\|_\infty^{1/p-1})^{p-1}\right)\phi_q(T)T \\
&\le \phi_q\left((aA)^{p-1} + (\|h\|_\infty^{1/p-1})^{p-1}\right)\phi_q(T)(T+1).
\end{aligned}
$$

Using the elementary inequality

$$x^p + y^p \le 2^{p-1}(x+y)^p,$$

and the form of A, it follows that

$$\|F(u)\| \le \phi_q(T+1)(2^{p-2})(aA + \|h\|_\infty^{1/p-1}) \le \|u\| = a.$$

Therefore, $\|Fu\| \le \|u\|$ for all $u \in \partial\Omega_1$. Then, by Theorem 1,

$$i(F, \Omega_1, K) = 1. \tag{3}$$

On the other hand, for $u \in \partial\Omega_2$ we have:

$$
\begin{aligned}
\|F(u)\| &\ge \phi_q\left(\int_\eta^T f(u(r)\,\nabla r\right) + \int_0^t \phi_q\left(\int_s^T (f(u(r))\,\nabla r\right)\triangle s. \\
&\ge B\phi_q(T-\eta) \ge b = \|u\|(\text{ since } B = \phi_p(T-\eta)).
\end{aligned}
$$

Therefore, $\|Fu\| \ge \|u\|$ for all $u \in \partial\Omega_2$. By Theorem 1,

$$i(F, \Omega_2, K) = 0. \tag{4}$$

It follows by (3) and (4) that

$$i(F, \Omega_1 \backslash \overline{\Omega_2}, K) = 1.$$

Then T has a fixed point $u \in \Omega_1 \backslash \overline{\Omega_2}$. Obviously, u is a positive solution of problem (1) and $b < \|u\| < a$. The proof of Theorem 3 is complete. $\square$

4. Multiplicity

By multiplicity we mean the existence of an arbitrary number of solutions. We now obtain results on the multiplicity of positive solutions for (1) under the following assumptions: we suppose that there exist positive real numbers $0 < a_1 < a_2 < \ldots < a_{k+1}$, such that:

(i) $\max_{0\le u\le a_{2i-1}} f(u) \le \phi_p(a_{2i-1}A), i = 1, \ldots [\frac{k+2}{2}]$,

(ii) $\min_{0\le u\le a_{2i}} f(u) \ge \phi_p(a_{2i}B), i = 1, \ldots [\frac{k+1}{2}]$,

where $[n]$ denote the integer part of n.

Theorem 4. *Assume that (i)-(ii) hold. Then, problem* (1) *has at least* k *positive solutions* $u_1, \ldots u_k$ *such that*

$$a_i < \|u_i\| < a_{i+1}, \quad i = 1, \ldots k\,.$$

Proof. By continuity, there exist

$$0 < b_1 < a_1 < c_1 < b_2 < a_2 < c_2 < \ldots c_k < b_{k+1} < a_{k+1} < +\infty$$

such that

$$\min_{0\le u\le b_{2i-1}} f(u) \ge \phi_p(b_{2i-1}B)\,, \qquad \min_{0\le u\le c_{2i-1}} f(u) \ge \phi_p(c_{2i-1}B),$$

for $i = 1, \ldots [\frac{k+2}{2}]$, and

$$\max_{0\le u\le c_{2i}} f(u) \le \phi_p(c_{2i}A)\,, \qquad \max_{0\le u\le b_{2i}} f(u) \le \phi_p(b_{2i}A),$$

for $i = 1, \ldots [\frac{k+1}{2}]$. Then, calling Theorem 3 to each interval $(c_i, b_{i+1}), i = 1, \ldots k$, we obtain the intended result. □

5. Infinite Solvability

Theorem 5. *If the following two conditions hold:*

(i) $\liminf_{a\to 0} \frac{\max_{0\le u\le a}\{f(u)\}}{a^{p-1}} \le \phi_p(A)$,

(ii) $\limsup_{b\to 0} \frac{\max_{0\le u\le b}\{f(u)\}}{b^{p-1}} \ge \phi_p(B)$,

then, problem (1) *has a sequence of positive solutions* $(u_k)_{k\ge 1}$ *such that* $\|u_k\| \to 0$ *as* $k \to \infty$.

Proof. From (i) and (ii), there exists a sequence of pairs of positive numbers $(a_k, b_k)_{k\geq 1}$ convergent to $(0,0)$ such that

$$\max_{0\leq u\leq a_k} f(u) \leq \phi_p(a_k A),$$

$$\min_{0\leq u\leq b_k} f(u) \geq \phi_p(b_k B).$$

Suppose that

$$a_1 > b_1 > a_2 > b_2 > \dots a_k > b_k > \dots$$

Calling Theorem 3 on each pair $(a_k, b_k)_{k\geq 1}$, we conclude that (1) has a sequence of positive solutions $(u_k)_{k\geq 1}$ such that $b_k \leq \|u_k\| \leq a_k$. $\square$

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References

[1] R.P. Agarwal, M. Bohner, Basic calculus on time scales and some of its applications, *Results Math.*, **35**, No. 1-2 (1999), 3-22.

[2] R.P. Agarwal, M. Bohner, D. O'Regan, A. Peterson, Dynamic equations on time scales: a survey, *J. Comput. Appl. Math.*, **141**, No. 1-2 (2002), 1-26.

[3] R.P. Agarwal, M. Bohner, P.J. Y. Wong, Sturm-Liouville eigenvalue problems on time scales, *Appl. Math. Comput.*, **99**, No. 2-3 (1999), 153-166.

[4] R.P. Agarwal, D. O'Regan, Nonlinear boundary value problems on time scales, *Nonlinear Anal., Ser. A: Theory Methods and Appl.*, **44**, No. 4 (2001), 527-535.

[5] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis*, **14** (1973), 349-381.

[6] D.R. Anderson, Solutions to second-order three-point problems on time scales, *J. Difference Equ. Appl.*, **8**, No. 8 (2002), 673-688.

[7] F.M. Atici, G. Sh. Guseinov, On Green's functions and positive solutions for boundary value problems on time scales, *J. Comput. Appl. Math.*, **141**, No. 1-2 (2002), 75-99.

[8] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales–An Introduction with Applications*, Birkhäuser Boston, Boston, MA (2001).

[9] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston, Boston, MA (2003).

[10] A. Castro, A. Kurepa, Energy analysis of a nonlinear singular differential equation and applications, *Rev. Colombiana Mat.*, **21**, No. 1 (1987), 155-166.

[11] A. El Hachimi, F. de Thelin, Infinitely many radially symmetric solutions for a quasilinear elliptic problem in a ball, *J. Differential Equations*, **128**, No. 1 (1996), 78-102.

[12] A. El Hachimi, J.-P. Gossez, A note on a nonresonance condition for a quasilinear elliptic problem, *Nonlinear Anal.*, **22**, No. 2 (1994), 229-236.

[13] D.J. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, MA (1988).

[14] D. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Acad. Publ., Dordrecht (1996).

[15] S. Hilger, Analysis on measure chains–a unified approach to continuous and discrete calculus, *Results Math.*, **18**, No. 1-2 (1990), 18-56.

[16] S. Hilger, Differential and difference calculus–unified!, *Nonlinear Anal.*, **30**, No. 5 (1997), 2683-2694.

[17] E.R. Kaufmann, Positive solutions of a three-point boundary-value problem on a time scale, *Electron. J. Differential Equations*, No. 82 (2003).

[18] D.-B. Wang, Existence, multiplicity and infinite solvability of positive solutions for p-Laplacian dynamic equations on time scales, *Electron. J. Differential Equations*, No. 96 (2006).