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P(EC)^{N}E-MODES FOR SOLVING K-TH ORDER FUZZY DIFFERENTIAL EQUATIONS

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Abstract: In this paper a particular numerical algorithm of P(EC)^{N}E-modes for solving initial value problem (IVP) of k-th order fuzzy differential equations (FDE) is presented. First of all we will transform an IVP of k-th order fuzzy differential equation to an IVP of the system of first order fuzzy differential equations, then we solve this system, numerically. After that we obtain the approximate solution of the k-th order FDE. Also we will prove that the algorithm converges to the exact solution as the stepsize goes to zero, and at the end the validity of the algorithm is illustrated by solving some examples.

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1. Introduction

A fuzzy subset of $\mathcal{R}$, the set of all real numbers, is a function $u : \mathcal{R} \to [0, 1]$. For each $r \in (0, 1]$ the $r-$ level set of u is defined by

$$[u]_r = \{x \in \mathcal{R} : u(x) \geq r\},$$

and the support of u is defined as the closure of the union of all its level sets,

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that is

$$[u]_0 = \overline{\bigcup_{r\in(0,1]} [u]_r}.$$

We denote by E the set of all fuzzy subset of $\mathcal{R}$ which are normal, convex and upper semicontinuous with bounded $r-$ level sets [5]. It means that, if $u \in E$, then $[u]_r$ is a closed bounded interval which is denoted by

$$[u]_r = [\underline{u}(r), \overline{u}(r)], \quad r \in [0,1].$$

The elements of E are called fuzzy numbers. Assume that I is a subinterval of $\mathcal{R}$. The mapping $y : I \to E$ is called a fuzzy process and for each $x \in I$ its $r-$ level set is denoted by

$$[y(x)]_r = [\underline{y}(x;r), \overline{y}(x;r)], \quad r \in [0,1].$$

The first derivative $y'(x)$ of $y(x)$ is defined by

$$[y'(x)]_r = [\underline{y}'(x;r), \overline{y}'(x;r)], \quad x \in I,\ r \in [0,1],$$

provided that this equation determines a fuzzy number, according to Seikkala [9]. Similarly the j-th derivative, $y^{(j)}$, of y is defined, where j is a natural number. In this paper we will use triangular and triangular shape fuzzy numbers (see [2,5]).

In [6] and [7] the IVP of first-order fuzzy differential equations has solved, numerically, by the Euler method. In this paper we will solve the IVP of k-th order fuzzy differential equations by the P(EC)^N E-modes.

Consider the IVP of k-th order fuzzy differential equation given by

$$\begin{cases} y^{(k)}(x) = f(x, y(x), y'(x), \ldots, y^{(k-1)}(x)), & x \in [x_0, \overline{x}], \\ y(x_0) = \alpha_1, \ldots, y^{(k-1)}(x_0) = \alpha_k, \end{cases} \tag{1.1}$$

where $y(x)$ is a fuzzy function of the crisp variable x and for $1 \le j \le k$, $y^{(j)}$ is the j-th fuzzy derivative of y, α_j is a fuzzy number and $f(x, y, y', \ldots, y^{(k-1)})$ is a fuzzy function of the crisp variable x and fuzzy variables $y, y', \cdots, y^{(k-1)}$.

Since the FIVP (1.1) can be transformed to an IVP of a system of first-order FDE (see [10]), it is sufficient to solve the IVP of the system of first-order FDE, in general; that is

$$\begin{cases} u_1'(x) &= f_1(x, u_1(x), \ldots, u_k(x)), \\ & \vdots \\ u_k'(x) &= f_k(x, u_1(x), \ldots, u_k(x)), \\ u_1(x_0) = \alpha_1, \ldots, u_k(x_0) = \alpha_k, \end{cases} \tag{1.2}$$

where $x \in [x_0, \overline{x}]$ and for $j, 1 \le j \le k, u_j(x)$ is a fuzzy function of crisp variable

x, $f_j(x, u_1, \ldots, u_k)$ is a fuzzy function of the crisp variable x and fuzzy variables u_j, α_j is a fuzzy number and $u_j^{'}$ is the derivative of fuzzy function u_j.

We will denote a fuzzy k-vector as $a = (a_1, \ldots, a_k)^t$, where a_j, for $1 \leq j \leq k$, is a fuzzy number and its lower $r-$ level set is denoted as

$$\underline{a}(r) = (\underline{a}_1(r), \ldots, \underline{a}_k(r))^t, \quad 0 \leq r \leq 1.$$

Similarly for r, $0 \leq r \leq 1$, $\overline{a}(r) = (\overline{a}_1(r), \ldots, \overline{a}_k(r))^t$ is the upper $r-$ level set of fuzzy $k-$ vector a. Now, by the above notations, (1.2) will be as the following FIVP:

$$\begin{cases} u'(x) = f(x, u(x)), & x \in [x_0, \overline{x}], \\ u(x_0) = \alpha, \end{cases} \tag{1.3}$$

where $u = (u_1, \ldots, u_k)^t$, $u^{'} = (u_1^{'}, \ldots, u_k^{'})^t$, $f = (f_1, \ldots, f_k)^t$, $\alpha = (\alpha_1, \ldots, \alpha_k)^t$.

By using r-levels (1.3) can be written as:

$$\begin{cases} \underline{u}'(x; r) = \underline{\underline{f}}(x, u(x); r), & \underline{u}(x_0; r) = \underline{\alpha}(r), \\ \overline{u}'(x; r) = \overline{\overline{f}}(x, u(x); r), & \overline{u}(x_0; r) = \overline{\alpha}(r). \end{cases}$$

Then we have

$$\begin{cases} \underline{u}_1^{'}(x; r) = \underline{f}_1(x, u_1(x), \ldots, u_k(x); r), \\ \qquad \vdots \\ \underline{u}_k^{'}(x; r) = \underline{\underline{f}}_k(x, u_1(x), \ldots, u_k(x); r), \\ \overline{u}_1^{'}(x; r) = \overline{\overline{f}}_1(x, u_1(x), \ldots, u_k(x); r), \\ \qquad \vdots \\ \overline{u}_k^{'}(x; r) = \overline{f}_k(x, u_1(x), \ldots, u_k(x); r). \end{cases} \tag{1.4}$$

By the extension principle [2], there exist functions $F_1, \ldots, F_k, G_1, \ldots, G_k$ such that (1.4) can be written as:

$$\begin{cases} \underline{u}_1^{'}(x; r) = F_1(x, \underline{u}_1(x; r), \ldots, \underline{u}_k(x; r), \overline{u}_1(x; r), \ldots, \overline{u}_k(x; r)), \\ \qquad \vdots \\ \underline{u}_k^{'}(x; r) = F_k(x, \underline{u}_1(x; r), \ldots, \underline{u}_k(x; r), \overline{u}_1(x; r), \ldots, \overline{u}_k(x; r)), \\ \overline{u}_1^{'}(x; r) = G_1(x, \underline{u}_1(x; r), \ldots, \underline{u}_k(x; r), \overline{u}_1(x; r), \ldots, \overline{u}_k(x; r)), \\ \qquad \vdots \\ \overline{u}_k^{'}(x; r) = G_k(x, \underline{u}_1(x; r), \ldots, \underline{u}_k(x; r), \overline{u}_1(x; r), \ldots, \overline{u}_k(x; r)). \end{cases}$$

Therefore, we obtain

$$\begin{cases} \underline{u}'(x; r) = F(x, \underline{u}(x; r), \overline{u}(x; r)), \\ \overline{u}'(x; r) = G(x, \underline{u}(x; r), \overline{u}(x; r)), \\ \underline{u}(x_0; r) = \underline{\alpha}(r), \quad \overline{u}(x_0; r) = \overline{\alpha}(r), \end{cases} \tag{1.5}$$

where $F = (F_1, \ldots, F_k)^t$, $G = (G_1, \ldots, G_k)^t$.

Assume that $\widetilde{N}$ is a natural number and $h = \frac{\overline{x}-x_0}{\widetilde{N}}$. Remember that the IVP of the system of first-order ordinary differential equations given by

$$\begin{cases} u'(x) = f(x, u(x)), & x \in [x_0, \overline{x}], \\ u(x_0) = \alpha, \end{cases}$$

can be solved, numerically, by 2-step P(EC)^N E-modes (see [4]).

The 2-step P(EC)^N E-modes consist of a pair of linear 2-step methods [4] as the following:

Predictor method:

$$u_n^{(0)} = \alpha_1 u_{n-1} + \alpha_2 u_{n-2} + h\beta_1 f(x_{n-1}, u_{n-1}) + h\beta_2 f(x_{n-2}, u_{n-2}), \quad (1.6)$$

Corrector method:

$$u_n^{(j)} = \widetilde{\alpha}_1 u_{n-1} + \widetilde{\alpha}_2 u_{n-2} + h\widetilde{\beta}_0 f(x_n, u_n^{(j-1)}) + h\widetilde{\beta}_1 f(x_{n-1}, u_{n-1}) + h\widetilde{\beta}_2 f(x_{n-2}, u_{n-2}), \quad (1.7)$$

where N is a natural number, $1 \le j \le N$, and $u_n = u_n^{(N)}, 2 \le n \le \widetilde{N}$. In this paper we will use the 2-step P(EC)^N E-modes given by (1.6)-(1.7) for solving FIVP, where $\alpha_i, \beta_i, \widetilde{\alpha}_i, \widetilde{\beta}_i$, $1 \le i \le 2$, are all nonnegative and $\widetilde{\beta}_0 > 0$. Note that a linear 2-step method of the form

$$u_n = \alpha_1 u_{n-1} + \alpha_2 u_{n-2} + h\beta_0 f(x_n, u_n) + h\beta_1 f(x_{n-1}, u_{n-1}) + h\beta_2 f(x_{n-2}, u_{n-2}), \quad (1.8)$$

is of order q [4] if

$$\alpha_1 + \alpha_2 = 1, \quad \alpha_1 + 2\alpha_2 = \beta_0 + \beta_1 + \beta_2,$$
$$\alpha_1 + 2^j \alpha_2 = j(\beta_1 + 2^{j-1}\beta_1), \quad j = 2, \ldots, q.$$

At the end of this section we state two lemmas which are employed in Section 2.

Lemma 1.1. *Let a sequence $\{P_n\}_{n=0}^{\widetilde{N}}$ of nonnegative numbers satisfies*

$$P_n \le AP_{n-1} + BP_{n-2} + C,$$

for some given positive constant A, B and C. Then

$$P_n + (\alpha - A)P_{n-1} \le \alpha^{n-1}[P_1 + (\alpha - A)P_0] + C\frac{\alpha^{n-1} - 1}{\alpha - 1},$$

where $\alpha = \frac{(A^2+4B)^{1/2}+A}{2}$.

Proof. See [8]. □

Lemma 1.2. *If the vector function $u \in C^{q+1}[x_{n-2}, x_n]$ and the order of the linear 2-step method given by (1.8) is q, then*

$$u(x) = \alpha_1 u(x_{n-1}) + \alpha_2 u(x_{n-2}) + h\beta_0 u'(x_n) + h\beta_1 u'(x_{n-1}) + h\beta_2 u'(x_{n-2})$$
$$- \sum_{i=1}^{2} \alpha_i \frac{u^{(q+1)}(\xi_i) p_{q+1}(x_{n-i})}{(q+1)!} - h \sum_{i=1}^{2} \beta_i \frac{u^{(q+1)}(\eta_i) p'_{q+1}(x_{n-i})}{(q+1)!}$$

where $p_{q+1}(x) = (x - x_n)^{q+1}$, ξ_i and η_i are between x_{n-i} and x_n, $1 \le i \le 2$.

Proof. See [4]. □

2. 2-Step $\mathrm{P(EC)}^N$ E-Modes for Solving the System of First-Order Fuzzy Differential Equations

Consider the IVP of the system of first-order FDE given by

$$\begin{cases} u'(x) = f(x, u(x)), & x \in [x_0, \overline{x}], \\ u(x_0) = \alpha. \end{cases} \tag{2.1}$$

First of all note that the equations (1.6) and (1.7) are fuzzified by equations

$$\underline{u}_n^{(0)}(r) = \alpha_1 \underline{u}_{n-1}(r) + \alpha_2 \underline{u}_{n-2}(r) + h\beta_1 \underline{f}(x_{n-1}, u_{n-1}; r) + h\beta_2 \underline{f}(x_{n-2}, u_{n-2}; r), \tag{2.2}$$

$$\overline{u}_n^{(0)}(r) = \alpha_1 \overline{u}_{n-1}(r) + \alpha_2 \overline{u}_{n-2}(r) + h\beta_1 \overline{f}(x_{n-1}, u_{n-1}; r) + h\beta_2 \overline{f}(x_{n-2}, u_{n-2}; r), \tag{2.3}$$

$$\underline{u}_n^{(j)}(r) = \widetilde{\alpha}_1 \underline{u}_{n-1}(r) + \widetilde{\alpha}_2 \underline{u}_{n-2}(r) + h\widetilde{\beta}_0 \underline{f}(x_n, u_n^{(j-1)}; r) + h\widetilde{\beta}_1 \underline{f}(x_{n-1}, u_{n-1}; r) + h\widetilde{\beta}_2 \underline{f}(x_{n-2}, u_{n-2}; r), \tag{2.4}$$

$$\overline{u}_n^{(j)}(r) = \widetilde{\alpha}_1 \overline{u}_{n-1}(r) + \widetilde{\alpha}_2 \overline{u}_{n-2}(r) + h\widetilde{\beta}_0 \overline{f}(x_n, u_n^{(j-1)}; r) + h\widetilde{\beta}_1 \overline{f}(x_{n-1}, u_{n-1}; r) + h\widetilde{\beta}_2 \overline{f}(x_{n-2}, u_{n-2}; r), \tag{2.5}$$

where $1 \le j \le N$, $\underline{u}_n(r) = \underline{u}_n^{(N)}(r)$, and $\overline{u}_n(r) = \overline{u}_n^{(N)}(r)$, $n = 2, \cdots, \widetilde{N}$.

In order to start with the 2-step $\mathrm{P(EC)}^N$ E-modes we will use the fuzzy Euler method [6] to find $\underline{u}_1(r)$ and $\overline{u}_1(r)$. Now we prove the convergence of the 2-step $\mathrm{P(EC)}^N$ E-modes for solving FIVP.

Theorem 2.1. *Let $K = \{(x, u_1, \ldots, u_k, v_1, \ldots, v_k) : x_0 \le x \le \overline{x}, -\infty < u_j \le v_j, -\infty < v_j < \infty, 1 \le j \le k\}$ and suppose that $F(x, U, V)$ and $G(x, U, V)$*

are in $C^{\overline{N}}(K)$*, where* $U = (u_1, \ldots, u_k), V = (v_1, \ldots, v_k)$*,* $\overline{N} = \max(m, \widetilde{m})$*,* m *and* $\widetilde{m}$ *are the orders of predictor and corrector methods given by (1.6) and (1.7), respectively. Also assume that the partial derivatives of* F *and* G *are bounded on* K*, and*

$$\begin{cases} \| F(x, \underline{U}, \overline{U}) - F(x, \underline{V}, \overline{V}) \| \leq L \max\{\| \underline{U} - \underline{V} \|, \| \overline{U} - \overline{V} \|\}, \\ \| G(x, \underline{U}, \overline{U}) - G(x, \underline{V}, \overline{V}) \| \leq L \max\{\| \underline{U} - \underline{V} \|, \| \overline{U} - \overline{V} \|\}, \end{cases} \tag{2.6}$$

where L is a positive constant and the norm in use is $\ell_\infty-$ *norm. Then for fixed r,* $0 \leq r \leq 1$*,*

$$\lim_{h \to 0} \underline{u}_{\widetilde{N}}(r) = \underline{u}(x_{\widetilde{N}}; r), \ \lim_{h \to 0} \overline{u}_{\widetilde{N}}(r) = \overline{u}(x_{\widetilde{N}}; r).$$

Proof. Assume that $w_n = \|\underline{u}_n(r) - \underline{u}(x_n; r)\|$, $v_n = \|\overline{u}_n(r) - \overline{u}(x_n; r)\|$ and $z_n = v_n + w_n$, where $0 \leq r \leq 1$ and $n = 0, 1, \ldots, \widetilde{N}$. Suppose that $u(x)$ is the exact solution of FIVP (2.1). Then by Lemma 1.2 for corrector method, (1.5) and (2.4) we have

$$\begin{aligned} &\| \underline{u}_n^{(1)}(r) - \underline{u}(x_n; r) \| \\ &\quad \leq \widetilde{\alpha}_1 \| \underline{u}_{n-1}(r) - \underline{u}(x_{n-1}; r) \| + \widetilde{\alpha}_2 \| \underline{u}_{n-2}(r) - \underline{u}(x_{n-2}; r) \| \\ &\quad + h\widetilde{\beta}_0 \| F(x_n, \underline{u}_n^{(0)}(r), \overline{u}_n^{(0)}(r)) - F(x_n, \underline{u}(x_n; r), \overline{u}(x_n; r)) \| \\ &\quad + h\widetilde{\beta}_1 \| F(x_{n-1}, \underline{u}_{n-1}(r), \overline{u}_{n-1}(r)) - F(x_{n-1}, \underline{u}(x_{n-1}; r), \overline{u}(x_{n-1}; r)) \| \\ &\quad + h\widetilde{\beta}_2 \| F(x_{n-2}, \underline{u}_{n-2}(r), \overline{u}_{n-2}(r)) - F(x_{n-2}, \underline{u}(x_{n-2}; r), \overline{u}(x_{n-2}; r)) \| \\ &\quad + \sup_{x \in [x_{n-2}, x_n]} \| \underline{u}^{(\widetilde{m}+1)}(x; r) \| \widetilde{M}.H^{\widetilde{m}+1}, \end{aligned}$$

where $\widetilde{M} = \frac{1}{(\widetilde{m}+1)!} \sum_{i=1}^{2} (\widetilde{\alpha}_i) i^{\widetilde{m}+1} + \frac{1}{\widetilde{m}!} \sum_{i=1}^{2} (\widetilde{\beta}_i) i^{\widetilde{m}}$. Then by (2.6) we obtain

$$\begin{aligned} \| \underline{u}_n^{(1)}(r) - \underline{u}(x_n; r) \| &\leq \widetilde{\alpha}_1 w_{n-1} + \widetilde{\alpha}_2 w_{n-2} \\ &+ hL\widetilde{\beta}_0 \max(\| \underline{u}_n^{(0)}(r) - \underline{u}(x_n; r) \|, \| \overline{u}_n^{(0)}(r) - \overline{u}(x_n; r) \|) \\ &+ hL\widetilde{\beta}_1 \max(\| \underline{u}_{n-1}(r) - \underline{u}(x_{n-1}; r) \|, \| \overline{u}_{n-1}(r) - \overline{u}(x_{n-1}; r) \|) \\ &+ hL\widetilde{\beta}_2 \max(\| \underline{u}_{n-2}(r) - \underline{u}(x_{n-2}; r) \|, \| \overline{u}_{n-2}(r) - \overline{u}(x_{n-2}; r) \|) \\ &+ \underline{M}(r).\widetilde{M}.H^{\widetilde{m}+1}, \end{aligned}$$

where $\underline{M}(r) = \sup_{x \in [x_{n-2}, x_n]} \| \underline{u}^{(\widetilde{m}+1)}(x; r) \|$. Therefore we have

$$\begin{aligned} &\| \underline{u}_n^{(1)}(r) - \underline{u}(x_n; r) \| \leq \widetilde{\alpha}_1 w_{n-1} + \widetilde{\alpha}_2 w_{n-2} + hL\widetilde{\beta}_0 \{\| \underline{u}_n^{(0)}(r) - \underline{u}(x_n; r) \| \\ &\quad + \| \overline{u}_n^{(0)}(r) - \overline{u}(x_n; r) \|\} + hL\widetilde{\beta}_1 z_{n-1} + hL\widetilde{\beta}_2 z_{n-2} + \underline{M}(r).\widetilde{M}.H^{\widetilde{m}+1}. \end{aligned} \tag{2.7}$$

Similarly by Lemma 1.2 for corrector method, (1.5), (2.5) and (2.6) we obtain

$$\| \overline{u}_n^{(1)}(r) - \overline{u}(x_n;r) \| \leq \widetilde{\alpha}_1 v_{n-1} + \widetilde{\alpha}_2 v_{n-2} + hL\widetilde{\beta}_0 \{ \| \underline{u}_n^{(0)}(r) - \underline{u}(x_n;r) \| + \| \overline{u}_n^{(0)}(r) - \overline{u}(x_n;r) \| \} + hL\widetilde{\beta}_1 z_{n-1} + hL\widetilde{\beta}_2 z_{n-2} + \overline{M}(r).\widetilde{M}.H^{\widetilde{m}+1}, \quad (2.8)$$

where $\overline{M}(r) = \sup_{x\in[x_{n-2},x_n]} \| \overline{u}^{(\widetilde{m}+1)}(x;r) \|$. By (2.7) and (2.8) we have

$$\| \underline{u}_n^{(1)}(r) - \underline{u}(x_n;r) \| + \| \overline{u}_n^{(1)}(r) - \overline{u}(x_n;r) \| \leq (\widetilde{\alpha}_1 + 2hL\widetilde{\beta}_1) z_{n-1} + (\widetilde{\alpha}_2 + 2hL\widetilde{\beta}_2) z_{n-2} + 2hL\widetilde{\beta}_0 \{ \| \underline{u}_n^{(0)}(r) - \underline{u}(x_n;r) \| + \| \overline{u}_n^{(0)}(r) - \overline{u}(x_n;r) \| \} + M(r).\widetilde{M}.H^{\widetilde{m}+1},$$

where $M(r) = \underline{M}(r) + \overline{M}(r)$. By induction, we can easily prove that

$$\begin{aligned}\| \underline{u}_n^{(N)}(r) - \underline{u}(x_n;r) \| &+ \| \overline{u}_n^{(N)}(r) - \overline{u}(x_n;r) \| \\ &\leq \{1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}\}(\widetilde{\alpha}_1 + 2hL\widetilde{\beta}_1) z_{n-1} \\ &+ \{1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}\}(\widetilde{\alpha}_2 + 2hL\widetilde{\beta}_2) z_{n-2} \\ &+ (2hL\widetilde{\beta}_0)^N \{ \| \underline{u}_n^{(0)}(r) - \underline{u}(x_n;r) \| + \| \overline{u}_n^{(0)}(r) - \overline{u}(x_n;r) \| \} \\ &+ \{1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}\} M(r).\widetilde{M}.H^{\widetilde{m}+1}. \quad (2.9)\end{aligned}$$

By Lemma 1.2 for predictor method, (1.5) and (2.2) we obtain

$$\begin{aligned}\| \underline{u}_n^{(0)}(r) - \underline{u}(x_n;r) \| &\leq \alpha_1 w_{n-1} + \alpha_2 w_{n-2} \\ &+ h\beta_1 \| F(x_{n-1}, \underline{u}_{n-1}(r), \overline{u}_{n-1}(r)) - F(x_{n-1}, \underline{u}(x_{n-1};r), \overline{u}(x_{n-1};r)) \| \\ &+ h\beta_2 \| F(x_{n-2}, \underline{u}_{n-2}(r), \overline{u}_{n-2}(r)) - F(x_{n-2}, \underline{u}(x_{n-2};r), \overline{u}(x_{n-2};r)) \| \\ &+ \sup_{x\in[x_{n-2},x_n]} \| \underline{u}^{(m+1)}(x;r) \| .H.H^{m+1},\end{aligned}$$

where $H = \frac{1}{(m+1)!}\sum_{i=1}^{2}(\alpha_i) i^{m+1} + \frac{1}{m!}\sum_{i=1}^{2}(\beta_i) i^m$.

Assume that $\underline{J}(r) = \sup_{x\in[x_{n-2},x_n]} \| \underline{u}^{(m+1)}(x;r) \|$, then by (2.6) we have

$$\begin{aligned}\| \underline{u}_n^{(0)}(r) - \underline{u}(x_n;r) \| &\leq \alpha_1 w_{n-1} + \alpha_2 w_{n-2} \\ &+ hL\beta_1 \max(\| \underline{u}_{n-1}(r) - \underline{u}(x_{n-1};r) \|, \| \overline{u}_{n-1}(r) - \overline{u}(x_{n-1};r) \|) + hL\beta_2 \\ &\times \max(\| \underline{u}_{n-2}(r) - \underline{u}(x_{n-2};r) \|, \| \overline{u}_{n-2}(r) - \overline{u}(x_{n-2};r) \|) + \underline{J}(r).H.H^{m+1}.\end{aligned}$$

Hence we get

$$\| \underline{u}_n^{(0)}(r) - \underline{u}(x_n;r) \| \leq \alpha_1 w_{n-1} + \alpha_2 w_{n-2} + hL\beta_1(w_{n-1} + v_{n-1}) + hL\beta_2(w_{n-2} + v_{n-2}) + \underline{J}(r).H.H^{m+1},$$

and then we obtain

$$\| \underline{u}_n^{(0)}(r) - \underline{u}(x_n;r) \| \leq \alpha_1 w_{n-1} + \alpha_2 w_{n-2} + hL\beta_1 z_{n-1} + hL\beta_2 z_{n-2} + \underline{J}(r).H.H^{m+1}. \quad (2.10)$$

Similarly, from Lemma 1.2 for predictor method, (1.5), (2.3) and (2.6) we obtain

$$\| \overline{u}_n^{(0)}(r) - \overline{u}(x_n;r) \| \leq \alpha_1 v_{n-1} + \alpha_2 v_{n-2} + hL\beta_1 z_{n-1} + hL\beta_2 z_{n-2} + \overline{J}(r).H.H^{m+1}, \quad (2.11)$$

where $\overline{J}(r) = \sup_{x\in[x_{n-2},x_n]} |\overline{u}^{(m+1)}(x;r)|$. Therefore by (2.10) and (2.11) we get

$$\| \underline{u}_n^{(0)}(r) - \underline{u}(x_n;r) \| + \| \overline{u}_n^{(0)}(r) - \overline{u}(x_n;r) \| \leq (\alpha_1 + 2hL\beta_1)z_{n-1} + (\alpha_2 + 2hL\beta_2)z_{n-2} + J(r).H.H^{m+1}, \quad (2.12)$$

where $J(r) = \underline{J}(r) + \overline{J}(r)$. Now (2.9) and (2.12) give the following

$$\begin{aligned}
&\| \underline{u}_n^{(N)}(r) - \underline{u}(x_n;r) \| + \| \overline{u}_n^{(N)}(r) - \overline{u}(x_n;r) \| \\
&\leq \{1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}\}(\widetilde{\alpha}_1 + 2hL\widetilde{\beta}_1)z_{n-1} \\
&+ \{1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}\}(\widetilde{\alpha}_2 + 2hL\widetilde{\beta}_2)z_{n-2} \\
&+ (2hL\widetilde{\beta}_0)^N\{(\alpha_1 + 2hL\beta_1)z_{n-1} + (\alpha_2 + 2hL\beta_2)z_{n-2} + J(r).H.H^{m+1}\} \\
&+ \{1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}\}M(r).\widetilde{M}.H^{\widetilde{m}+1}.
\end{aligned}$$

Since $\underline{u}_n(r) = \underline{u}_n^{(N)}(r)$ and $\overline{u}_n(r) = \overline{u}_n^{(N)}(r)$, then we have

$$\begin{aligned}
z_n \leq\ &\{[1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}] \\
&\times (\widetilde{\alpha}_1 + 2hL\widetilde{\beta}_1) + (2hL\widetilde{\beta}_0)^N(\alpha_1 + 2hL\beta_1)\}z_{n-1} \\
&+ \{[1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}] \\
&\times (\widetilde{\alpha}_2 + 2hL\widetilde{\beta}_2) + (2hL\widetilde{\beta}_0)^N(\alpha_2 + 2hL\beta_2)\}z_{n-2} \\
&+ (2hL\widetilde{\beta}_0)^N J(r).H.H^{m+1} + \{1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}\}M(r).\widetilde{M}.H^{\widetilde{m}+1},
\end{aligned}$$

and this inequality, If $n = \widetilde{N}$, yields that

$$\begin{aligned}
z_{\widetilde{N}} \leq\ &\{[1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}] \\
&\times (\widetilde{\alpha}_1 + 2hL\widetilde{\beta}_1) + (2hL\widetilde{\beta}_0)^N(\alpha_1 + 2hL\beta_1)\}z_{\widetilde{N}-1} \\
&+ \{[1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}] \\
&\times (\widetilde{\alpha}_2 + 2hL\widetilde{\beta}_2) + (2hL\widetilde{\beta}_0)^N(\alpha_2 + 2hL\beta_2)\}z_{\widetilde{N}-2} \\
&+ (2hL\widetilde{\beta}_0)^N J(r).H.H^{m+1} + \{1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}\}M(r).\widetilde{M}.H^{\widetilde{m}+1}.
\end{aligned}$$

Suppose that

$$A = \{[1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}](\widetilde{\alpha}_1 + 2hL\widetilde{\beta}_1)$$

$$+ (2hL\widetilde{\beta}_0)^N(\alpha_1 + 2hL\beta_1)\},$$

$$B = \{[1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}](\widetilde{\alpha}_2 + 2hL\widetilde{\beta}_2) + (2hL\widetilde{\beta}_0)^N(\alpha_2 + 2hL\beta_2)\},$$

$$C = (2hL\widetilde{\beta}_0)^N J(r).H.H^{m+1} + \{1 + (2hL\widetilde{\beta}_0) + \cdots + (2hL\widetilde{\beta}_0)^{N-1}\}M(r).\widetilde{M}.H^{\widetilde{m}+1}.$$

Then by Lemma 1.1 we get

$$0 \le z_{\widetilde{N}} + (\alpha - A)z_{\widetilde{N}-1} \le \alpha^{\widetilde{N}-1}[z_1 + (\alpha - A)z_0] + C\frac{\alpha^{\widetilde{N}-1} - 1}{\alpha - 1},$$

where $\alpha = \frac{(A^2+4B)^{1/2}+A}{2}$. Since $\lim_{h\to 0}\{\alpha^{\widetilde{N}-1}[z_1 + (\alpha - A)z_0] + C\frac{\alpha^{\widetilde{N}-1}-1}{\alpha-1}\} = 0$, we have

$$\lim_{h\to 0}\{z_{\widetilde{N}} + (\alpha - A)z_{\widetilde{N}-1}\} = 0.$$

Thus for r, $0 \le r \le 1$, we obtain

$$\lim_{h\to 0} \underline{u}_{\widetilde{N}}(r) = \underline{u}(x_{\widetilde{N}}; r), \quad \lim_{h\to 0} \overline{u}_{\widetilde{N}}(r) = \overline{u}(x_{\widetilde{N}}; r),$$

and the proof is completed. □

Corollary 2.1. *The approximate solution of fuzzy initial value problem (1.1) converges to the exact solution as the step size goes to zero.*

3. Numerical Results

In this section we will solve some examples by using the following 2-step P(EC)^N E-mode, where the orders of predictor and corrector methods are two.

Predictor method:

$$\begin{cases} \underline{u}_n^{(0)}(r) = \underline{u}_{n-2}(r) + 2h\underline{f}(x_{n-1}, u_{n-1}; r), \\ \overline{u}_n^{(0)}(r) = \overline{u}_{n-2}(r) + 2h\overline{f}(x_{n-1}, u_{n-1}; r). \end{cases} \tag{3.1}$$

Corrector method:

$$\begin{cases} \underline{u}_n^{(j)}(r) = \underline{u}_{n-2}(r) + (\frac{h}{2})[\underline{f}(x_n, u_n^{(j-1)}; r) + 2\underline{f}(x_{n-1}, u_{n-1}; r) \\ \qquad + \underline{f}(x_{n-2}, u_{n-2}; r)], \\ \overline{u}_n^{(j)}(r) = \overline{u}_{n-2}(r) + (\frac{h}{2})[\overline{f}(x_n, u_n^{(j-1)}; r) + 2\overline{f}(x_{n-1}, u_{n-1}; r) \\ \qquad + \overline{f}(x_{n-1}, u_{n-1}; r)], \end{cases} \tag{3.2}$$

where $1 \le j \le N$, $\underline{u}_n(r) = \underline{u}_n^{(N)}(r)$ and $\overline{u}_n(r) = \overline{u}_n^{(N)}(r)$, $n = 2, \ldots, \widetilde{N}$.

Example 3.1. Consider the FIVP given by

$$\begin{cases} y''(x) = 2xy'(x) + y(x), & x \in [0,1], \\ \underline{y}(0;r) = 0.4\exp(r) - 0.3, & \underline{y}'(0;r) = 0.5r^{0.5} - 0.3, \\ \overline{y}(0;r) = 0.4\exp(2-r) - 0.3, & \overline{y}'(0;r) = 0.2(1-r)^{0.5} + 0.2, \end{cases} \tag{3.3}$$

where $0 \le r \le 1$. By using series the exact solution of the above FIVP is

$$\underline{y}(x;r) = \underline{y}(0;r)[1 + \sum_{j=0}^{\infty} \frac{(4j+1)\times\cdots\times 9\times 5\times 1}{(2j+2)!} x^{2j+2}] + \underline{y}'(0;r)[x + \sum_{j=1}^{\infty} \frac{(4j-1)\times\cdots\times 7\times 3}{(2j+1)!} x^{2j+1}],$$

$$\overline{y}(x;r) = \overline{y}(0;r)[1 + \sum_{j=0}^{\infty} \frac{(4j+1)\times\cdots\times 9\times 5\times 1}{(2j+2)!} x^{2j+2}] + \overline{y}'(0;r)[x + \sum_{j=1}^{\infty} \frac{(4j-1)\times\cdots\times 7\times 3}{(2j+1)!} x^{2j+1}],$$

where $x \in [0,1]$, $0 \le r \le 1$. In order to obtain the approximate solution of (3.3), suppose that

$$u_1(x) = y(x), \qquad u_2(x) = y'(x). \tag{3.4}$$

Then the FDE of (3.3) is transformed to the system of first-order FDE

$$u_1'(x) = u_2(x), \qquad u_2'(x) = 2xu_2(x) + u_1(x).$$

Now by using $r-$ level sets we have

$$\begin{cases} \underline{u}_1'(x;r) = \underline{u}_2(x;r), & \underline{u}_1(0;r) = 0.4\exp(r) - 0.3, \\ \overline{u}_1'(x;r) = \overline{u}_2(x;r), & \overline{u}_1(0;r) = 0.4\exp(2-r) - 0.3, \\ \underline{u}_2'(x;r) = 2x\underline{u}_2(x;r) + \underline{u}_1(x;r), & \underline{u}_2(0;r) = 0.5r^{0.5} - 0.3, \\ \overline{u}_2'(x;r) = 2x\overline{u}_2(x;r) + \overline{u}_1(x;r), & \overline{u}_2(0;r) = 0.2(1-r)^{0.5} + 0.2. \end{cases} \tag{3.5}$$

After computing the approximate solution of (3.5) by the 2-step P(EC)^N E-mode given by (3.1)-(3.2), the approximate solution of (3.3), that is $[y_{\widetilde{N}}]_r = [\underline{y}_{\widetilde{N}}(r), \overline{y}_{\widetilde{N}}(r)]$, $0 \le r \le 1$, according to (3.4) is obtained.

For some values of r, $0 \le r \le 1$, the approximate and exact solutions of (3.3) at $x = 1$ with $N = 2$ and $\widetilde{N} = 150$ are given in Table 1. Also their graphs are given in Figure 1.

r	$\underline{y}_{\widetilde{N}}(r)$	$\underline{y}(x_{\widetilde{N}};r)$	$\overline{y}_{\widetilde{N}}(r)$	$\overline{y}(x_{\widetilde{N}};r)$
0	-0.3408339721	-0.3408342109	5.4428528938	5.4427351756
0.2	0.2049281880	0.2049212245	4.4479935218	4.4478987597
0.4	0.5588514434	0.5588386689	3.6219014377	3.6218255593
0.6	0.9182780853	0.9182587731	2.9302988439	2.9302385436
0.8	1.3106996987	1.3106727594	2.3402026875	2.3401553246
1	1.7546750942	1.7546391105	1.7546750942	1.7546391105

Table 1: Approximate and exact solutions

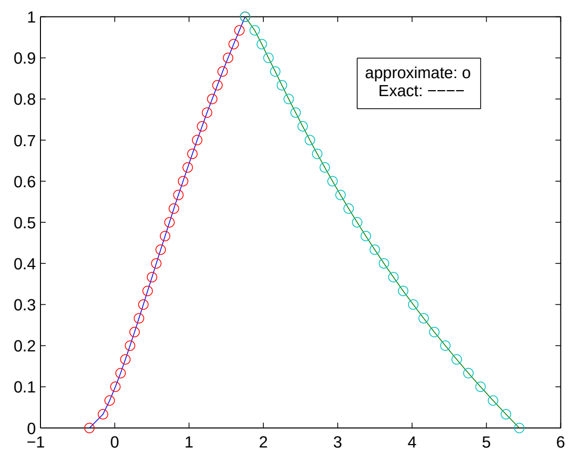

Figure 1: $h = \frac{1}{\widetilde{N}}$

r	$\underline{y}_{\widetilde{N}}(r)$	$\underline{y}(x_{\widetilde{N}};r)$	$\overline{y}_{\widetilde{N}}(r)$	$\overline{y}(x_{\widetilde{N}};r)$
0	-4.3845948915	-4.3845894449	3.2892374955	3.2892652794
0.2	-3.8740644462	-3.8740571543	1.8665266998	1.8665515281
0.4	-3.4547255021	-3.4547163572	0.6860915764	0.6861137041
0.6	-2.9980393231	-2.9980282693	-0.2982706587	-0.2982510237
0.8	-2.4794430946	-2.4794300587	-1.1274381612	-1.1274208502
1	-1.8782504957	-1.8782353859	-1.8782504957	-1.8782353859

Table 2: Approximate and exact solutions

Example 3.2. Consider the FIVP given by

$$\begin{cases} y^{(4)}(x) = y'''(x) + y''(x) + y'(x) + 2y(x), & x \in [0,1], \\ \underline{y}(0;r) = 0.4\exp(r) - 5, & \overline{y}(0;r) = 0.4\exp(2-r) - 5, \\ \underline{y}'(0;r) = 0.5r^{0.5} - 0.3, & \overline{y}'(0;r) = 0.2(1-r)^{0.5} + 0.2, \\ \underline{y}''(0;r) = \exp(r), & \overline{y}''(0;r) = \exp(2-r), \\ \underline{y}'''(0;r) = r + 2, & \overline{y}'''(0;r) = 4 - r, \end{cases} \tag{3.6}$$

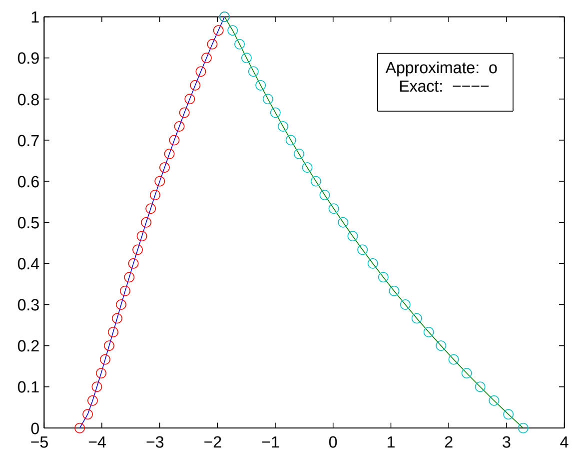

Figure 2: $h = \frac{1}{\tilde{N}}$

where $0 \leq r \leq 1$. The exact solution of the above FIVP is

$$\begin{aligned}
\underline{y}(x;r) &= ((10\underline{y}(0;r) - 5\underline{y}'(0;r) + 10\underline{y}''(0;r) - 5\underline{y}'''(0;r))/30)\exp(-1) \\
&+ ((\underline{y}(0;r) + \underline{y}'(0;r) + \underline{y}''(0;r) + \underline{y}'''(0;r))/15)\exp(2) \\
&+ ((18\underline{y}(0;r) + 3\underline{y}'(0;r) - 12\underline{y}''(0;r) + 3\underline{y}'''(0;r))/30)\cos(1) \\
&+ ((6\underline{y}(0;r) + 21\underline{y}'(0;r) + 6\underline{y}''(0;r) - 9\underline{y}'''(0;r))/30)\sin(1); \\
\overline{y}(x;r) &= ((10\overline{y}(0;r) - 5\overline{y}'(0;r) + 10\overline{y}''(0;r) - 5\overline{y}'''(0;r))/30)\exp(-1) \\
&+ ((\overline{y}(0;r) + \overline{y}'(0;r) + \overline{y}''(0;r) + \overline{y}'''(0;r))/15)\exp(2) \\
&+ ((18\overline{y}(0;r) + 3\overline{y}'(0;r) - 12\overline{y}''(0;r) + 3\overline{y}'''(0;r))/30)\cos(1) \\
&+ ((6\overline{y}(0;r) + 21\overline{y}'(0;r) + 6\overline{y}''(0;r) - 9\overline{y}'''(0;r))/30)\sin(1);
\end{aligned}$$

where $x \in [0,1]$, $0 \leq r \leq 1$. In order to obtain the approximate solution of (3.6), suppose that

$$\begin{cases} u_1(x) = y(x), & u_2(x) = y'(x), \\ u_3(x) = y''(x), & u_4(x) = y'''(x). \end{cases} \tag{3.7}$$

Then the FDE of (3.6) is transformed to the following system of first-order FDE

$$\begin{cases} u_1'(x) = u_2(x), & u_2'(x) = u_3(x), \\ u_3'(x) = u_4(x), & u_4'(x) = u_4(x) + u_3(x) + u_2(x) + 2u_1(x). \end{cases}$$

Now by using $r-$ level sets we have

$$\begin{cases} \underline{u}_1'(x;r) = \underline{u}_2(x;r), \underline{u}_1(0;r) = 0.4\exp(r) - 5, \\ \overline{u}_1'(x;r) = \overline{u}_2(x;r), \overline{u}_1(0;r) = 0.4\exp(2-r) - 5, \\ \underline{u}_2'(x;r) = \underline{u}_3(x;r), \underline{u}_2(0;r) = 0.5r^{0.5} - 0.3, \\ \overline{u}_2'(x;r) = \overline{u}_3(x;r), \overline{u}_2(0;r) = 0.2(1-r)^{0.5} + 0.2, \\ \underline{u}_3'(x;r) = \underline{u}_4(x;r), \underline{u}_3(0;r) = \exp(r), \\ \overline{u}_3'(x;r) = \overline{u}_4(x;r), \overline{u}_3(0;r) = \exp(2-r), \\ \underline{u}_4'(x;r) = \underline{u}_4(x;r) + \underline{u}_3(x;r) + \underline{u}_2(x;r) + 2\underline{u}_1(x;r), \underline{u}_4(0;r) = r+2, \\ \overline{u}_4'(x;r) = \overline{u}_4(x;r) + \overline{u}_3(x;r) + \overline{u}_2(x;r) + 2\overline{u}_1(x;r), \overline{u}_4(0;r) = 4-r. \end{cases} \tag{3.8}$$

After computing the approximate solution of (3.8) by the 2-step P(EC)^N E-mode given by (3.1)-(3.2), the approximate solution of (3.6), according to (3.7) is obtained.

For some values of r, $0 \leq r \leq 1$, the approximate and exact solutions of (3.6) at $x = 1$ with $N = 2$ and $\widetilde{N} = 150$ are given in Table 2. Also their graphs are given in Figure 2.

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