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p-FAT POINTS OF A PROJECTIVE SPACE

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Abstract: Fix an algebraically closed base field $\mathbb{K}$ such that $p := \mathrm{char}(\mathbb{K}) > 0$, a p-power q and any $P \in \mathbf{P}^n$. Let $[q,n]P$ denote the closed subscheme of $\mathbf{P}^n$ whose ideal is generated by all L^q, where L is a homogeneous degree 1 form vanishing at P. Hence $([q,n]P)_{red} = \{P\}$ and $\mathrm{length}([q,n]P) = q^n$. Here we study the postulation of zero-dimensional schemes $Z = \sqcup [q_i,n]P_i$.

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Fix an algebraically closed base field $\mathbb{K}$ such that $p := \mathrm{char}(\mathbb{K}) > 0$, a p-power q and any $P \in \mathbf{P}^n$. Let $[q,n]P$ denote the closed subscheme of $\mathbf{P}^n$ whose ideal is generated by all L^q, where L is a homogeneous degree 1 form vanishing at P. Hence $([q,n]P)_{red} = \{P\}$. Since $(L+M)^q = L^q + M^q$ for all linear forms L, M, $[q,n]P$ is defined by the vanishing of the q-powers of any n linearly independent linear forms vanishing at P. Fix homogeneous coordinates $x_0, \dots, x_n$ such that $P = (1;0,\dots;0)$. We get that $[q,n]P$ is defined by the monomial equations ${x_i}^q = 0$, $1 \le i \le q$. Thus $\mathrm{length}([q,n]P) = q^n$. We will say that $[q,n]P$ is a p-fat point of $\mathbf{P}^n$ with multiplicity q. Let H be a hyperplane containing P. For any scheme $Z \subset \mathbf{P}^n$ let $\mathrm{Res}_H(Z)$ denote the residual scheme of Z with respect to H. If Z is zero-dimensional, then $\mathrm{lenght}(Z) = \mathrm{length}(Z \cap H) + \mathrm{length}(\mathrm{Res}_H(Z))$. Set $\mathrm{Res}^0_H(Z) = Z$ and $\mathrm{Res}^1_H(Z) = \mathrm{Res}_H(Z)$. For every integer $t \ge 2$ define the closed subscheme $\mathrm{Res}^t_H(Z)$ of $\mathbf{P}^n$ using the inductive formula $\mathrm{Res}^t_H(Z) := \mathrm{Res}_H(\mathrm{Res}^{t-1}_H(Z))$. For every integer t such that $0 \le t \le$

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$q-1$ we have length$(\mathrm{Res}^t_H([q,n]P) = q^n - tq^{n-1}$ and length$(\mathrm{Res}^t_H(H \cap [q,n]P) = q^{n-1}$. $\mathrm{Res}^q_H([q,n]P) = \emptyset$. Here we prove the following results.

Theorem 1. *Fix integers $n \geq 2$, $s > 0$ and s p-powers $q_1 \geq \cdots \geq q_s > 0$. Let $Z \subset \mathbf{P}^n$ be any disjoint union of s p-fat points with multiplicity $q_1, \ldots, q_s$. Then $h^1(\mathbf{P}^n, \mathcal{O}_Z(t)) = 0$ for all $t \geq 2q_1 + \cdots + q_s - 2$.*

Theorem 2. *Fix an integer $s > 0$ and a p-power q. Let $Z \subset \mathbf{P}^2$ be a general union of s p-fat points of $\mathbf{P}^2$ with multiplicity q. Let c be the minimal posive integer such that $c(c+1)/2 \geq s$. Then $h^1(\mathbf{P}^2, \mathcal{O}_Z(t)) = 0$ for all $t \geq (c+1)q - 2$.*

Proof of Theorem 1. First assume the existence of a line D such that $Z_{red} \subset D$. Fix a hyperplane H such that $D \subseteq H$ and use q_1 times Horace Lemma with respect to H. We use an inductive statement with respect to cubical fat points with respect to H. Notice that $\mathrm{Res}^{q_1}_H(Z) = \emptyset$. Now consider the general case. Fix a general $P \in \mathbf{P}^n$ and a hyperplane $M \subset \mathbf{P}^n$ such that $P \notin M$. In particular we assume $P \notin Z_{red}$. Fix homogeneous coordinates $x_0, \ldots, x_n$ such that $P = (1; 0, \cdots, 0)$ and $H = \{x_0 = 0\}$. For every $\lambda \in \mathbb{K}\backslash\{0\}$ let $h_\lambda : \mathbf{P}^n \to \mathbf{P}^n$ by defined by the formula $h_\lambda(x_0; x_1; \ldots; x_n) = (\lambda x_0; x_1; \cdots; x_n)$. Since Z and $h_\lambda(Z)$ are projectively equivalent, their ideal sheaves have the same cohomology. The flat family $\{h_\lambda(Z)\}_{\lambda \in \mathbb{K}\backslash\{0\}}$ has a flat limit for λ going to 0 a disjoint union W of s p-fat points of $\mathbf{P}^n$ with multiplicities $q_1, \ldots, q_s$ and $W_{red} \subset M$. By semicontinuity we have $h^1(\mathbf{P}^n, \mathcal{O}_Z(t)) \leq h^1(\mathbf{P}^n, \mathcal{O}_W(t))$ for all t. If $n = 2$, then we are done. If $n \geq 3$, we use $n-2$ further linear projections and flat limits to reduce to the case in which the support is contained in a line. □

Proof of Theorem 2. Fix a line $D \subset \mathbf{P}^2$ and call W the union c p-fat points of $\mathbf{P}^2$ with multiplicity q and whose support is contained in D. Hence $h^1(\mathbf{P}^2, \mathcal{I}_W(t)) = 0$ for all $t \geq (c+1)q - 2$ (Theorem 1) and $\mathrm{Res}^q_D(W) = \emptyset$. Then we insert a union $c-1$ p-fat points of $\mathbf{P}^2$ with multiplicity q and whose support is contained in D and apply again Horace Lemma q times. And so on. □

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