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ON NECESSARY AND SUFFICIENT CONDITIONS FOR CONVERGENCE TO THE GEOMETRIC DISTRIBUTION

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Abstract: By using Lévy representation of characteristic function we show necessary and sufficient conditions for convergence of a sequence of distribution functions of sums of infinitesimal system of random variables to the geometric distribution.

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1. Introduction

It is useful in mathematical statistics or applied statistics to know necessary and sufficient conditions under which a sequence of distribution functions of sums of independent random variables converges to some limit distributions, especially normal distribution or Poisson distribution, and there are many research papers which have done about necessary and sufficient conditions for convergence to these distributions. There are also many papers, among others, [2], [4], [5] and the literature [1], [3], [6], [7] which are studying the geometric distribution from point of view of infinitely divisibile probability distribution. In this paper we deal with necessary and sufficient conditions that a sequence

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of distribution functions of sums of independent random variables converges to geometric distribution since we could not see what conditions are needed in that case and we think this result is necessary for further study in the other field related to geometric distribution. We make use of several theorems in Tucker's book [8] to treat the geometric distribution with infinitesimal system of independent random variables. Let us denote the probability of success event or appearing head in coin tossing by $q = 1-p$, where p is the probability of fail event or appearing tail in coin tossing and $0 < p < 1$. With this notation and under infinitesimal system of independent random variables we prove the main theorem in Section 3 and give an example in Section 4.

2. Canonical Representation of Infinitely Divisible Characteristic Function

Let X_{nj} be a random variable with indices $n,\ j$ and let us denote distribution function of a random variable X_{nj} by $F_{nj}(x) = P(X_{nj} \leq x)$. We call the system of random variables $\{X_{nj} : n = 1,\ 2, \cdots ; j = 1, 2, \cdots , k_n\}$ a triangular array when $\{X_{nj} : j = 1, 2, \cdots , k_n\}$ are independent random variables for each n. We call an infinitesimal system or null array of random variables when the triangular array $\{X_{nj} : n = 1,\ 2, \cdots ; j = 1, 2, \cdots , k_n\}$ satisfies

$$\lim_{n\to\infty} \max_{1\leq j\leq k_n} P(|X_{nj}| \geq \epsilon) = 0$$

for every $\epsilon > 0$. We mostly make use of the canonical representation of characteristic function of an infinitely divisible distribution. In what follows, we describe theorems which are necessary to obatin the results.

Theorem i. *A function ϕ is the characteristic function of an infinitely divisible distribution function if and only if there exists a real number γ and a bounded nondecreasing function G defined over R such that:*

$$\phi(u) = \exp\Big\{i\gamma u - \frac{\sigma^2u^2}{2} + \int_{R-\{0\}} \Big(e^{iux} - 1 - \frac{iux}{1+x^2}\Big)\frac{1+x^2}{x^2}dG(x)\Big\},$$

where we let $\sigma^2 = G(+0) - G(-0)$. This representation is unique.

Theorem ii. *A function ϕ is a characteristic function of an infinitely divisible distribution function if and only if there exists constants γ and $\sigma \geq 0$ and a function M defined over $R - \{0\}$ which is nondecreasing over $(-\infty, 0)$ and over $(0, \infty)$ and satisfies*

$$M(-\infty) = M(+\infty) = 0$$

and

$$\int_{-1}^{-0} + \int_{+0}^{1} x^2 dM(x) < \infty,$$

and such that

$$\phi(u) = \exp\Big\{i\gamma u - \frac{\sigma^2 u^2}{2} + \int_{R-\{0\}} \Big(e^{iux} - 1 - \frac{iux}{1+x^2}\Big) dM(x)\Big\}.$$

Theorem iii. *Let $\{F_n\}$ be a sequence of infinitely divisible distributions whose corresponding representations are determined by the sequence of triples $\{(\gamma_n, \sigma_n^2, M_n)\}$. If F_n converges completely (in law) to F, where F uniquely determines (γ, σ^2, M), then:*

(a) $\gamma_n \to \gamma$,

(b) $M_n(x) \to M(x)$ in wide sense for $x \in$ Cont M, and

(c) $$\lim_{\epsilon\downarrow 0}\Big[\lim\sup_{n\to\infty}\Big\{\int_{\{|x|<\epsilon\}-\{0\}} x^2\, dM_n(x) + \sigma_n^2\Big\}\Big] = \lim_{\epsilon\downarrow 0}\Big[\lim\inf_{n\to\infty}\Big\{\int_{\{|x|<\epsilon\}-\{0\}} x^2\, dM_n(x) + \sigma_n^2\Big\}\Big] = \sigma^2.$$

Conversely, if there exists a triple (γ, σ^2, M) such that (a), (b), and (c) hold and M satisfies the conditions of Theorem ii then $F_n \to F$ completely, where F is an infinitely divisible distribution function determined by (γ, σ^2, M).

Theorem iv. *Let $X = \{\{X_{nj}\}\}$ be an infinitesimal system of random variables, and let $\{c_n\}$ be a sequence of real numbers, and let*

$$Z_n = X_{n1} + \ \ldots \ + X_{nk_n} - c_n.$$

In order that there exists a distribution function F such that

$$F_{Z_n} \longrightarrow F$$

completely as $n \to \infty$ it is necessary and sufficient that there exists constants γ and $\sigma^2 \geq 0$ and a Lévy spectral function M such that:

(A) $$M(x) = \lim_{n\to\infty}\sum_{j=1}^{k_n} F_{nj}(x) \quad \text{if} \quad 0 > x \in \text{Cont } M$$

$$= \lim_{n\to\infty}\sum_{j=1}^{k_n}\Big(F_{nj}(x) - 1\Big) \quad \text{if } 0 < x \in \text{ Cont } M,$$

$$(B)\ \lim_{\epsilon\downarrow 0}\Big[\lim\ \sup_{n\to\infty}\sum_{j=1}^{k_n}\Big\{\int_{|x|<\epsilon} x^2\, dF_{nj}(x) - \Big(\int_{|x|<\epsilon} x dF_{nj}(x)\Big)^2\Big\}\Big]$$

$$= \lim_{\epsilon\downarrow 0}\Big[\lim\ \inf_{n\to\infty}\sum_{j=1}^{k_n}\Big\{\int_{|x|<\epsilon} x^2\, dF_{nj}(x) - \Big(\int_{|x|<\epsilon} x dF_{nj}(x)\Big)^2\Big\}\Big] = \sigma^2,$$

and

$$(C)\ [\sum_{j=1}^{k_n} a_{nj} - c_n \ \rightarrow\ \gamma + \int_{\{|x|<\tau\}-\{0\}} \frac{x^3}{1+x^2} dM(x) - \int_{\{|x|\geq\tau\}} \frac{x}{1+x^2} dM(x)]$$

as $n \to \infty$, *where*

$$a_{nj} = \int_{|x|<\tau} x dF_{nj}(x),$$

$\tau > 0$, *and* $+\tau, -\tau \in$ Cont M.

The distribution function F is infinitely divisible and $\{\gamma, \sigma^2, M\}$ determines F as in the Theorem ii. The notation Cont M *means the set of all continuity points of the function $M(x)$.*

3. Convergence to the Geometric Distribution

In this section we will state the main result.

Theorem 1. *Let $X = \{\{X_{nj}\}\}$ be an infinitesimal system of random variables, and let $Z_n = X_{n1} + \ \dots\ + X_{nk_n}$. In order that $F_{Z_n} \longrightarrow F$ completely as $n \to \infty$, where F is the geometric probability distribution function with expectation p/q, it is necessary and sufficient that, for $0 < \varepsilon \leq 1/2$ and $0 < \varepsilon_i \leq 1/2$, , $i = 1, 2, 3, 4$,*

c1) α) $\displaystyle\sum_{j=1}^{k_n} F_{nj}(-\epsilon_1) \to 0,$

and

c1) β)] $\displaystyle\sum_{j=1}^{k_n} \big(F_{nj}(\epsilon_2) - 1\big) \to -\sum_{r=1}^{\infty} \frac{p^r}{r},$

c2) α′) for every positive integer $r = 1, 2, \dots,$

$$\sum_{j=1}^{k_n}\big(F_{nj}(r-\epsilon_3)-F_{nj}(r-1+\epsilon_3)\big)\to 0,$$

and

c2) β') $\displaystyle\sum_{j=1}^{k_n}\big(F_{nj}(r+\epsilon_3)-F_{nj}(r-\epsilon_3)\big)\to\frac{p^r}{r},$

c3) $\displaystyle\sum_{j=1}^{k_n}\int_{|x|<\epsilon_4} x\,dF_{nj}(x)\to 0,$

and

c4) $\displaystyle\sum_{j=1}^{k_n}\Big(\int_{|x|<\epsilon} x^2\,dF_{nj}(x)-\Big(\int_{|x|<\epsilon} x\,dF_{nj}(x)\Big)^2\Big)\to 0,$ as $n\to\infty$.

Proof. We first recall that the characteristic function of the geometric distribution with expectation p/q is

$$\begin{aligned}\phi(u) &= \frac{q}{1-p\exp[iu]}=\exp\big[\sum_{n=1}^{\infty}\frac{p^n}{n}(\exp[iun]-1)\big]\\ &= \exp\big[\int_{+0}^{\infty}(\exp[iux]-1)L(dx)\big],\end{aligned}$$

where $L(\{n\})=p^n/n$ for $n=1,2,\dots$, that is, the measure $L(dx)$ has only point mass with weight p^n/n at positive integer n. By using Theorem i in Section 2 about the canonical representation of characteristic function, we obtain

$$G(x)=\int_{-\infty}^{x}\frac{t^2}{1+t^2}L(dt)=\sum_{1\le r\le x}\frac{r}{1+r^2}p^r,$$

$$\gamma=\int_{+0}^{\infty}\frac{1}{x}dG(x)=\sum_{n=1}^{\infty}\frac{p^n}{1+n^2},$$

and

$$\sigma^2=G(+0)-G(-0)=0.$$

We obtain the Lévy representation of $\phi(u)$ as follows,

$$\phi(u)=\exp\Big[iu\sum_{n=1}^{\infty}\frac{p^n}{1+n^2}+\int_{+0}^{\infty}\Big(\exp[iux]-1-\frac{iux}{1+x^2}\Big)dM(x)\Big],$$

where $M(x) = 0$ for $x < 0$, and

$$M(x) = -\int_{(x,+\infty)} \frac{1+y^2}{y^2} dG(y) = -\sum_{x<r} \frac{p^r}{r}$$

for $x > 0$. $M(x)$ is a right continuous step function,

$$M(+0) = -\sum_{r=1}^{\infty} \frac{p^r}{r},\ M(1) = -\sum_{r=2}^{\infty} \frac{p^r}{r},\ M(2) = -\sum_{r=3}^{\infty} \frac{p^r}{r},\ \cdots .$$

Let us denote

$$M_n(x) = \begin{cases} \sum_{j=1}^{k_n} F_{nj}(x) & \text{if } x < 0, \\ \sum_{j=1}^{k_n} (F_{nj}(x) - 1) & \text{if } x > 0. \end{cases}$$

Let F be the geometric distribution function and $\phi(u)$ its characteristic function. By Theorem iv (A), (B), (C) in Section 2, $F_{z_n} \to F$ completely if and only if

$$M_n(x) \to M(x) \text{ as } n \to \infty \tag{1}$$

for $x \neq 0, 1, 2, \ldots$, and

$$\lim_{\epsilon \downarrow 0} \Big[\limsup_{n\to\infty} \sum_{j=1}^{k_n} \Big(\int_{|x|<\epsilon} x^2 \, dF_{nj}(x) - \Big(\int_{|x|<\epsilon} x \, dF_{nj}(x) \Big)^2 \Big) \Big]$$

$$= \lim_{\epsilon \downarrow 0} \Big[\liminf_{n\to\infty} \sum_{j=1}^{k_n} \Big(\int_{|x|<\epsilon} x^2 \, dF_{nj}(x) - \Big(\int_{|x|<\epsilon} x \, dF_{nj}(x) \Big)^2 \Big) \Big] = 0, \tag{2}$$

and

$$\sum_{j=1}^{k_n} \int_{|x|<\tau} x \, dF_{nj}(x)$$

$$\to \sum_{n=1}^{\infty} \frac{p^n}{1+n^2} + \int_{\{|x|<\tau\}-\{0\}} \frac{x^3}{1+x^2} dM(x) - \int_{\{|x|\geq\tau\}} \frac{x}{1+x^2} dM(x) \tag{3}$$

as $n \to \infty$ and $0 < \tau$, $+\tau, -\tau \in \text{Cont } M$. We show that the conditions (1), (2) and (3) are equivalent to the conditons c1), c2), c3), c4). Suppose that the conditions (1), (2), (3) hold. By (1) we see that c1) α) hold since $-\epsilon_1 \in \text{Cont } M$.

By (1) we have

$$\begin{aligned}
-M_n(\epsilon_2) &= \sum_{j=1}^{k_n}\int_{\cup_{r=0}^{\infty}(r+\epsilon_2,r+1-\epsilon_2]} dF_{nj}(x) \\
&+ \sum_{j=1}^{k_n}\int_{\cup_{r=1}^{\infty}(r-\epsilon_2,r+\epsilon_2]} dF_{nj}(x) \\
&\to -M(\epsilon_2) = \sum_{r=1}^{\infty}\frac{p^r}{r},
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
-M_n(m+\epsilon_3) &= \sum_{j=1}^{k_n}\int_{\cup_{r=m}^{\infty}(r+\epsilon_3,r+1-\epsilon_3]} dF_{nj}(x) \\
&+ \sum_{j=1}^{k_n}\int_{\cup_{r=m}^{\infty}(r+1-\epsilon_3,r+1+\epsilon_3]} dF_{nj}(x) \\
&\to -M(m+\epsilon_3) = \sum_{r=m+1}^{\infty}\frac{p^r}{r},\ m=1,\ 2,\ \ldots,
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
-M_n(m-\epsilon_3) &= \sum_{j=1}^{k_n}\int_{\cup_{r=m}^{\infty}(r+\epsilon_3,r+1-\epsilon_3]} dF_{nj}(x) \\
&+ \sum_{j=1}^{k_n}\int_{\cup_{r=m}^{\infty}(r-\epsilon_3,r+\epsilon_3]} dF_{nj}(x) \\
&\to -M(m-\epsilon_3) = \sum_{r=m}^{\infty}\frac{p^r}{r},\ m=1,\ 2,\ \ldots,
\end{aligned} \tag{6}$$

as $n\to\infty$. By (1) we see that

$$\begin{aligned}
&\sum_{j=1}^{k_n}\int_{(m+\epsilon_3,m+1-\epsilon_3]} dF_{nj}(x) \\
&= \sum_{j=1}^{k_n}(F_{nj}(m+1-\epsilon_3)-1)-(F_{nj}(m+\epsilon_3)-1) \\
&= M_n(m+1-\epsilon_3)-M_n(m+\epsilon_3) \\
&\to M(m+1-\epsilon_3)-M(m+\epsilon_3)=0,\ m=0,\ 1,\ 2,\ \ldots,
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
&\sum_{j=1}^{k_n} \int_{(m-\epsilon_3, m+\epsilon_3]} dF_{nj}(x) \\
&= \sum_{j=1}^{k_n} (F_{nj}(m+\epsilon_3) - 1) - (F_{nj}(m-\epsilon_3) - 1) \\
&= M_n(m+\epsilon_3) - M_n(m-\epsilon_3) \\
&\to M(m+\epsilon_3) - M(m-\epsilon_3) = \frac{p^m}{m},\ m = 1,\ 2,\ \ldots. \qquad (8)
\end{aligned}$$

These yield c1) β), c2) α') and c2) β'). Next, we show c4). Let us denote

$$J_n(\epsilon) = \sum_{j=1}^{k_n} \Big(\int_{|x|<\epsilon} x^2\, dF_{nj}(x) - \Big(\int_{|x|<\epsilon} x\, dF_{nj}(x) \Big)^2 \Big),$$

and let $0 < \delta < \epsilon \le 1/2$. Then

$$J_n(\epsilon) = J_n(\delta) + J_n(\delta, \epsilon) - 2L_n(\delta, \epsilon),$$

where

$$J_n(\delta, \epsilon) = \sum_{j=1}^{k_n} \Big(\int_{\delta \le |x| < \epsilon} x^2\, dF_{nj}(x) - \Big(\int_{\delta \le |x| < \epsilon} x\, dF_{nj}(x) \Big)^2 \Big),$$

and

$$L_n(\delta, \epsilon) = \sum_{j=1}^{k_n} \Big(\int_{|x|<\delta} x\, dF_{nj}(x) \Big) \Big(\int_{\delta \le |x| < \epsilon} x\, dF_{nj}(x) \Big).$$

We see by (1) that

$$0 \le J_n(\delta, \epsilon) \le \sum_{j=1}^{k_n} \int_{\delta \le |x| < \epsilon} x^2\, dF_{nj}(x) \le \epsilon^2 \sum_{j=1}^{k_n} \int_{\delta \le |x| < \epsilon} dF_{nj}(x) \to 0$$

as $n \to \infty$. In fact, we see by the right continuity of the distribution functions F_{nj} and by (1) that for $0 < \delta_1 < \delta$,

$$\begin{aligned}
0 &\le \sum_{j=1}^{k_n} \int_{\delta \le |x| < \epsilon} dF_{nj}(x) \le \sum_{j=1}^{k_n} \Big(\int_{\delta_1 < x \le \epsilon} + \int_{-\epsilon < x \le -\delta} dF_{nj}(x) \Big) \\
&= \sum_{j=1}^{k_n} \Big(F_{nj}(\epsilon) - F_{nj}(\delta_1) + F_{nj}(-\delta) - F_{nj}(-\epsilon) \Big) \to 0. \qquad (9)
\end{aligned}$$

We also have by (1) that

$$|L_n(\delta,\epsilon)| \le \delta\epsilon \sum_{j=1}^{k_n} \int_{\delta\le|x|<\epsilon} dF_{nj}(x) \to 0,$$

as $n \to \infty$. Hence

$$|J_n(\epsilon) - J_n(\delta)| \to 0$$

as $n \to \infty$ and so we see by the fact $0 \le J_n(\epsilon)$ and by (2) that the value of $\limsup_{n\to\infty} J_n(\epsilon)$ is finite for sufficiently small ϵ and that

$$\limsup_{n\to\infty} J_n(\epsilon) = \limsup_{n\to\infty} J_n(\delta)$$

and

$$\liminf_{n\to\infty} J_n(\epsilon) = \liminf_{n\to\infty} J_n(\delta).$$

Hence we see that $\limsup_{n\to\infty} J_n(\epsilon)$ and $\liminf_{n\to\infty} J_n(\epsilon)$ do not depend on the value of ϵ. By (2) both $\limsup_{n\to\infty} J_n(\epsilon)$ and $\liminf_{n\to\infty} J_n(\epsilon)$ of the quantity

$$\sum_{j=1}^{k_n}\Big(\int_{|x|<\epsilon} x^2\, dF_{nj}(x) - \Big(\int_{|x|<\epsilon} x\, dF_{nj}(x)\Big)^2\Big)$$

are equal to 0. Therefore we obtain c4). Next, we show c3). If τ belongs to the interval $(0,1/2]$ and if we take τ as ϵ_4 we see by (3) that

$$\sum_{j=1}^{k_n} \int_{|x|<\epsilon_4} x\, dF_{nj}(x) \to \sum_{n=1}^{\infty} \frac{p^n}{1+n^2} + 0 - \sum_{n=1}^{\infty} \frac{p^n}{1+n^2} = 0$$

as $n \to \infty$. We obtain the condition c3) if $0 < \tau \le 1/2$. If τ belongs to the interval $(1/2, 1)$ we also see by (3) that

$$\sum_{j=1}^{k_n} \int_{|x|<\tau} x\, dF_{nj}(x) \to 0$$

as $n \to \infty$. When $1 < \tau$ and $l < \tau < l+1$, we write

$$\sum_{j=1}^{k_n} \int_{|x|<\tau} x\, dF_{nj}(x) = A_n + B_n,$$

where

$$A_n = \sum_{j=1}^{k_n} \int_{|x|<\epsilon_4} x\, dF_{nj}(x) \quad\text{and}\quad B_n = \sum_{j=1}^{k_n} \int_{\epsilon_4\le|x|<\tau} x\, dF_{nj}(x).$$

One easily computes that

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{p^n}{1+n^2} + \int_{|x|<\tau} \frac{x^3}{1+x^2} dM(x) - \int_{|x|\geq\tau} \frac{x}{1+x^2} dM(x) \\ = &\sum_{n=1}^{\infty} \frac{p^n}{1+n^2} + \sum_{n=1}^{l} \frac{n^2 p^n}{1+n^2} - \sum_{n=l+1}^{\infty} \frac{p^n}{1+n^2} = \sum_{n=1}^{l} p^n. \end{aligned} \tag{10}$$

If (3) holds then $A_n \to 0$, which has been already proven as the condition c3), and by (10),

$$B_n \to \sum_{n=1}^{l} p^n$$

as $n \to \infty$. We write B_n in the following two integrals,

$$\begin{aligned} B_n &= \sum_{j=1}^{k_n} \int_{\epsilon_4 \leq |x| < \tau} x\ dF_{nj}(x) \\ &= \sum_{j=1}^{k_n} \Big(\int_{\epsilon_4 \leq x < \tau} x\ dF_{nj}(x) + \int_{-\tau < x \leq -\epsilon_4} x\ dF_{nj}(x) \Big). \end{aligned}$$

We see by (1) that

$$|\sum_{j=1}^{k_n} \int_{-\tau < x \leq -\epsilon_4} x\ dF_{nj}(x)| \to 0$$

and obtain from the above B_n that

$$\sum_{j=1}^{k_n} \int_{\epsilon_4 < x \leq \tau} x\ dF_{nj}(x) \to \sum_{k=1}^{l} p^k$$

as $n \to \infty$.

Conversely, suppose that c1), c2), c3), c4) hold. By the c1) α) we obtain $M_n(y) \to M(y) = 0$ for $y < 0$. We see by c1) β) that $M_n(y)$ converges to a constant function

$$M(y) = -\sum_{r=1}^{\infty} \frac{p^r}{r}$$

for y in the interval $(0, 1/2]$ as $n \to \infty$. When $1/2 < y < 1$, take ϵ_2 such that $y = 1 - \epsilon_2$. By c2) α') we see that

$$\sum_{j=1}^{k_n} \int_{(\epsilon_2, 1-\epsilon_2]} dF_{nj}(x) \to 0$$

and by c1) β) we obtain

$$\begin{aligned} M_n(1-\epsilon_2) &= -\sum_{j=1}^{k_n}\int_{(\epsilon_2,\infty)} dF_{nj}(x) \\ &+\sum_{j=1}^{k_n}\int_{(\epsilon_2,1-\epsilon_2]} dF_{nj}(x) \to M(1-\epsilon_2) = -\sum_{r=1}^{\infty}\frac{p^r}{r}, \quad (11) \end{aligned}$$

and so we obtain

$$M_n(y) \to M(y) = -\sum_{r=1}^{\infty}\frac{p^r}{r}$$

for $0 < y < 1$ as $n \to \infty$. In general, we see by c2) β') and by the same discussion as the above that

$$M_n(y) \to M(y) = -\sum_{r=m+1}^{\infty}\frac{p^r}{r}$$

for $m < y < m+1$, $m = 1, 2, \ldots$ and so (1) holds. By c4) and the fact $0 \leq J_n(\epsilon)$ and by (1) just proved we obtain (2). Next, we show (3). If τ belongs to the interval $(0, 1/2]$ and if we take τ as ϵ_4 we see by c3) that

$$\sum_{j=1}^{k_n}\int_{|x|<\epsilon_4} x\, dF_{nj}(x) \to 0 = \sum_{n=1}^{\infty}\frac{p^n}{1+n^2} + 0 - \sum_{n=1}^{\infty}\frac{p^n}{1+n^2}$$

as $n \to \infty$. We obtain the condition (3) for $0 < \tau \leq 1/2$. If τ belongs to the interval $(1/2, 1)$, we also see by (1) that

$$\begin{aligned} \sum_{j=1}^{k_n}\int_{|x|<\tau} x\, dF_{nj}(x) &= \sum_{j=1}^{k_n}\int_{|x|<\epsilon_4} x\, dF_{nj}(x) + \sum_{j=1}^{k_n}\int_{\epsilon_4\leq x<\tau} x\, dF_{nj}(x) \\ &+\sum_{j=1}^{k_n}\int_{-\tau<x\leq-\epsilon_4} x\, dF_{nj}(x) \to 0 \quad (12) \end{aligned}$$

as $n \to \infty$. When $1 < \tau$ and $l < \tau < l+1$, we write

$$\sum_{j=1}^{k_n}\int_{|x|<\tau} x\, dF_{nj}(x) = A_n + B_n,$$

where

$$A_n = \sum_{j=1}^{k_n}\int_{|x|<\epsilon_4} x\, dF_{nj}(x) \quad\text{and}\quad B_n = \sum_{j=1}^{k_n}\int_{\epsilon_4\leq|x|<\tau} x\, dF_{nj}(x).$$

By c3) we have $A_n \to 0$. In order to prove (3) we show

$$B_n \to \sum_{n=1}^{l} p^n$$

as $n \to \infty$. We write B_n as follows,

$$\begin{aligned} B_n &= \sum_{j=1}^{k_n} \int_{\epsilon_4 \le |x| < \tau} x \, dF_{nj}(x) \\ &= \sum_{j=1}^{k_n} \int_{(-\tau,-\epsilon_4]} x \, dF_{nj}(x) + \sum_{j=1}^{k_n} \int_{[\epsilon_4,\tau)} x \, dF_{nj}(x). \end{aligned} \tag{13}$$

We see by (1) that

$$|\sum_{j=1}^{k_n} \int_{(-\tau,-\epsilon_4]} x \, dF_{nj}(x)| \le \tau \sum_{j=1}^{k_n} \big(F_{nj}(-\epsilon_4) - F_{nj}(-\tau)\big) \to 0,$$

and

$$0 \le \sum_{j=1}^{k_n} \int_{(\epsilon_4,1-\epsilon_2]} x \, dF_{nj}(x) \le (1-\epsilon_2) \sum_{j=1}^{k_n} \big(F_{nj}(1-\epsilon_2) - F_{nj}(\epsilon_4)\big) \to 0,$$

and

$$0 \le \sum_{j=1}^{k_n} \int_{(l+\epsilon_2,\tau)} x \, dF_{nj}(x) \le \tau \sum_{j=1}^{k_n} \big(F_{nj}(\tau) - F_{nj}(l+\epsilon_2)\big) \to 0,$$

where let $l + \epsilon_2 < \tau$, and if $m = 1, 2, \dots, l$,

$$\begin{aligned} &\sum_{j=1}^{k_n} \int_{(m+\epsilon_3,m+1-\epsilon_3]} x \, dF_{nj}(x) \\ &\le (m+1-\epsilon_3) \sum_{j=1}^{k_n} \big(F_{nj}(m+1-\epsilon_3) - F_{nj}(m+\epsilon_3)\big) \to 0, \end{aligned} \tag{14}$$

as $n \to \infty$. By (1) we obtain $\sum_{j=1}^{k_n} \int_{(m-\epsilon_3,m+\epsilon_3]} x \, dF_{nj}(x) \to p^m$, $n \to \infty$. In fact, we see by (1) that for sufficiently small s

$$\begin{aligned} \sum_{j=1}^{k_n} \int_{(m-\epsilon_3,m+\epsilon_3]} x \, dF_{nj}(x) &= \sum_{j=1}^{k_n} \int_{(m-\epsilon_3,m-s]} x \, dF_{nj}(x) \\ &+ \sum_{j=1}^{k_n} \int_{(m-s,m+s]} x \, dF_{nj}(x) + \sum_{j=1}^{k_n} \int_{(m+s,m+\epsilon_3]} x \, dF_{nj}(x). \end{aligned} \tag{15}$$

From the fact that

$$(m-s)\sum_{j=1}^{k_n}\int_{(m-s,m+s]} dF_{nj}(x) \le \sum_{j=1}^{k_n}\int_{(m-s,m+s]} x\ dF_{nj}(x)$$
$$\le (m+s)\sum_{j=1}^{k_n}\int_{(m-s,m+s]} dF_{nj}(x)$$

we obtain

$$(m-s)\frac{p^m}{m} \le \lim\inf_{n\to\infty}\sum_{j=1}^{k_n}\int_{(m-\epsilon_3,m+\epsilon_3]} x\ dF_{nj}(x)$$
$$\le \lim\sup_{n\to\infty}\sum_{j=1}^{k_n}\int_{(m-\epsilon_3,m+\epsilon_3]} x\ dF_{nj}(x) \le (m+s)\frac{p^m}{m},$$

and letting $s \downarrow 0$, then

$$\lim_{n\to\infty}\sum_{j=1}^{k_n}\int_{(m-\epsilon_3,m+\epsilon_3]} x\ dF_{nj}(x) = p^m.$$

From the above facts we see that

$$\sum_{j=1}^{k_n}\int_{\epsilon_4\le x<\tau} x\ dF_{nj}(x) = \sum_{j=1}^{k_n}\int_{(\epsilon_4,1-\epsilon_3]} x\ dF_{nj}(x) + \sum_{j=1}^{k_n}\int_{(1-\epsilon_3,1+\epsilon_3]} x\ dF_{nj}(x)$$
$$+\sum_{j=1}^{k_n}\int_{(1+\epsilon_3,2-\epsilon_3]} x\ dF_{nj}(x) + \sum_{j=1}^{k_n}\int_{(2-\epsilon_3,2+\epsilon_3]} x\ dF_{nj}(x) + \cdots$$
$$+\sum_{j=1}^{k_n}\int_{(l-1+\epsilon_3,l-\epsilon_3]} x\ dF_{nj}(x) + \sum_{j=1}^{k_n}\int_{(l-\epsilon_3,l+\epsilon_3]} x\ dF_{nj}(x)$$
$$+\sum_{j=1}^{k_n}\int_{(l+\epsilon_2,\tau)} x\ dF_{nj}(x) \to \sum_{m=1}^{l} p^m, \quad (16)$$

for $\epsilon_4 < 1-\epsilon_3$ and $l+\epsilon_2 < \tau$ as $n\to\infty$. This shows that

$$B_n \to \sum_{n=1}^{l} p^n$$

as $n \to \infty$ and this implies (3). Hence we obtain necessary and sufficient condtions c1), c2), c3), c4) for convergence of distribution function of the infinitesimal system of random variables to the geometric distribution. $\square$

4. An Example of Infinitesimal System of Random Variables

In this section we show an example of infinitesimal system of random variables of which sum converges to the geometric distributed random variable. Let $\{\{X_{nj}\}_{j=1,\cdots,n}\}$ be a sequence of row-wise independent identically distributed random variables with the negative-binomial distribution as the following probabilities,

$$P\{X_{nj} = r\} = \nu(\nu+1)\cdots(\nu+r-1)\frac{p^r}{r!}q^\nu,$$

where $r = 0, 1, 2, \cdots$ and $\nu = 1/n$ for positive integer n and $0 < p < 1,\ q = 1-p$. It is seen that for every $\eta > 0$

$$\max_{j=1,\ldots,n} P\{|X_{nj}| \geq \eta\} = 1 - q^\nu \to 0$$

as $n \to \infty$.

Theorem 2. *The system of independent identically distributed random variables $\{\{X_{nj}\}_{j=1,\cdots,n}\}_{n=1,2,\cdots}$ is the infinitesimal system of random variables and the sequence of distribution functions of sums of independent random variables*

$$Z_n = X_{n1} + X_{n2} + \cdots + X_{nn}$$

converges completely to the geometric distribution.

Proof. We will show that the conditions in Theorem 1 hold. Let us denote the negative-binomial distribution function by

$$F(n;x) = q^\nu \sum_{0\leq r\leq x} \nu(\nu+1)\cdots(\nu+r-1)\frac{p^r}{r!},\ (r = 0, 1, 2, \cdots).$$

On the condition c1) α) we obtain

$$\sum_{j=1}^{n} F_{nj}(-\epsilon_1) = nF(n; -\epsilon_1) = 0$$

for every ϵ_1, where $0 < \epsilon_1 \leq 1/2$. On the condition c1) β) we obtain that for every ϵ_2, where $0 < \epsilon_2 \leq 1/2$,

$$\begin{aligned} M_n(\epsilon_2) &= \sum_{j=1}^{n}(F_{nj}(\epsilon_2) - 1) = -n(1 - F(n; \epsilon_2)) \\ &= -n(1 - q^{1/n}) \to \log\ q = -\sum_{r=1}^{\infty}\frac{p^r}{r} = M(\epsilon_2) \quad (17) \end{aligned}$$

as $n \to\infty$. We see that for every ϵ_3, where $0 < \epsilon_3 \leq 1/2$, and $r = 1, 2, \ldots,$

$$\sum_{j=1}^{n}\Big(F_{nj}(r+1-\epsilon_3) - F_{nj}(r+\epsilon_3)\Big) = n\Big(F(n; r+1-\epsilon_3) - F(n; r+\epsilon_3)\Big) = 0.$$

From the above facts we see that c2) α') hold. On the condition c2) β') we obtain

$$\begin{aligned}\sum_{j=1}^{n}\Big(F_{nj}(m+\epsilon_3) - F_{nj}(m-\epsilon_3)\Big) &= n\Big(F(n; m+\epsilon_3) - F(n; m-\epsilon_3)\Big)\\ &= nq^{\nu}\nu(\nu+1)\cdots(\nu+m-1)\frac{p^m}{m!} \to \frac{p^m}{m},\end{aligned} \tag{18}$$

for every ϵ_3, where $0 < \epsilon_3 \leq 1/2$, as $n \to \infty$. On the condition c3) we see that

$$\sum_{j=1}^{n}\int_{|x|<\epsilon_4} x\, dF_{nj}(x) = n\int_{|x|<\epsilon_4} x\, dF(n; x) = n \cdot 0 \cdot q^{\nu} = 0.$$

On the condition c4), from the above fact, we see that for every ϵ $(0 < \epsilon \leq 1/2)$

$$\begin{aligned}&\sum_{j=1}^{n}\Big(\int_{|x|<\epsilon} x^2\, dF_{nj}(x) - \Big(\int_{|x|<\epsilon} x\, dF_{nj}(x)\Big)^2\Big)\\ &\qquad = n\Big(\int_{|x|<\epsilon} x^2\, dF(n; x) - \Big(\int_{|x|<\epsilon} x\, dF(n; x)\Big)^2\Big) = 0.\end{aligned}$$

We see that the conditions c1), c2), c3), c4) are satisfied, and we see that the sequence of probability distributions of the above infinitesimal system of independent random variables converges to the geometric distribution. □

Let us denote

$$\begin{aligned}G_n(x) &= n\int_{-\infty}^{x} \frac{t^2}{1+t^2} dF(n; t)\\ &= n\sum_{1\leq r\leq x} \frac{r^2}{1+r^2} q^{\nu}\nu(\nu+1)\cdots(\nu+r-1)\frac{p^r}{r!}\end{aligned} \tag{19}$$

and

$$\gamma_n = \int_{+0}^{\infty} \frac{1}{x} dG_n(x) = n\sum_{1\leq r<\infty} \frac{r}{1+r^2} q^{\nu}\nu(\nu+1)\cdots(\nu+r-1)\frac{p^r}{r!}. \tag{20}$$

Then we can easily show that for $x \in \operatorname{Cont} G$

$$G_n(x) \to G(x) = \sum_{1\leq r\leq x} \frac{r}{1+r^2} p^r$$

and

$$\gamma_n \to \gamma$$

as $n \to \infty$. This also shows that $F_{Z_n} \longrightarrow F$ completely as $n \to \infty$, where F is the geometric distribution function with expectation p/q.

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