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ON THE NUMBER OF k-INDEPENDENT SETS IN SOME PRODUCTS OF GRAPHS

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Abstract: A subset $S \subset V(G)$ is k-independent if for each two distinct vertices from S the distance between them is at least k. In this paper we determine the number of all k-independent sets in some product of graphs.

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1. Introduction

In general we use the standard terminology and notation of graph theory, see Berge [1] and Diestel [3]. Only simple, undirected, connected graphs are considered. $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The length of the shortest path joining vertices x and y in G will be denoted by $d_G(x, y)$. By P_n, $n \geq 2$ we mean the graph with the vertex set $V(P_n) = \{t_1, \ldots, t_n\}$ and the edge set $E(P_n) = \{\{t_i, t_{i+1}\};\ i = 1, \ldots, n-1\},\ n \geq 1$. Moreover P_1 is a graph with $V(P_1) = \{t_1\}$ and P_0 is a graph with $V(P_0) = \emptyset$. By K_n we will denote the complete graph on n vertices, $n \geq 1$. Let G be a graph on $V(G) = \{t_1, \ldots, t_n\},\ n \geq 1$ and H be a graph on $V(H) = \{y_1, \ldots, y_m\},\ m \geq 1$. By Cartesian product of two graphs G and H we mean a graph $G \times H$ such that $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{\{(t_i, y_p), (t_j, y_q)\};\ t_i = t_j$ and $\{y_p, y_q\} \in E(H)$ or $\{t_i, t_j\} \in E(G)$ and $y_p = y_q\}$. Let G be a graph on $V(G) = \{t_1, \ldots, t_n\}$, $n \geq 2$ and $h_n = (H_i)_{i \in \{1,\ldots,n\}}$ be a sequence of vertex

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disjoint graphs on $V(H_i) = V = \{y_1, \dots, y_x\},\ x \geq 1$. By generalized lexicographic product of G and $h_n = (H_i)_{i\in\{1,\dots,n\}}$ we mean a graph $G[h_n]$ such that $V(G[h_n]) = V(G)\times V$ and $E(G[h_n]) = \{\{(t_i, y_p), (t_j, y_q)\};\ t_i = t_j$ and $\{y_p, y_q\} \in E(H_i)$ or $\{t_i, t_j\} \in E(G)\}$. If $H_i = H,\ i = 1, \dots, n$, then $G[h_n] = G[H]$, where $G[H]$ is a lexicographic products of two graphs.

A subset $S \subseteq V(G)$ is said to be k-independent of G if for each two distinct vertices $x, y \in S, d_G(x, y) \geq k$. In addition the empty set and a subset containing only one vertex also are meant as a k-independent sets of G. Note that for $k = 2$ the definition reduces to the definition of an independent set of the graph G. The k-independent sets and the total number of k-independent sets of graph were studied in [4]-[10]. Prodinger et al [7] initiated the study of the number of independent sets in a graph. The problem of counting the number of independent sets in graph is NP-complete (see for instance Roth [8]). However for certain types of graphs the problem of determining their numbers of independent sets is polynomial. The literature includes many papers dealing with the theory of counting of k-independent sets in graphs, see [2, 5, 6, 9].

The total number of independent sets of graph G was named by Prodinger et al [7] as Fibonacci number of a graph G. They denote it by $F(G)$. Let $|V(G)| = n$. If $f_G(n, p)$ denotes the number of all p-elements independent sets of G, then $F(G) = \sum_{p\geq 0} f_G(n, p)$. It is interesting to know that $F(P_n) = F_n = \sum_{p\geq 0} \binom{n-p+1}{p}$, so it is equal to the Fibonacci numbers, see Berge [1]. The Fibonacci numbers has also the recurrence form $F_n = F_{n-1} + F_{n-2}$ with the initial conditions $F_0 = 1,\ F_1 = 2$. Kwaśnik et al [5] defined more general concept, namely generalized Fibonacci number of a graph G. This number was defined as the total number of k-independent sets of a graph G and it was denoted by $F_k(G)$. If $f_G(k, n, p)$ denotes the number of all p-elements k-independent sets of G, then $F_k(G) = \sum_{p\geq 0} f_G(k, n, p)$. It was proved:

Theorem 1. (see [5]) *Let $k \geq 2,\ n \geq 0,\ 0 \leq p \leq n$ be integers. Then*

$$F_k(P_n) = \sum_{p\geq 0} f_{P_n}(k, n, p) = \sum_{p\geq 0} \binom{n-p-(p-1)(k-2)+1}{p}.$$

The $F_k(P_n)$ generalize the Fibonacci numbers F_n and we put notation $F_k(P_n) = F(k, n)$. Evidently $F(2, n) = F_n$. The numbers $F(k, n)$ has also the recurrence form:

Theorem 2. (see [5]) *Let $k \geq 2, n \geq 0$ be integers. Then numbers $F(k, n)$*

satisfy the following recurrence: $F(k,n) = F(k,n-1) + F(k,n-k)$ *for* $n \geq k$ *with the initial conditions* $F(k,n) = n+1$ *for* $n = 0, 1, \ldots, k-1$.

For others classes of graphs the total number of k-independent sets were determined, see [4]-[10].

Theorem 3. (see [4]) *Let* $n \geq 0,\ p \geq 0,\ x \geq 1$ *be integers. Then for an arbitrary graph* G *on* n *vertices* $F(G[K_x]) = \sum_{p \geq 0} f_G(n,p)x^p$.

Theorem 4. (see [10]) *Let* $k \geq 3,\ x \geq 1,\ p \geq 0$ *be integers. Then for an arbitrary graph* G *on* $n, n \geq 2$, *vertices and for an arbitrary sequence of vertex disjoint graphs* $h_n = (H_i)_{i \in \{1,\ldots,n\}}$ *such that* $|V(H_i)| = x$ *for* $i = 1, \ldots, n$, $F_k(G[h_n]) = \sum_{p \geq 0} f_G(k,n,p)x^p$.

2. Main Results

Now we consider the graph $P_n \times K_m, n \geq 0,\ m \geq 1$ and we present numbers $F((P_n \times K_m)[K_x])$ and $F_k((P_n \times K_m)[h_n])$, where $h_n = (H_i)_{i \in \{1,\ldots,n\}}$ is an arbitrary sequence of graphs. Firstly we calculate the numbers $f_{P_n \times K_m}(k, mn, p)$. In this paper for convenience we denote these numbers by $f(k,n,p)$.

Theorem 5. *Let* $k \geq 2, p \geq 2$ *be integers. If* $n < (p-1)\tau + 1$, *then* $f(k,n,p) = 0$, *where* $\tau = \begin{cases} k-1 \text{ if } m > 1\,, \\ k \text{ if } m = 1\,. \end{cases}$

Proof. Let $n < (p-1)\tau + 1$. We shall prove that $f(k,n,p) = 0$. Assume on the contrary that $f(k,n,p) > 0$. This means that there exists a p-element k-independent set of $P_n \times K_m$. If $m = 1$, then it is clear that graph $P_n \times K_1$ is isomorphic to P_n and to construct a k-independent set S of P_n having p-elements, $p \geq 2$, we need at least $(p-1)k+1$ vertices, hence $S = \{t_1, t_{k+1}, \ldots, t_{(p-1)k+1}\}$, contradiction the assumption. If $m > 1$, then from the definition of graph $P_n \times K_m$ and by fact that K_m is a complete graph on m vertices we deduce that to construct a k-independent set S' of $P_n \times K_m$ we need at least $(p-1)(k-1)+1$ vertices. Then S' has the following form $S' = \{(t_1, y_i), (t_k, y_j), \ldots, (t_{(p-1)(k-1)+1}, y_q)\}$, where if $(t_r, y_i), (t_s, y_j) \in S'$ and $s = r + k - 1$, then $i \neq j$. Hence $n \geq (p-1)(k-1)+1$, contradiction. Consequently from the above cases if $n < (p-1)\tau + 1$, then $f(k,n,p) = 0$, where $\tau = \begin{cases} k-1 \text{ if } m > 1\,, \\ k \text{ if } m = 1\,. \end{cases}$ Thus the theorem is proved. $\square$

Theorem 6. *Let $k \geq 2, n \geq 0, m \geq 1, p \geq 0$ be integers. Then the numbers $f(k,n,p)$ satisfy the following recurrence relations: $f(k,n,0) = 1$, $f(k,n,1) = mn$, for $p \geq 2$ $f(k,n,p) = 0$ if $n < (p-1)\tau + 1$ and for $n \geq (p-1)\tau+1$ we have $f(k,n,p) = f(k,n-1,p) + mB_n^p$ and $B_n^p = f(k,n-k,p-1) + (m-1)B_{n-(k-1)}^{p-1}$, where $B_n^1 = 1$ and $\tau = \begin{cases} k-1 \text{ if } m>1\,, \\ k \text{ if } m=1\,. \end{cases}$*

Proof. Let k,n,p,m be as in the statement of the theorem. If $p=0$, then the empty set is the unique k-independent set of the graph $P_n \times K_m$. So $f(k,n,0) = 1$. If $p=1$, then every vertex of the graph $P_n \times K_m$ is a k-independent set of the graph $P_n \times K_m$. Consequently $f(k,n,1) = m \cdot n$. Let now $p \geq 2$. If $n < (p-1)\tau+1$, where $\tau = \begin{cases} k-1 \text{ if } m>1\,, \\ k \text{ if } m=1\,, \end{cases}$ then by Theorem 5 we have that $f(k,n,p) = 0$. Assume that $n \geq (p-1)\tau+1$. Let S_1 be a family of all p-element k-independent sets $S \subseteq V(P_n \times K_m)$ such that $(t_n, y_m) \notin S$ and let S_2 be a family of all p-elements k-independent sets $S \subseteq V(P_n \times K_m)$ such that $(t_n, y_m) \in S$. By the general rule for counting k-independent sets $f(k,n,p) = |S_1| + |S_2|$. Assume that the number of all p-element k-independent sets of $P_n \times K_m$ containing a vertex (t_n, y_i), where i is one from $1,\ldots,m$ is equal to B_{n,y_i}^p. Moreover by the definition of the Cartesian product and by fact that K_m is a complete graph we deduce that for every $1 \leq i,j \leq m$, $B_{n,y_i}^p = B_{n,y_j}^p$. Consequently we put notation $B_{n,y_i}^p = B_n^p$ for every $i = 1,\ldots,m$. Of course $B_n^1 = 1$. Let $S \in S_1$. If $(t_n, y_i) \notin S$, $i = 1,\ldots,m-1$, then $S = S^*$, where S^* is an arbitrary p-element k-independent set of the graph $P_{n-1} \times K_m$. Hence we have $f(k,n-1,p)$ k-independent sets S having p-elements. If there exists $1 \leq i \leq m-1$ such that $(t_n, y_i) \in S$ by our assumption we have B_n^p such subsets. By the fact that we can choose the vertex (t_n, y_i) belonging to S on $(m-1)$ ways we obtain that there are $(m-1)B_n^p$ k-independent sets S with p-elements. So $|S_1| = f(k,n-1,p) + (m-1)B_n^p$ in this case. Now we calculate the number $|S_2|$. By previous considerations $|S_2| = B_n^p$, so we have to determine the number B_n^p. Since $(t_n, y_m) \in S$, then $(t_n, y_i) \notin S$, $i = 1,\ldots,m-1$, $(t_r, y_j) \notin S$, where $r = n-1,\ldots,n-(k-2)$, $j = 1,\ldots,m$ and $(t_{n-(k-1)}, y_m) \notin S$. If $(t_{n-(k-1)}, y_i) \notin S$, $i = 1,\ldots,m-1$, then $S = S' \cup \{(t_n, y_m)\}$, where S' is an arbitrary $(p-1)$-element k-independent set of the graph $P_{n-k} \times K_m$, so we have $f(k,n-k,p-1)$ such sets S. If there exists $1 \leq i \leq m-1$ such that $(t_{n-(k-1)}, y_i) \in S$, then by our assumption we have $B_{n-(k-1)}^{p-1}$ such subsets. By fact that we can choose the vertex $(t_{n-(k-1)}, y_i)$ on $(m-1)$ ways we obtain $(m-1)B_{n-(k-1)}^{p-1}$ p-element k-independent sets S. Consequently $B_n^p = f(k,n-k,p-1) + (m-1)B_{n-(k-1)}^{p-1}$. Finally from the above casses for $p \geq 2$ we have that

$f(k,n,p) = |S_1| + |S_2| = f(k,n-1,p) + (m-1)B_n^p + B_n^p = f(k,n-1,p) + mB_n^p$, where $B_n^p = f(k,n-k,p-1) + (m-1)B_{n-(k-1)}^{p-1}$. Thus the theorem is proved. □

Corollary 1. *Let $n \geq 0$, $p \geq 0$, $k \geq 2$ be integers. If $m = 1$, then $\sum_{p\geq 0} f(k,n,p) = F(k,n)$.*

Proof. If $0 \leq n \leq k$, then $\sum_{p\geq 0} f(k,n,p) = f(k,n,0) + f(k,n,1) = n+1 = F(k,n)$ in this case. If $n \geq k+1$, then

$$\begin{aligned}
\sum_{p\geq 0} f(k,n,p) &= f(k,n,0) + f(k,n,1) + \sum_{p\geq 2} f(k,n,p) \\
&= 1 + n + \sum_{p\geq 2}(f(k,n-1,p) + f(k,n-k,p-1)) \\
&= 1 + n + \sum_{p\geq 2} f(k,n-1,p) + \sum_{p\geq 2} f(k,n-k,p-1) \\
&= (n-1) + 1 + \sum_{p\geq 2} f(k,n-1,p) + 1 + \sum_{r=p-1\geq 1} f(k,n-k,r) \\
&= f(k,n-1,0) + f(k,n-1,1) + \sum_{p\geq 2} f(k,n-1,p) + f(k,n-k,0) \\
&\quad + \sum_{r\geq 1} f(k,n-k,r) = \sum_{p\geq 0} f(k,n-1,p) + \sum_{r\geq 0} f(k,n-k,r) \\
&= F(k,n-1) + F(k,n-k) = F(k,n),
\end{aligned}$$

which ends the proof. □

Corollary 2. *Let $n \geq 1$, $m \geq 1$, $x \geq 1$ be integers. Then $F((P_n \times K_m)[K_x]) = \sum_{p\geq 0} f(2,n,p)x^p$.*

Corollary 3. *Let $k \geq 3$, $n \geq 2$, $m \geq 1$, $x \geq 1$ be integers. Then for an arbitrary sequence of vertex disjoint graphs $h_n = (H_i)_{i\in\{1,\ldots,n\}}$ such that $|V(H_i)| = x$, for $i = 1,\ldots,n$ we have $F_k((P_n \times K_m)[h_n]) = \sum_{p\geq 0} f(k,n,p)x^p$.*

Theorem 7. *Let $k \geq 2$, $n \geq 0$, $m \geq 1$. Then for $n \geq k$ numbers $F_k(P_n \times K_m)$ satisfy the following recurrence relations: $F_k(P_n \times K_m) = F_k(P_{n-1} \times K_m) + mB_m$ and $B_n = F_k(P_{n-k} \times K_m) + (m-1)B_{n-(k-1)}$, with the initial conditions: $F_k(P_n \times K_m) = mn + 1$, for $n = 0,1,\ldots,k-1$, $B_n = 1$, $n = 1,\ldots,k-1$.*

Proof. If $n = 0$, then also $p = 0$ and this implies $F_k(P_n \times K_m) = f(k, 0, 0) = 1$, by the definition of $F_k(P_n \times K_m)$.

If $n = 1, \dots, k-1$, then $p = 0$ or $p = 1$ so $F_k(P_n \times K_m) = f(k, n, 0) + f(k, n, 1)$ and using Theorem 6 we have $F_k(P_n \times K_m) = 1 + mn$.

Let $n \geq k$. Then using Theorem 6, $F_k(P_n \times K_m) = \sum_{p \geq 0} f(k, n, p) = \sum_{p \geq 0} (f(k, n-1, p) + mB_n^p) = \sum_{p \geq 0} f(k, n-1, p) + m \sum_{p \geq 0} B_n^p = F_k(P_{n-1} \times K_m) + m \sum_{p \geq 0} B_n^p$.

Let $\sum_{p \geq 0} B_n^p = B_n$. Evidently if $n = 1, \dots, k-1$, then $p = 1$ and $B_n = B_n^1 = 1$. In our case $n \geq k$ we obtain $F_k(P_n \times K_m) = F_k(P_{n-1} \times K_m) + mB_n$. Moreover by Theorem 6, $B_n = \sum_{p \geq 0} B_n^p = \sum_{p \geq 0} (f(k, n-k, p-1) + (m-1)B_{n-(k-1)}^{p-1}) = \sum_{p \geq 0} f(k, n-k, p-1) + (m-1) \sum_{p \geq 0} B_{n-(k-1)}^{p-1}$. Because for $p = 0$ the numbers $f(k, n-k, p-1)$ and $B_{n-(k-1)}^{p-1}$ there no exist we can put $B_n = \sum_{r=p-1 \geq 0} f(k, n-k, r) + (m-1) \sum_{r=p-1 \geq 0} B_{n-(k-1)}^r = F_k(P_{n-k} \times K_m) + (m-1)B_{n-(k-1)}$, which ends the proof. $\square$

References

[1] C. Berge, *Principles of Combinatorics*, Academic Press, New York and London (1971).

[2] M.J. Chou, G.J. Chang, Survey on counting maximal independent sets, In: *Proceedings of the Second Asian Mathematical Conference* (Ed-s: S. Tangmance, E. Schulz), Word Scientific, Singapore (1995), 265-275.

[3] R. Diestel, *Graph Theory*, Springer-Verlag, Heidelberg, New-York (2005).

[4] G. Hopkins, W. Staton, Some idenities arising from the Fibonacci numbers of certain graphs, *The Fibonacci Quarterly* (1984), 225-228.

[5] M. Kwaśnik, I. Włoch, The total number of generalized stable sets and kernels of graphs, *Ars Combinatoria*, **55** (2000), 139-146.

[6] A.S. Pedersen, P.D. Vestergaard, The number of independent sets in unicyclic graphs, *Discrete Appl. Math.*, **152** (2005) 246-256.

[7] H. Prodinger, R.F. Tichy, Fibonacci numbers of graphs, *The Fibonacci Quarterly*, **20** (1982), 16-21.

[8] D. Roth, On the hardness of aproximate reasoning, *Artif. Intell.*, **82** (1996) 273-302.

[9] A. Włoch, I. Włoch, The total number of maximal k-independent sets in the generalized lexicographical product of graph, *Ars Combinatoria*, **75** (2005), 163-170.

[10] I. Włoch, Generalized Fibonacci polynomial of graphs, *Ars Combinatoria*, **68** (2003), 49-55.