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AN APPROXIMATE SOLUTION OF BURGER'S EQUATION USING ADOMIAN'S DECOMPOSITION METHOD

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Abstract: The aim of this paper is to find an approximate solution of Burger's equation using the decomposition method, which has been developed by George Adomian [2]. The advantage of this method focuses on avoiding simplifications and restrictions which change the non-linear problem to mathematically tractable one, whose solution is not consistent with physical solution. Theoretical analysis and all calculations are done. The results are compared with the analytical solution and discussed.

AMS Subject Classification: 65C20
Key Words: Burger's equation, Adomian decomposition method, Adomian polynomials

1. Introduction

The one dimension non-linear differential equation, which is similar to the one dimension Navier-Stokes equation without the stress term, and was presented for the first time in a paper in 1940 from Burger's, is the model for the solution of Navier-Stokes equation and is applied to laminar and turbulence flows as well.

The Burger's equation was first studied by Cole [13] who gave a theoretical solution, based on Fourier series analysis, using the appropriate initial and boundary conditions. Another theoretical solution was given by Madsen and Sincovec [14], based on the "test and trial" method, using the appropriate initial and boundary conditions. In Benton and Platzman [9], almost 35 dinstinct

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solutions of Burger's equation and Agas [8] tried to get approximate solutions of Burger's equation using numerical analysis are mentioned. He tried the method of finite differences and the method of lines in finite elements. The problem he faced was that these methods could not give solutions for big values of the Reynolds number. He also found some problems in convergence.

In this paper, we will find solutions using the Adomian decomposition method. Even though decomposition method is a non-numerical method, it can be adapted for solving nonlinear differential equations and gives a computable and accurate solution of the problem for a small number of terms.

2. Formulation of the Problem and Analytical Solution of Burger's's Equation

The equation of motion in one dimension has the following form [10]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}\,. \tag{1}$$

The boundary conditions are defined as follows:

$$u(0,t) = u(1,t) \quad \text{for} \quad t \geq 0 \tag{2}$$

and the initial condition:

$$u(x,0) = 4x\left(1-x\right)\,. \tag{3}$$

The analytical solution of Burger's's equation with boundary conditions (2) and initial condition $u(x,0) = f(x)$ is given by Cole [8], [13] and has the form

$$u\left(x,t\right) = \frac{2\pi\sum\limits_{m=1}^{\infty} mA_m \sin\left(m\pi x\right)\exp\left(\frac{-m^2\pi^2 t}{\mathrm{Re}}\right)}{\mathrm{Re}\left\{A_0 + \sum\limits_{m=1}^{\infty} A_m \cos\left(m\pi x\right)\exp\left(\frac{-m^2\pi^2 t}{\mathrm{Re}}\right)\right\}}\,, \tag{4}$$

where

$$\begin{aligned} A_m &= 2\int_0^1 \cos\left(m\pi x\right)\exp\left(-\frac{\mathrm{Re}}{2}\int_0^x f\left(x'\right)dx'\right)dx \quad \left(m = 1,2,3,\dots\right)\,, \\ A_0 &= \int_0^1 \exp\left(-\frac{\mathrm{Re}}{2}\int_0^x f\left(x'\right)dx'\right)dx\,, \end{aligned} \tag{5}$$

Re = Reynolds number.

For the case, where $f(x) = 4x(1-x)$ terms (5) take the form

$$
\begin{aligned}
A_m &= 2\int_0^1 \cos(m\pi x)\exp\left(-\frac{\mathrm{Re}\cdot x^2}{3}(3-2x)\right)dx \quad (m = 1,2,3,\dots)\ , \\
A_0 &= \int_0^1 \exp\left(-\frac{\mathrm{Re}\cdot x^2}{3}(3-2x)\right)dx\,.
\end{aligned} \tag{6}
$$

For small values of Re (i.e. Re=1) the coefficients A_m can be calculated quite easily, which means that we can achieve better accuracy. In contrast with big values of Re (Re>10), the convergence is very slow, so, this is the reason we use other approximative methods. We now proceed in decomposition method.

3. Theoretic Approach of Decomposition Method

In equation (1) the first term is the linear, the second is the non-linear and the third is the highest order term.

If we define

$$L_t u = \frac{\partial u}{\partial t} = Ru, \quad L_x u = \frac{\partial^2 u}{\partial x^2} = Lu, \quad Nu = u\frac{\partial u}{\partial x}\,, \tag{7}$$

where Nu represents the non-linear term, Lu the highest order term, and Ru the rest of the equation, equation (1) takes the form

$$Ru + Nu = \nu Lu\,.$$

We solve equation (1) for $L_t u$ and $L_x u$ separately and we get

$$L_t u = \nu L_x u - Nu\,, \tag{8}$$

$$L_x u = \nu^{-1}(L_t u + Nu)\ . \tag{9}$$

Let L_t^{-1} and L_x^{-1} be the inverse operators of $L_t u$ and $L_x u$ respectively, given by the form [6], [7]:

$$L_t^{-1} = \int(\cdot)\,dt \quad \text{and} \quad L_x^{-1} = \int\int(\cdot)\,dxdx \tag{10}$$

Then operating both sides of equations (8) and (9) with the inverse operators (10) we obtain.

$$u = u_0 + L_t^{-1}\left(\nu\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}\right), \tag{11}$$

$$u = u_0 + \nu^{-1}L_x^{-1}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right), \tag{12}$$

where u_0 is to be determined from the corresponding initial condition, so

$$u_0 = 4x\,(1-x)\,. \tag{13}$$

Now, adding (11) and (12) and dividing by 2, we get the following form:

$$\begin{aligned} u &= u_0 + L_t^{-1}\left(\nu\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}\right) + \nu^{-1}L_x^{-1}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) \\ &= 4x\,(1-x) + \frac{1}{2}\left[L_t^{-1}\left(\nu\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}\right) + \nu^{-1}L_x^{-1}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\right]. \end{aligned} \tag{14}$$

After that, we write the parametrized form of (14) which is:

$$u = u_0 + \lambda\frac{1}{2}\left[\nu^{-1}L_x^{-1}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + L_t^{-1}\left(\nu\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}\right)\right] \tag{15}$$

and the parametrized decomposition forms of u and Nu as

$$u = \sum_{n=0}^{\infty}\lambda^n u_n\,, \tag{16}$$

$$u\frac{\partial u}{\partial x} = Nu = \sum_{n=0}^{\infty}\lambda^n A_n\,, \tag{17}$$

where A_n are the Adomian's special polynomials [1], [3] to be determined. Here the parameter λ looks like a perturbation parameter; but actually is not a perturbation parameter; it is used only for grouping the terms.

Now substitution of (16) and (17) into (15) gives

$$\sum_{n=0}^{\infty}\lambda^n u_n = u_0 + \lambda\frac{1}{2}\times$$

$$\left[\nu^{-1}L_x^{-1}\left(\frac{\partial\sum\limits_{n=0}^{\infty}\lambda^n u_n}{\partial t}+\sum_{n=0}^{\infty}\lambda^n A_n\right)+L_t^{-1}\left(\nu\frac{\partial^2\sum\limits_{n=0}^{\infty}\lambda^n u_n}{\partial x^2}-\sum_{n=0}^{\infty}\lambda^n A_n\right)\right]. \tag{18}$$

If we compare like-power terms of λ from both sides of equation (18), and taking under consideration that parameter λ is being proved that has the unique value $\lambda = 1$ (see [11], [12]), we get

$$\begin{aligned}
u_0 &= 4x\,(1-x)\ , \\
u_1 &= \frac{1}{2}\left[\nu^{-1}L_x^{-1}\left(\frac{\partial u_0}{\partial t}+A_0\right)+L_t^{-1}\left(\nu\frac{\partial^2 u_0}{\partial x^2}-A_0\right)\right]\ , \\
u_2 &= \frac{1}{2}\left[\nu^{-1}L_x^{-1}\left(\frac{\partial u_1}{\partial t}+A_1\right)+L_t^{-1}\left(\nu\frac{\partial^2 u_1}{\partial x^2}-A_1\right)\right]\ , \\
&\ \vdots \\
u_{n+1} &= \frac{1}{2}\left[\nu^{-1}L_x^{-1}\left(\frac{\partial u_n}{\partial t}+A_n\right)+L_t^{-1}\left(\nu\frac{\partial^2 u_n}{\partial x^2}-A_n\right)\right],
\end{aligned} \tag{19}$$

where $n = 0, 1, 2, \ldots, n$.

Next, we proceed to determine Adomian's special polynomials A_n.

4. Determination of Adomian's Special Polynomials

The A_n polynomials are defined in such a way that each A_n depents only on $u_0, u_1, \ldots, u_n$ for $n = 0, 1, 2, \ldots, n$, i.e., $A_0 = A\,(u_0)\,,\ A_1 = A_1\,(u_0, u_1)\,,\ A_2 = A_2\,(u_0, u_1, u_2)$, etc. In order to do this we substitute (16) into (17) and we have

$$\begin{aligned}
Nu &= u\frac{\partial u}{\partial x} \\
&= \left(u_0+\lambda u_1+\lambda^2 u_2+\lambda^3 u_3+\ldots\right)\left(\frac{\partial u_0}{\partial x}+\lambda\frac{\partial u_1}{\partial x}+\lambda^2\frac{\partial u_2}{\partial x}+\lambda^3\frac{\partial u_3}{\partial x}+\ldots\right) \\
&= u_0\frac{\partial u_0}{\partial x}+\lambda\left(u_0\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_0}{\partial x}\right)+\lambda^2\left(u_0\frac{\partial u_2}{\partial x}+u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_0}{\partial x}\right) \\
&\quad +\lambda^3\left(u_0\frac{\partial u_3}{\partial x}+u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_1}{\partial x}+u_3\frac{\partial u_0}{\partial x}\right)+\lambda^4\left(\ldots\right).
\end{aligned} \tag{20}$$

From (20) we conclude that the Adomian Polynomials have the following form:

$$A_0 = u_0\frac{\partial u_0}{\partial x}\,,$$

$$\begin{aligned} A_1 &= u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x}\,, \\ A_2 &= u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x}\,, \\ &\vdots \end{aligned} \tag{21}$$

Hence, the polynomial A_0 has the following form:

$$A_0 = u_0 \frac{\partial u_0}{\partial x} = 4\left(4x - 12x^2 + 8x^3\right)\,. \tag{22}$$

Using (13) and A_0 from (22) into the expression of u_1 in (21) and then performing the integrations with respect to t and x respectively, we have

$$u_1 = 2\left[\nu^{-1}\left(\frac{4x^3}{6} - \frac{4x^4}{4} + \frac{4x^5}{10}\right) - \left(4\nu + 4x - 12x^2 + 8x^3\right)t\right]\,. \tag{23}$$

The method continues the same way to calculate the next term u_2. If we suggest as a solution of u an approximation of only three terms then from (13), (23) and the calculated u_2, we have the solution

$$u = u_0 + u_1 + u_2\,. \tag{24}$$

5. Tables of Results, Diagrams and Discussion

For the solution of this equation the initial conditions $u(x) = 4x(1-x)$ were used without restricting the generality. The boundary conditions were $u(0,t) = u(1,t) = 0 \quad$ for $\quad t \geq 0$, The compared methods were the analytic solution given by Cole, and the three term solution using the Adomian's decomposition method.

To compare with the results of other published papers, $\Delta x = 0.25$ and time amplitude $0.01 \leq t \leq 0.25$ were used. In the figures below numerical results of Burger's equation are registered for different values of v. For comparison reasons, the viscosity values $v = 1, 0.1, 0.01$ were used.

We give below the tables of the results and some figures, where we compare these two methods in three cases. For $\nu = 1, x = 0.25$ and $0.01 \leq t \leq 0.25$, for $\nu = 0.1, x = 0.75$ and $0.01 \leq t \leq 0.25$, and for $\nu = 0.01, x = 0.50$ and $0.01 \leq t \leq 0.25$.

From the diagrams it is obvious how powerful this method is. Using only three terms we can obtain similar results. Of course, in some cases the present

x=.25	v=1		x=.75/t	v=0.1		x=.50/t	v=0.01	
t	**Analytic**	**Decomp**	**t**	**Analytic**	**Decomp**	**t**	**Analytic**	**Decomp**
0,01	0,6724	0,6487	**0,01**	0,7417	0,7538	**0,01**	0,9992	1,0003
0,05	0,4356	0,4809	**0,05**	0,7663	0,7659	**0,05**	0,996	0,9987
0,1	0,2751	0,3081	**0,1**	0,7882	0,7782	**0,1**	0,992	0,9967
0,15	0,1794	0,1762	**0,15**	0,7999	0,7875	**0,15**	0,988	0,9947
0,2	0,1191	0,0852	**0,2**	0,802	0,7936	**0,2**	0,984	0,9927
0,25	0,0807	0,0352	**0,25**	0,7955	0,7968	**0,25**	0,98	0,9907

Table 1: Comparison results for Burger's equation for initial condition $4x(1-x)$ with $v = 1, 0.1, 0.01,\ \Delta x = 0.25;\ \Delta t = 0.05$

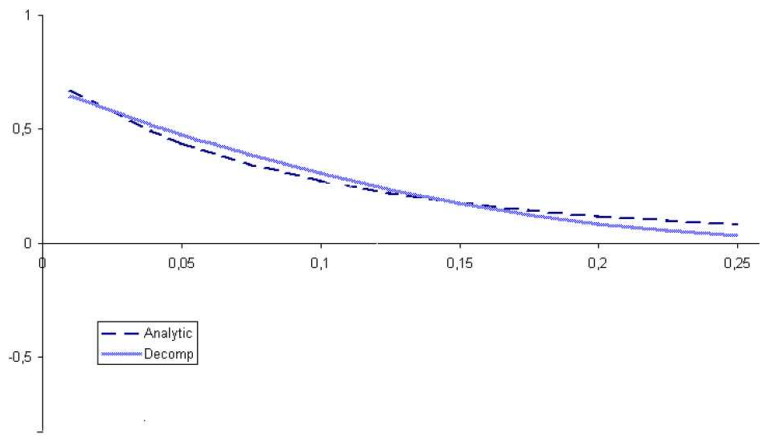

Figure 1: Comparison results with $v = 1$, $x = 0.25$, $0.01 \le t \le 0.25$

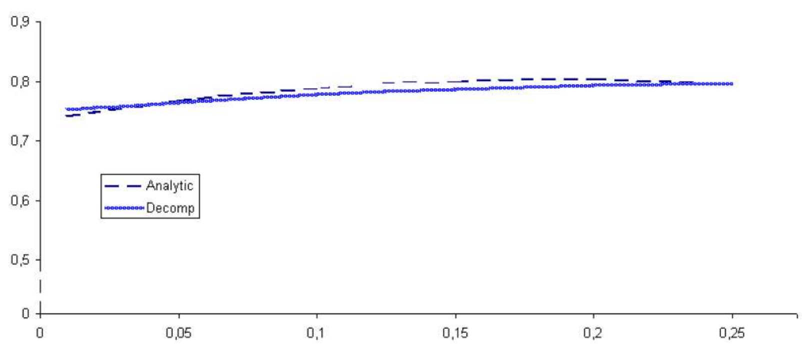

Figure 2: Comparison results with $v = 0.1$, $x = 0.75$, $0.01 \le t \le 0.25$

solutions deviate from the solutions given in the table. The decomposition solution can be furtherly improved if more-term approximations of the solution

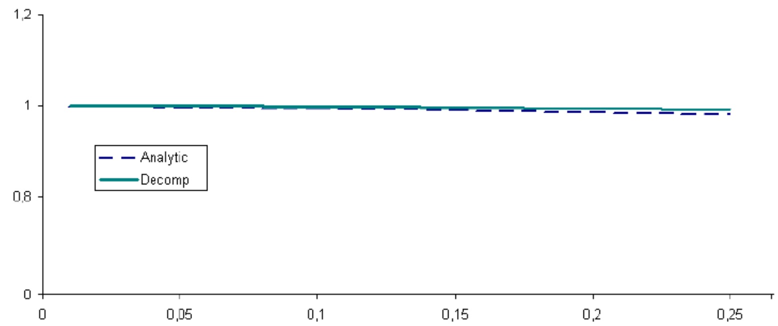

Figure 3: Comparison results with $v = 0.01$, $x = 0.5$, $0.01 \leq t \leq 0.25$

are obtained.

As far as accurate results are concerned, computational experience has shown that they can be obtained easily by taking half a dozen terms. In case we do not have sufficiently high precision by using a few of the A_n, then accordingly to R. Rach [15] there are two alternatives. One is to compute additional terms by any of the available procedures. The second approach is to use the Adomian-Malakian "convergence acceleration" procedure, see [5]. This unique approach conveniently yields the error-damping effect of calculating many more terms of the A_n to determine whether further calculation is justified.

The advantage of this global methodology is that it leads to an analytical continuous approximated solution that is very rapidly convergent, see [3], [11], [12]. This method does not take any help of linearization or any other simplifications for handling the non-linear terms. Since the decomposition parameter in general is not a perturbation parameter, it follows that the non-linearities in the operator equation can be handled easily and accurate solution may be obtained for any physical problem.

References

[1] G. Adomian, *Nonlinear Stochastic Operator Equations*, Academic Press (1986)

[2] G. Adomian, Application of the decomposition method to the Navier-Stokes equations, *J. Math. Anal. Appl.*, **119** (1986), 340-360.

[3] G. Adomian, *Nonlinear Stochastic Systems Theory and Applications to Physics*, Kluwer Academic Publishers (1989)

[4] G. Adomian, *Appld. Math. Lett.*, **6**, No. 5 (1993), 35-3.

[5] G. Adomian, Malakian, Self-correcting approximate solutions by the iterative method for nonlinear stochastic differential equations, *J. Math. Anal. Appl.*, **76** (1980), 309-32.

[6] G. Adomian, Malakian, Existence of the inverse of a linear stochastic operator, *J. Math. Anal. Appl.*, **114** (1986), 55-5.

[7] G. Adomian, R. Rach, Inversion of nonlinear stochastic operators, *J. Math. Anal. Appl.*, **91** (1983), 39-4.

[8] K. Agas, *The Effect of Kinematic Viscosity in the Numerical Solution of Burger's Equation*, Thessaloniki (1998).

[9] E.R. Benton, G.W. Platzman, A table of solutions of the one-dimensional Burger's equation, *Quart. Appl. Math.* (1972).

[10] J.M. Burger, *The Nonlinear Diffusion Equation*, D. Reidel Publishing Company, Univ. of Maryland, USA (1974).

[11] Y. Cherruault, *Kybernetes*, **18**, No. 2 (1989), 31-3.

[12] Y. Cherruault, *Math. Comp. Modelling*, **16**, No. 2 (1992), 85-9.

[13] J.D. Cole, On a quasilinear parabolic equation occuring in aerodynamics, *A. Appl. Maths.*, **9** (1951), 225-23.

[14] N.K. Madsen, R.F. Sincovec, General software for partial differential equations, In: *Numerical Methods for Differential System* (Ed-s: L. Lapidus, W.E. Schiesser), Academic Press (1976).

[15] R. Rach, A convenient computational form of the Adomian polynomials, *J. Math. Anal. Appl.*, **102** (1984), 415-41.