International Journal of Pure and Applied Mathematics
Volume 39 No. 2 2007, 217-233

PORTFOLIO OPTIMIZATION UNDER LIQUIDITY COSTS

Rudi Zagst[1 §], Dieter Kalin[2]
[1]HVB-Institute of Mathematical Finance
Munich University of Technology
Boltzmannstr. 3, Garching, D-85748, GERMANY
e-mai: zagst@ma.tum.de
[2]Insitute of Optimization and Operations Research
University of Ulm
Helmholtzstr. 18, Ulm, D-89069, GERMANY
e-mail: dieter.kalin@uni-ulm.de

Abstract: In this paper we examine the problem of optimally structuring a portfolio of assets with respect to transaction costs and liquidity effects. We claim that the intention of the portfolio manager is to maximize the expected net return of his portfolio, i.e. the expected return after costs, under a given limit for the portfolio risk. We show how this problem can be characterized by a convex optimization problem and that it can be solved by an equivalent quadratic optimization problem minimizing the portfolio risk under a given minimum level for the expected net return. The liquidity cost is estimated using intraday data of the German stock market. A case study shows how the results can be applied to practical trading problems.

AMS Subject Classification: 91B28
Key Words: portfolio optimization, transaction costs, liquidity effects

1. Introduction

The process of performing an optimal asset allocation basically deals with the problem of finding a portfolio that maximizes the expected utility of the investor or portfolio manager. As long as it is supposed that the returns of the portfolio assets follow a normal distribution, the return distribution of any portfolio considered will also be normal. In this case, as is done throughout the

Received: May 14, 2007

§Correspondence author

traditional portfolio theory introduced by Markowitz [22] and Sharpe [26], the problem of finding an expected utility-maximizing portfolio for a risk-averse investor, represented by a concave utility function, can be restricted to finding an optimal combination of the two parameters mean and variance. This dramatically simplifies the whole asset allocation process and is known as mean-variance analysis. It is the aim of the portfolio manager to find a portfolio that maximizes his expected return under a given level of risk or a portfolio that minimizes his risk under a given return level. Risk in this case is measured by the variance of the portfolio return. Unfortunately, a portfolio manager or trader also faces transaction costs reducing the net return of his portfolio. Placing a trade with a broker for execution, the portfolio manager must pay a direct cost of trading no matter if he buys or sells the position. This cost is due to broker commissions, custodial fees, etc. and is also called the explicit cost (EC). Within an optimzation program, EC can be considered by introducing a portfolio turnover constraint, a cost factor that is proportional to the amount of assets bought or sold (see, e.g., Best and Hlouskova [3], Cornuejols [8] and Pogue [24]), a fixed transaction cost (see, e.g., Brennan [5]), a fixed plus a proportional cost, or a proportional cost with lower limit (see, e.g., Lobo et al [19], Maringer [20] and Maringer [21]). Mitchell and Braun [23], Isaenko [11], and Isaenko [12] consider convex (linear, piecewise linear and quadratic) transaction costs with a special focus on illiquid stocks. However, the total cost of the trade also depends on the size of the trade and the broker's ability to place the required trading volume in the market. If the trading volume is too high, i.e. above a critical trading level, the price of the share may rise (fall) between the investment decision and the complete trade execution if the share is to be bought (sold). Thus, this additional cost occurs because the transaction itself may change the market price of the traded asset and is implied by the actual liquidity situation in the market or the broker's ability to trade. The difference between the transaction price and what the market price would have been in the absence of the transaction is termed the market impact cost (MIC), per share, of the transaction.

The vast amount of data generated by electronic trading raised quite some research both, in academic and industrial communities to understand and quantify the MIC. Representative for many others, BARRA [2] analyzes MIC based on the assets of the S&P 500, Bikker et al [4] on equity trades by a Dutch pension fund, Chan and Lakonishok [6] and [7] examine the price impact of single institutional trades and trade packages performed by large institutional money management firms, Gallagher and Looi [9] examine the MIC incurred by active Australian equity managers, Almgren et al [1] examine MIC based on US stock trades executed by Citigroup equity trading desk. The major findings are that MIC is not negligible and can make up a large proportion of the execution costs and that, beside others, MIC is influenced by the stock price volatility, the size of the trade, the average daily trading volume, the relative package size, and the investment style. Dependence on the investment style of the portfolio manager was also stated in Keim and Madhavan [13] and Keim and Madhavan [14]. However, information about the investment style of a portfolio manager or trader is usually not available in the market and will therefore be neglected here. Furthermore, for the sake of simplicity, we assume that we are dealing within one currency, i.e. that all costs and share prices are already reported in local currency.

Despite all the work on MIC and on portfolio optimization with (explicit) transaction cost, very little studies concentrated on the combination of the both, i.e. the optimization of portfolios under the special consideration of MIC. Konno and Wijayanayake [15], [16] and [17] consider non-convex transaction costs and minimal transaction unit constraints. They also allow for the unit price to increase beyond a certain point due to illiquidity and market impact effects. They propose a branch-and-bound algorithm for the resulting d.c. (difference of two convex functions) optimization program and linear programming subproblems for an efficient solution. It is the focus of this paper to propose an easy quadratic optimization model to derive portfolios under the consideration of explicit and market impact costs. In Section 2 we introduce the notion of explicit and implicit transaction cost and define what we understand under the market impact cost. We will show how the market impact cost can be estimated using intraday data of the German stock market in Section 3. In Section 4 we introduce a portfolio optimization problem to find a portfolio with maximum expected net return including explicit and implicit cost under a given maximum level of risk. It is also shown that we can always define an equivalent quadratic optimization problem minimizing risk under a given minimum level of expected net return and hence find an efficient frontier according to the well-known mean-variance theory. We conclude with a practical case study in Section 5.

2. Transaction Costs

As already stated, we decompose the total transaction costs into explicit and implicit costs. Explicit costs are directly observable such as broker commissions or custodial fees. Implicit costs are implied by market or liquidity restrictions and defined as the deviation of the transaction price from the "unperturbed price" that would have prevailed if the trade had not occurred. In other words, market impact cost is the additional price an investor pays for immediate execution. This cost is difficult to measure because the unperturbed price is not observable. Here, the corresponding quote just prior to the transaction is chosen as a proxy for the unperturbed price $S_i > 0$ per share $i = 1, \ldots, n$. Within the problem of portfolio optimization we will assume, for the sake of simplicity, that each share is traded at it is mid price. Therefore, we neglect the cost implied by the bid-ask spread. Hence, the market impact cost is the change in the stock price that only occurs when the number of stocks an investor desires to buy or sell exceeds the number other market participants are willing to buy ($x^+_{i,\min} \geq 0$) or sell ($x^-_{i,\min} \geq 0$) at that price. Typically, market impact cost would decrease over time because a trader with more time can split up the transaction into smaller transactions that individually exert little or no price pressure. We assume that there is a maximum execution time from which on the market impact cost vanishes. On the other hand, waiting for the complete execution may lead to a loss in oppotunities related to changing market prices or a decaying value of the information responsible for the original portfolio decision. This so-called opportunity cost tends to increase over time. The reflection principle introduced by Hafner [10] states that there is a trade-off between market impact and opportunity cost. It holds under the main assumptions that the liquidity demander and the liquidity provider have the same risk aversion and that liquidity is priced efficiently. For an immediate execution only market impact cost will occur as we have no opportunity cost. As time increases market impact cost will vanish leaving us with opportunity cost only. Due to the reflection principle there is an exact shift from market impact to opportunity cost, i.e. market impact cost for an immediate execution is equal to the opportunity cost for the maximum execution time. Therefore, market impact cost will include a factor for the time the broker needs to execute the position and a factor for the risk of the unknown asset price at which the position can be executed. The first factor will increase with the volume to be traded and decrease with the average (tick or daily) trade size $\overline{x}_i > 0$ as a measure for the liquidity of the share, the second factor is usually measured in terms of the share's intraday volatility $\sigma_{i,intraday} > 0$, $i = 1, \ldots, n$. We assume immediate execution and a market impact cost function of

$$c_{MI}\left(x_i^*\right) = \lambda_i \cdot S_i \cdot \sigma_{i,intraday} \cdot \frac{\left(x_i^* - x_{i,\min}^*\right)^+}{\overline{x}_i} = k_i \cdot S_i \cdot \left(x_i^* - x_{i,\min}^*\right)^+$$

with $* \in \{+,-\}$, x_i^* denoting the number of shares to be bought (+) or sold (−), and liquidity cost (factor)

$$k_i = \lambda_i \cdot \sigma_{i,intraday} \cdot \frac{1}{\overline{x}_i}.$$

We hereby assume that the market impact cost per share $i = 1, \dots, n$ is proportional to the excess traded volume above the critical trade size $x_{i,\min}^*$ as well as to the inverse of the average trade size and hence to the average time for execution $\left(x_i^* - x_{i,\min}^*\right)^+ / \overline{x}_i$. Furthermore, it is proportional to the volatility $\sigma_{i,intraday}$ as well as to a factor $\lambda_i \geq 0$ which we call the price of liquidity risk for share i and which may depend on the (excess) traded volume or the average trade size. Each choice of λ_i leads to a different model of liquidity risk. However, we have chosen λ_i to be constant for the sake of simplicity here. This approach corresponds to the empirical findings stated above and follows the economic approaches in BARRA [2] and Hafner [10].

3. Estimating the Cost of Liquidity

The sample data for estimating the model parameters consists of cleaned tick data from the German stock market between 17-th April 2001 and 5-th June 2001 as it was used by Hafner [10]. Each data record includes date, time (accurate to seconds), bid, ask and last price as well as the corresponding (critical) trade sizes. Only trades during normal market hours, i.e. after 9:00 a.m. and before 8:00 p.m. are considered. To be a trade the cumulative traded volume of the day must have changed. If the trade price is above (below) the latest mid price, the trade is considered as buyer- (seller-) initiated. By definition, the unperturbed price is the latest quote prior to the trade. To calculate the volatility of the stocks in a consistent way we had to consider that the data may be nonsynchronous because some shares were more frequently traded than others. Therefore, the time unit Δt is chosen such that each stock is traded at least once in each time interval. Given the different trades and their volumes in a specific time interval, the synthetic trade price of the corresponding stock is set to the volume-weighted average trade price and the trading volume to the average trade size in this interval. We then use this synthetic empirical price data to calculate the log-returns and their empirical variance assuming that the expected log-return equals zero. Dividing the variance by Δt and taking the square root we end up with the stock's volatility. Given the trade considered is buyer- (seller-) initiated and the trading volume exceeds the ask (bid) size, the market impact cost is defined to be the absolute difference between the trade price and the ask (bid) price just before the trade multiplied with the liquidity cost (factor). Thus, the only parameter missing is the price of liquidity risk λ_i, $i = 1, \ldots, n$. This parameter can now be determined using an OLS regression. The results are summarized in Table 1.

4. The Optimization Problem

In this section we state the optimization problem which maximizes the net profit over a given planning horizon T under a given maximum level of risk $\sigma_{\max} > 0$ for the portfolio return. As above, let $x_i^+ \geq 0$ $\left(x_i^- \geq 0\right)$ denote the number of stocks from asset $i = 1, \ldots, n$ which are to be bought (sold) for an optimal portfolio decision. The number of stocks $x = (x_1, \ldots, x_n)'$ in the portfolio is then given by $x_i = x_i^+ - x_i^-$, $i = 1, \ldots, n$. Furthermore, let $c = (c_1, \ldots, c_n)' \geq 0$ denote the proportional explicit cost per share, i.e. the explicit cost for a number of $x_i^\pm$ shares bought or sold at an unperturbed price $S_i > 0$, $i = 1, \ldots, n$, is given by

$$c_E\left(x_i^\pm\right) = c_i \cdot S_i \cdot x_i^\pm.$$

We assume that the portfolio decision is for an immediate execution resulting in an additional market impact cost if the optimal number of stocks to be bought or sold exceeds the critical trade size $x_{i,\min}^\pm$. The prices at the end of the planning horizon are given by the random vector $S(T) = (S_1(T), \ldots, S_n(T))'$ resulting in a corresponding vector $R = (R_1, \ldots, R_n)'$ for the rate of return with

$$R_i = \frac{S_i(T) - S_i}{S_i},\ i = 1, \ldots, n.$$

The expected rate of return is denoted by $\mu = (\mu_1, \ldots, \mu_n)'$ with $\mu_i = E[R_i]$, $i = 1, \ldots, n$, and the covariance matrix is given by $C = (\sigma_{ij})_{i,j=1,\ldots,n}$ with $\sigma_{ij} = \mathrm{Cov}[R_i, R_j]$ and $\sigma_i^2 := \sigma_{ii} > 0$, $i, j = 1, \ldots, n$. It is assumed that C is positive definit and that the total budget or trading volume is restricted to a cash amount of $B > 0$ where the part of the budget which is not used for a stock investment can be allocated at a constant rate of return $r > 0$. Hence, the total cost $TC(x, x^+, x^-)$ of the portfolio is limited by

$$TC\left(x, x^+, x^-\right) = \sum_{i=1}^{n} \left(x_i \cdot S_i + c_E\left(x_i^+ + x_i^-\right) + c_{MI}\left(x_i^+\right) + c_{MI}\left(x_i^-\right)\right) \leq B,$$

Share	Last	Volatility	Aver.	Liquidity
	Price		daily trade size	cost
ALLIANZ	321,45	0,0035%	782040	0,03%
TELEKOM	19,10	0,0124%	22696585	0,07%
MUNICH RE	328,10	0,0054%	455819	0,03%
DAIMLERCHRYSLER	52,50	0,0061%	2717134	0,05%
SIEMENS	59,90	0,0110%	4537083	0,07%
SAP	159,91	0,0052%	1657061	0,05%
DEUTSCHE BANK	74,70	0,0087%	3308851	0,08%
E.ON	60,60	0,0067%	1668693	0,01%
BASF	44,92	0,0044%	2599873	0,04%
RWE	47,75	0,0059%	1073789	0,06%
BAYER	36,10	0,0097%	8246891	0,01%
BMW	38,60	0,0059%	1916431	0,03%
HYPOVEREINSBANK	49,41	0,0089%	986790	0,10%
VOLKSWAGEN	52,22	0,0061%	1113849	0,04%
INFINEON	24,99	0,0230%	4168153	0,11%
METRO	44,80	0,0040%	802733	0,04%
COMMERZBANK	26,83	0,0092%	2465530	0,07%
SCHERING	59,39	0,0065%	441379	0,06%
HENKEL	73,25	0,0074%	393766	0,10%
DEUTSCHE POST	17,96	0,0036%	710135	0,01%
THYSSEN KRUPP	15,57	0,0045%	971003	0,04%
FRESENIUS	88,60	0,0038%	205062	0,09%
DT LUFTHANSA	18,70	0,0061%	1224505	0,07%
PREUSSAG	34,60	0,0080%	411352	0,14%
DEGUSSA	29,50	0,0026%	262865	0,05%
LINDE	47,50	0,0051%	186740	0,15%
MAN	26,63	0,0037%	427172	0,06%
ADIDAS SALOMON	75,15	0,0052%	205732	0,06%
EPCOS	49,74	0,0120%	373249	0,20%

Table 1: Liquidity cost and market information

or equivalently

$$e'\widetilde{x} + c'\left(\widetilde{x}^{+} + \widetilde{x}^{-}\right) + k'\left(\widetilde{x}^{+} - \widetilde{x}^{+}_{\min}\right)^{+} + k'\left(\widetilde{x}^{-} - \widetilde{x}^{-}_{\min}\right)^{+} \leq 1$$

with

$$\widetilde{x}_i = \frac{x_i \cdot S_i}{B},\ \widetilde{x}_i^{\pm} = \frac{x_i^{\pm} \cdot S_i}{B}, \text{ and } \widetilde{x}_{i,\min}^{\pm} = \frac{x_{i,\min}^{\pm} \cdot S_i}{B},\ i = 1, \ldots, n,$$

and $\widetilde{x}^{\pm}_{\min} = \left(\widetilde{x}^{\pm}_{1,\min 1}, \ldots, \widetilde{x}^{\pm}_{n,\min}\right)'$. Furthermore, the (net) return $R(x, x^{+}, x^{-})$

of the portfolio is given by

$$\begin{aligned} R\left(x,x^{+},x^{-}\right) &= \frac{\sum_{i=1}^{n} x_i \cdot S_i\left(T\right) + \left(B - TC\left(x,x^{+},x^{-}\right)\right)\cdot(1+r) - B}{B} \\ &= R'\widetilde{x} + r\cdot\left(1-e'\widetilde{x}\right) - (1+r)\cdot c\left(\widetilde{x},\widetilde{x}^{+},\widetilde{x}^{-}\right) \end{aligned}$$

with $e = (1,\ldots,1)'$ and

$$c\left(\widetilde{x},\widetilde{x}^{+},\widetilde{x}^{-}\right) = c'\left(\widetilde{x}^{+}+\widetilde{x}^{-}\right) + k'\left(\widetilde{x}^{+}-\widetilde{x}^{+}_{\min}\right)^{+} + k'\left(\widetilde{x}^{-}-\widetilde{x}^{-}_{\min}\right)^{+}.$$

Consequently, the expected portfolio return is

$$\mu\left(\widetilde{x},\widetilde{x}^{+},\widetilde{x}^{-}\right) = \mu'\widetilde{x} + r\cdot\left(1-e'\widetilde{x}\right) - (1+r)\cdot c\left(\widetilde{x},\widetilde{x}^{+},\widetilde{x}^{-}\right)$$

and the variance of the portfolio return is given by

$$\sigma^2\left(\widetilde{x}\right) = \widetilde{x}'C\widetilde{x}.$$

Replacing $\widetilde{x}^{+} = \widetilde{x} + \widetilde{x}^{-}$ we consider the following optimization problem

$$P_1\left(\sigma^2_{\max}\right)\begin{cases} \widehat{\mu}'\widetilde{x} + r\cdot(1-e'\widetilde{x}) - \widehat{c}'\widetilde{x}^{-} - \widehat{k}'\left(y^{+}+y^{-}\right) \to \max\,, \\ \quad \widetilde{x}'C\widetilde{x} \le \sigma^2_{\max}\,, \\ \quad (e+c)'\,\widetilde{x} + 2\cdot c'\widetilde{x}^{-} + k'\left(y^{+}+y^{-}\right) \le 1\,, \\ \quad \widetilde{x} + \widetilde{x}^{-} - \widetilde{x}^{+}_{\min} \le y^{+}\,, \\ \quad \widetilde{x}^{-} - \widetilde{x}^{-}_{\min} \le y^{-}\,, \\ \quad \widetilde{x} + \widetilde{x}^{-} \ge 0, \widetilde{x}^{-} \ge 0, y^{+} \ge 0, y^{-} \ge 0\,, \end{cases}$$

with $\widehat{\mu} := \mu - (1+r)\cdot c$, $\widehat{c} := 2\cdot(1+r)\cdot c$, and $\widehat{k} := (1+r)\cdot k$. Let I_n denote the $n-$ dimensional identity matrix, 0_n the $n-$dimensional matrix filled with zeros,

$$A_1 = \begin{pmatrix} (e+c)' \\ I_n \\ 0_n \\ -I_n \\ 0_n \\ 0_n \\ 0_n \end{pmatrix}, A_2 = \begin{pmatrix} 2\cdot c' \\ I_n \\ I_n \\ -I_n \\ -I_n \\ 0_n \\ 0_n \end{pmatrix}, A_3 = \begin{pmatrix} k' \\ -I_n \\ 0_n \\ 0_n \\ 0_n \\ -I_n \\ 0_n \end{pmatrix}, A_4 = \begin{pmatrix} k' \\ 0_n \\ -I_n \\ 0_n \\ 0_n \\ 0_n \\ -I_n \end{pmatrix}$$

and $b = \left(1, \left(\widetilde{x}^{+}_{\min}\right)', \left(\widetilde{x}^{-}_{\min}\right)', 0', 0', 0', 0'\right)'$. Then we can reformulate our optimization problem to

$$P_1\left(\sigma^2_{\max}\right)\begin{cases} \widehat{\mu}'\widetilde{x} + r\cdot(1-e'\widetilde{x}) - \widehat{c}'\widetilde{x}^{-} - \widehat{k}'\left(y^{+}+y^{-}\right) \to \max\,, \\ \quad \widetilde{x}'C\widetilde{x} \le \sigma^2_{\max}\,, \\ \quad A_1\widetilde{x} + A_2\widetilde{x}^{-} + A_3 y^{+} + A_4 y^{-} \le b\,. \end{cases}$$

We generally assume that the expected excess rate of return after cost exceeds the interest we pay for financing the transaction cost, i.e.

$$\mu_i - r - c_i - k_i > r \cdot (c_i + k_i) \quad \text{for all } i = 1, \ldots, n,$$

or equivalently

$$\widehat{\mu}_i > r + (1+r) \cdot k_i \quad \text{for all } i = 1, \ldots, n.$$

In the special case of no transaction cost this reduces to the well-known assumption that $\mu_i > r$ for all $i = 1, \ldots, n$.

Lemma 4.1. *Let $(\widetilde{x}', \widetilde{x}^{-\prime}, y^{+\prime}, y^{-\prime})'$ be an optimal solution for $P_1\left(\sigma^2_{\max}\right)$. Then,*

$$\widehat{\mu}'\widetilde{x} + r \cdot \left(1 - e'\widetilde{x}\right) - \widehat{c}'\widetilde{x}^- - \widehat{k}'\left(y^+ + y^-\right) > r$$

and for each $i \in \{1, \ldots, n\}$ with $c_i > 0$ we have

$$\widetilde{x}_i^+ = y_i^+ = 0 \quad \text{or} \quad \widetilde{x}_i^- = y_i^- = 0.$$

Proof. Let $(\widetilde{x}', \widetilde{x}^{-\prime}, y^{+\prime}, y^{-\prime})'$ be an optimal solution for $P_1\left(\sigma^2_{\max}\right)$. Furthermore, let $(x', x^{-\prime}, x', x^{-\prime})'$ be defined by

$$x_i := \begin{cases} \min\left\{\frac{\sigma_{\max}}{\sigma_1}, \frac{1}{1+c_1+k_1}\right\} & , \text{if } i = 1 \\ 0 & , \text{if } i \neq 1 \end{cases} \quad \text{and} \quad x^- \equiv 0.$$

Then, $(x', x^{-\prime}, x', x^{-\prime})'$ is a feasible solution for $P_1\left(\sigma^2_{\max}\right)$ with

$$\begin{aligned} \widehat{\mu}'x + r \cdot \left(1 - e'x\right) - \widehat{c}'x^- - \widehat{k}'\left(x + x^-\right) &= \widehat{\mu}_1 \cdot x_1 + r - r \cdot x_1 - \widehat{k}_1 \cdot x_1 \\ &= r + \underbrace{\left(\widehat{\mu}_1 - r - \widehat{k}_1\right)}_{>0} \cdot \underbrace{x_1}_{>0} > r. \end{aligned}$$

Due to the optimality of $(\widetilde{x}', \widetilde{x}^{-\prime}, y^{+\prime}, y^{-\prime})'$ we conclude that

$$\begin{aligned} r &< \widehat{\mu}'x + r \cdot \left(1 - e'x\right) - \widehat{c}'x^- - \widehat{k}'\left(x + x^-\right) \\ &\leq \widehat{\mu}'\widetilde{x} + r \cdot \left(1 - e'\widetilde{x}\right) - \widehat{c}'\widetilde{x}^- - \widehat{k}'\left(y^+ + y^-\right). \end{aligned}$$

Also due to the optimality of $(\widetilde{x}', \widetilde{x}^{-\prime}, y^{+\prime}, y^{-\prime})'$ it is straightforward that $y^\pm = \max\{\widetilde{x}^\pm - \widetilde{x}^\pm_{\min}; 0\}$ because $y^\pm \geq \widetilde{x}^\pm - \widetilde{x}^\pm_{\min}$ and $y^\pm \geq 0$. Assume that $\widetilde{x}_i^+ > 0$ and $\widetilde{x}_i^- > 0$ for some $i \in \{1, \ldots, n\}$. Define $(\widehat{x}, \widehat{x}^-, \widehat{y}^+, \widehat{y}^-)'$ by

$$\widehat{x}_j^+ := \begin{cases} \widetilde{x}_i^+ - \widetilde{x}_i^- & , \text{if } j = i, \widetilde{x}_i^+ \geq \widetilde{x}_i^-\,, \\ 0 & , \text{if } j = i, \widetilde{x}_i^+ < \widetilde{x}_i^-\,, \\ \widetilde{x}_j^+ & , \text{if } j \neq i\,, \end{cases}$$

and

$$\widehat{x}_j^- := \begin{cases} 0 & \text{, if } j = i, \widetilde{x}_i^+ \geq \widetilde{x}_i^- \,, \\ \widetilde{x}_i^- - \widetilde{x}_i^+ & \text{, if } j = i, \widetilde{x}_i^+ < \widetilde{x}_i^- \,, \\ \widetilde{x}_j^- & \text{, if } j \neq i \,, \end{cases}$$

for $j = 1, \ldots, n$ and $\widehat{x} := \widehat{x}^+ - \widehat{x}^-$. Then,

$$\widetilde{x} = \widehat{x},\ \widehat{x}^\pm < \widetilde{x}^\pm,\ \widehat{x}^+ + \widehat{x}^- < \widetilde{x}^+ + \widetilde{x}^-$$

and hence, $(\widehat{x}, \widehat{x}^-, y^+, y^-)'$ is a feasible solution for $P_1\left(\sigma^2_{\max}\right)$ with

$$\begin{aligned} &\widehat{\mu}'\widehat{x} + r \cdot (1 - e'\widehat{x}) - \widehat{c}'\widehat{x}^- - \widehat{k}'\left(y^+ + y^-\right) > \\ &\qquad > \widehat{\mu}'\widetilde{x} + r \cdot (1 - e'\widetilde{x}) - \widehat{c}'\widetilde{x}^- - \widehat{k}'\left(y^+ + y^-\right) \end{aligned}$$

which is a contradiction to the assumption that $(\widetilde{x}', \widetilde{x}^{-\prime}, y^{+\prime}, y^{-\prime})'$ is an optimal solution for $P_1\left(\sigma^2_{\max}\right)$. Hence, $\widetilde{x}_i^+ = 0$ or $\widetilde{x}_i^- = 0$ and consequently $y^+ = \left(\widetilde{x}^+ - \widetilde{x}^+_{\min}\right)^+ = \left(-\widetilde{x}^+_{\min}\right)^+ = 0$ or $y^- = \left(\widetilde{x}^- - \widetilde{x}^-_{\min}\right)^+ = \left(-\widetilde{x}^-_{\min}\right)^+ = 0$. □

According to the proof of Lemma 4.1 there is always an optimal solution $\widetilde{x}$ for $P_1\left(\sigma^2_{\max}\right)$, $\sigma_{\max} > 0$, with $\widetilde{x}_i^+ = y_i^+ = 0$ or $\widetilde{x}_i^- = y_i^- = 0$ for each $i \in \{1, \ldots, n\}$, even if the corresponding $c_i = 0$. Lemma 4.1 genaralizes the results of Lemma 2.1 in Best and Hlouskova [3], who assume $k \equiv 0$, for the case $k \geq 0$.

Lemma 4.2. *Let $(\widetilde{x}', \widetilde{x}^{-\prime}, y^{+\prime}, y^{-\prime})'$ be an optimal solution for $P_1\left(\sigma^2_{\max}\right)$. Then, $\widetilde{x}'C\widetilde{x} = \sigma^2_{\max}$.*

Proof. $(\widetilde{x}', \widetilde{x}^{-\prime}, y^{+\prime}, y^{-\prime})'$ is an optimal solution for $P_1\left(\sigma^2_{\max}\right)$ iff it is a feasible solution and there are non-negative $u_1, \widetilde{u}$ such that the following Kuhn-Tucker conditions are satisfied:

$$\begin{aligned} &(1) \quad \widehat{\mu} - r \cdot e + 2 \cdot u_1 \cdot C\widetilde{x} + A_1'\widetilde{u} = 0\,, \\ &(2) \quad -\widehat{c} + A_2'\widetilde{u} = 0\,, \\ &(3) \quad -\widehat{k} + A_3'\widetilde{u} = 0\,, \\ &(4) \quad -\widehat{k} + A_4'\widetilde{u} = 0\,, \\ &(5) \quad u_1 \cdot \left(\widetilde{x}'C\widetilde{x} - \sigma^2_{\max}\right) = 0\,, \\ &(6) \quad \widetilde{u}'\left(A_1\widetilde{x} + A_2\widetilde{x}^- + A_3 y^+ + A_4 y^- - b\right) = 0\,. \end{aligned}$$

Adding (5) and (6) we get

$$0 = -u_1 \cdot \sigma^2_{\max} + \left(u_1 \cdot \widetilde{x}'C \ + \widetilde{u}'A_1\right)\widetilde{x} + \widetilde{u}'A_2\widetilde{x}^- + \widetilde{u}'A_3 y^+ + \widetilde{u}'A_4 y^- - \widetilde{u}'b$$

and thus, using (1)-(4):

$$(7) \quad -u_1 \cdot \left(\widetilde{x}'C\widetilde{x} + \sigma^2_{\max}\right) - (\widehat{\mu} - r \cdot e)'\,\widetilde{x} + \widehat{c}'\widetilde{x}^- + \widehat{k}'\left(y^+ + y^-\right) - \widetilde{u}'b = 0.$$

Assume that $\widetilde{x}'C\widetilde{x} < \sigma^2_{\max}$. Then, using (5), we get $u_1 = 0$ and thus from (7):

$$(\widehat{\mu} - r \cdot e)' \, \widetilde{x} - \widehat{c}' \widetilde{x}^- - \widehat{k}' \left(y^+ + y^-\right) + \underbrace{\widetilde{u}' b}_{\geq 0} = 0 \, ,$$

which leads us to

$$\widehat{\mu}' \widetilde{x} + r \cdot \left(1 - e' \widetilde{x}\right) - \widehat{c}' \widetilde{x}^- - \widehat{k}' \left(y^+ + y^-\right) \leq r.$$

This is a contradiction to the statement in Lemma 4.1 and thus

$$\widetilde{x}'C\widetilde{x} = \sigma^2_{\max}. \qquad \square$$

Let us now fix a minimum level $\mu_{\min} > r$ for the expected portfolio return and consider the quadratic optimization problem

$$P_2\left(\mu_{\min}\right) \begin{cases} \widetilde{x}'C\widetilde{x} \to \min \, , \\ \quad \widehat{\mu}' \widetilde{x} + r \cdot (1 - e' \widetilde{x}) - \widehat{c}' \widetilde{x}^- - \widehat{k}' \left(y^+ + y^-\right) \geq \mu_{\min} \, , \\ \quad A_1 \widetilde{x} + A_2 \widetilde{x}^- + A_3 y^+ + A_4 y^- \leq b. \end{cases}$$

Then we can proof the following analogon to Lemma 4.2.

Lemma 4.3. *Let* $(\widehat{x}', \widehat{x}^{-\prime}, \widehat{y}^{+\prime}, \widehat{y}^{-\prime})'$ *be an optimal solution for* $P_2\left(\mu_{\min}\right)$. *Then,* $\widehat{x}'C\widehat{x} > 0$ *and*

$$\widehat{\mu}' \widehat{x} + r \cdot \left(1 - e' \widehat{x}\right) - \widehat{c}' \widehat{x}^- - \widehat{k}' \left(\widehat{y}^+ + \widehat{y}^-\right) = \mu_{\min}.$$

Proof. Let $(\widehat{x}', \widehat{x}^{-\prime}, \widehat{y}^{+\prime}, \widehat{y}^{-\prime})'$ be an optimal solution for $P_2\left(\mu_{\min}\right)$ and assume that $\widehat{x}'C\widehat{x} = 0$. Because C is positive definit this is equivalent to $\widehat{x} \equiv 0$. Thus,

$$\mu_{\min} \leq \underbrace{\widehat{\mu}' \widehat{x}}_{=0} + r \cdot \left(1 - \underbrace{e' \widehat{x}}_{=0}\right) - \underbrace{\widehat{c}' \widehat{x}^-}_{\geq 0} - \underbrace{\widehat{k}' \left(\widehat{y}^+ + \widehat{y}^-\right)}_{\geq 0} \leq r$$

which is a contradiction to our assumption $\mu_{\min} > r$. Now, $(\widehat{x}', \widehat{x}^{-\prime}, \widehat{y}^{+\prime}, \widehat{y}^{-\prime})'$ is an optimal solution for $P_2\left(\mu_{\min}\right)$ iff it is a feasible solution and there are non-negative $v_1, \widetilde{v}$ such that the following Kuhn-Tucker conditions are satisfied:

(1′) $2 \cdot C\widehat{x} - v_1 \cdot (\widehat{\mu} - r \cdot e) + A_1' \widetilde{v} = 0 \, ,$
(2′) $v_1 \cdot \widehat{c} + A_2' \widetilde{v} = 0 \, ,$
(3′) $v_1 \cdot \widehat{k} + A_3' \widetilde{v} = 0 \, ,$
(4′) $v_1 \cdot \widehat{k} + A_4' \widetilde{v} = 0 \, ,$
(5′) $v_1 \cdot \Big(\mu_{\min} - \widehat{\mu}' \widehat{x} - r \cdot (1 - e' \widehat{x}) + \widehat{c}' \widehat{x}^- + \widehat{k}' \left(\widehat{y}^+ + \widehat{y}^-\right)\Big) = 0 \, ,$
(6′) $\widetilde{v}' \left(A_1 \widehat{x} + A_2 \widehat{x}^- + A_3 \widehat{y}^+ + A_4 \widehat{y}^- - b\right) = 0 \, .$

Adding $(5')$ and $(6')$ we get

$$\begin{aligned} 0 &= v_1 \cdot (\mu_{\min} - r) + \left(\widetilde{v}' A_1 - v_1 \cdot (\widehat{\mu} - r \cdot e)'\right) \widehat{x} + \left(\widetilde{v}' A_2 + v_1 \cdot \widehat{c}'\right) \widehat{x}^- \\ &\quad + \left(\widetilde{v}' A_3 + v_1 \cdot \widehat{k}'\right) \widehat{y}^+ + \left(\widetilde{v}' A_4 + v_1 \cdot \widehat{k}'\right) \widehat{y}^- - \widetilde{v}' b \end{aligned}$$

and thus, using $(1')$-$(4')$:

$$(7') \quad v_1 \cdot (\mu_{\min} - r) - 2 \cdot \widehat{x}' C \widehat{x} - \widetilde{v}' b = 0.$$

Assume that $\mu_{\min} < \widehat{\mu}'\widehat{x} + r \cdot (1 - e'\widehat{x}) - \widehat{c}'\widehat{x}^- - \widehat{k}' (\widehat{y}^+ + \widehat{y}^-)$. Then, using $(5')$, we get $v_1 = 0$ and thus from $(7')$:

$$2 \cdot \underbrace{\widehat{x}' C \widehat{x}}_{\geq 0} + \underbrace{\widetilde{v}' b}_{\geq 0} = 0.$$

Because C is positive definite, we conclude that $\widehat{x} \equiv 0$ and thus

$$\mu_{\min} \leq \underbrace{\widehat{\mu}'\widehat{x}}_{=0} + r \cdot \left(1 - \underbrace{e'\widehat{x}}_{=0}\right) - \underbrace{\widehat{c}'\widehat{x}^-}_{\geq 0} - \underbrace{\widehat{k}' \left(\widehat{y}^+ + \widehat{y}^-\right)}_{\geq 0} \leq r$$

in contradiction to our assumption $\mu_{\min} > r$. Hence

$$\widehat{\mu}'\widehat{x} + r \cdot \left(1 - e'\widehat{x}\right) - \widehat{c}'\widehat{x}^- - \widehat{k}' \left(\widehat{y}^+ + \widehat{y}^-\right) = \mu_{\min}. \qquad \square$$

Theorem 4.4. *Let $\mu^* \left(\sigma^2_{\max}\right)$ denote the maximum value of the objective function in $P_1 \left(\sigma^2_{\max}\right)$ with $\sigma^2_{\max} > 0$. Furthermore, let $\sigma^{*2} (\mu_{\min})$ denote the minimum value of the objective function in $P_2 (\mu_{\min})$ with $\mu_{\min} > r$. Then,*

$$\mu^* \left(\sigma^{*2} (\mu_{\min})\right) = \mu_{\min} \quad \text{and} \quad \sigma^{*2} \left(\mu^* \left(\sigma^2_{\max}\right)\right) = \sigma^2_{\max}.$$

Proof. Let $(\widetilde{x}', \widetilde{x}^{-\prime}, y^{+\prime}, y^{-\prime})'$ be an optimal solution for $P_1 \left(\sigma^{*2} (\mu_{\min})\right)$. Then, using Lemma 4.2, $\widetilde{x}' C \widetilde{x} = \sigma^{*2} (\mu_{\min})$. Furthermore, let $(\widehat{x}', \widehat{x}^{-\prime}, \widehat{y}^{+\prime}, \widehat{y}^{-\prime})'$ be an optimal solution for $P_2 (\mu_{\min})$. Then, $(\widehat{x}', \widehat{x}^{-\prime}, \widehat{y}^{+\prime}, \widehat{y}^{-\prime})'$ is a feasible solution for $P_1 \left(\sigma^{*2} (\mu_{\min})\right)$ and, using Lemma 4.3,

$$\begin{aligned} \mu^* \left(\sigma^{*2} (\mu_{\min})\right) &= \widehat{\mu}'\widetilde{x} + r \cdot \left(1 - e'\widetilde{x}\right) - \widehat{c}'\widetilde{x}^- - \widehat{k}' \left(y^+ + y^-\right) \\ &\geq \widehat{\mu}'\widehat{x} + r \cdot \left(1 - e'\widehat{x}\right) - \widehat{c}'\widehat{x}^- - \widehat{k}' \left(\widehat{y}^+ + \widehat{y}^-\right) = \mu_{\min}. \end{aligned}$$

Hence, $(\widetilde{x}', \widetilde{x}^{-\prime}, y^{+\prime}, y^{-\prime})'$ is a feasible solution for $P_2 (\mu_{\min})$ with $\widetilde{x}' C \widetilde{x} = \sigma^{*2} (\mu_{\min})$ and thus an optimal solution for $P_2 (\mu_{\min})$. Therefore, again using Lemma 4.3,

$$\mu^* \left(\sigma^{*2} (\mu_{\min})\right) = \widehat{\mu}'\widetilde{x} + r \cdot \left(1 - e'\widetilde{x}\right) - \widehat{c}'\widetilde{x}^- - \widehat{k}' \left(y^+ + y^-\right) = \mu_{\min}.$$

Now, let $(\widehat{x}', \widehat{x}^{-\prime}, \widehat{y}^{+\prime}, \widehat{y}^{-\prime})'$ be an optimal solution for $P_2\left(\mu^*\left(\sigma^2_{\max}\right)\right)$. Then, using Lemma 4.3,

$$\widehat{\mu}'\widehat{x} + r \cdot \left(1 - e'\widehat{x}\right) - \widehat{c}'\widehat{x}^- - \widehat{k}'\left(\widehat{y}^+ + \widehat{y}^-\right) = \mu^*\left(\sigma^2_{\max}\right).$$

Furthermore, let $(\widetilde{x}', \widetilde{x}^{-\prime}, y^{+\prime}, y^{-\prime})'$ be an optimal solution for $P_1\left(\sigma^2_{\max}\right)$. Then, $(\widetilde{x}', \widetilde{x}^{-\prime}, y^{+\prime}, y^{-\prime})'$ is a feasible solution for $P_2\left(\mu^*\left(\sigma^2_{\max}\right)\right)$ and, using Lemma 4.2,

$$\sigma^{*2}\left(\mu^*\left(\sigma^2_{\max}\right)\right) = \widehat{x}'C\widehat{x} \leq \widetilde{x}'C\widetilde{x} = \sigma^2_{\max}.$$

Hence, $(\widehat{x}', \widehat{x}^{-\prime}, \widehat{y}^{+\prime}, \widehat{y}^{-\prime})'$ is a feasible solution for $P_1\left(\sigma^2_{\max}\right)$ with $\widehat{\mu}'\widehat{x} + r \cdot (1 - e'\widehat{x}) - \widehat{c}'\widehat{x}^- - \widehat{k}'\left(\widehat{y}^+ + \widehat{y}^-\right) = \mu^*\left(\sigma^2_{\max}\right)$ and thus an optimal solution for $P_1\left(\sigma^2_{\max}\right)$. Therefore, again using Lemma 4.2,

$$\sigma^{*2}\left(\mu^*\left(\sigma^2_{\max}\right)\right) = \widehat{x}'C\widehat{x} = \sigma^2_{\max}. \qquad \square$$

Theorem 4.4 is consistent to Theorem 3 in Krokhmal et al [18]. However, we had to give a separate proof because our parameters u_1 and v_1 may not be positive.

Theorem 4.5. *The efficient frontier $\mu_{\min} \to \sigma^*(\mu_{\min})$ is convex for all $\mu_{\min} > r$.*

Proof. Let $\lambda \in [0,1]$, $(\widehat{x}', \widehat{x}^{-\prime}, \widehat{y}^{+\prime}, \widehat{y}^{-\prime})'$ be an optimal solution for $P_2(\mu_{\min})$ and $(\overline{x}', \overline{x}^{-\prime}, \overline{y}^{+\prime}, \overline{y}^{-\prime})'$ be an optimal solution for $P_2(\overline{\mu}_{\min})$. Then,

$$\begin{pmatrix} x(\lambda) \\ x^-(\lambda) \\ y^+(\lambda) \\ y^-(\lambda) \end{pmatrix} = \lambda \cdot \begin{pmatrix} \widehat{x} \\ \widehat{x}^- \\ \widehat{y}^+ \\ \widehat{y}^- \end{pmatrix} + (1-\lambda) \cdot \begin{pmatrix} \overline{x} \\ \overline{x}^- \\ \overline{y}^+ \\ \overline{y}^- \end{pmatrix}$$

is a feasible solution for $P_2(\lambda \cdot \mu_{\min} + (1-\lambda) \cdot \overline{\mu}_{\min})$ and thus, using the inequality of Cauchy-Schwartz,

$$\begin{aligned} \sigma^{*2}(\lambda \cdot \mu_{\min} + (1-\lambda) \cdot \overline{\mu}_{\min}) &\leq x(\lambda)' C x(\lambda) \\ &= \lambda^2 \cdot \widehat{x}'C\widehat{x} + 2 \cdot \lambda \cdot (1-\lambda) \cdot \widehat{x}'C\overline{x} + (1-\lambda)^2 \cdot \overline{x}'C\overline{x} \\ &\leq \lambda^2 \cdot \widehat{x}'C\widehat{x} + 2 \cdot \lambda \cdot (1-\lambda) \cdot \sqrt{\widehat{x}'C\widehat{x}} \cdot \sqrt{\overline{x}'C\overline{x}} + (1-\lambda)^2 \cdot \overline{x}'C\overline{x} = \\ \left(\lambda \cdot \sqrt{\widehat{x}'C\widehat{x}} + (1-\lambda) \cdot \sqrt{\overline{x}'C\overline{x}}\right)^2 &= (\lambda \cdot \sigma^*(\mu_{\min}) + (1-\lambda) \cdot \sigma^*(\overline{\mu}_{\min}))^2. \qquad \square \end{aligned}$$

Setting $r = 0$ we can easily see that the statements of Lemmas 4.1 and 4.2 as well as those of Theorems 4.4 and 4.5 also hold if there is no possibility of a riskless investment.

5. Case Study

For studying the effect of market impact cost we use the same two-year time series of daily price data as Hafner [10] ending exactly at the same day for which the market impact cost was estimated, i.e. daily price data from 4-th June 1999 until 5-th June 2001. For the sake of simplicity we assume that the problem of the trader or portfolio manager is to decide on a portfolio consisting of the chemistry shares of BASF and BAYER and a riskless investment only. Given a maximum level for the volatility of 25% and a planning horizon of 1 year, the correlation matrix, the annualized standard deviation (STD), the expected rate of return (Exp. return) as well as the explicit cost (EC), the liquidity cost factor (Liq. cost), and the critical trade level (Crit. tr. level) are shown in Table 2.

Correlation	BASF	BAYER	STD	Exp. return	EC	Liq. cost	Crit. tr. level
BASF	1,00	0,66	30,56%	8,45%	0	0,04%	5100
BAYER	0,66	1,00	28,69%	7,87%	0	0,01%	200

Table 2: Market information

It is assumed that the critical trade level is the same, no matter if the stock is to be bought or sold. The riskless rate of return is 2% and the budget is increased fom 1000 EUR to 10 Mio. EUR by a factor of 10 for each step. If transaction cost is neglected, the structure of the optimal portfolio does not depend on the budget at all and is given by

$$(x_{BASF}; x_{BAYER}; x_{Riskless}) = (48,38\%; 44,10\%; 7,52\%)$$

with an expected rate of return equal to $7,71\%$.

If we consider liquidity cost, the optimal portfolio changes with increasing budget. For a budget of 1.000 and 10.000 EUR there are no liquidity costs. For a budget of 100.000 EUR there is liquidity cost for BAYER only due to the lower critical trade level. Therefore, the BASF share is overweighted relative to the optimal portfolio without liquidity cost and the weigth for BAYER is reduced. However, if we increase the budget to 1 Mio. EUR, there is liquidity cost for half of the BASF shares and nearly all BAYER shares. Nevertheless, the higher liquidity cost for BASF becomes dominant and BAYER is now overweighted instead of BASF. As we continue increasing the budget this effect decreases a

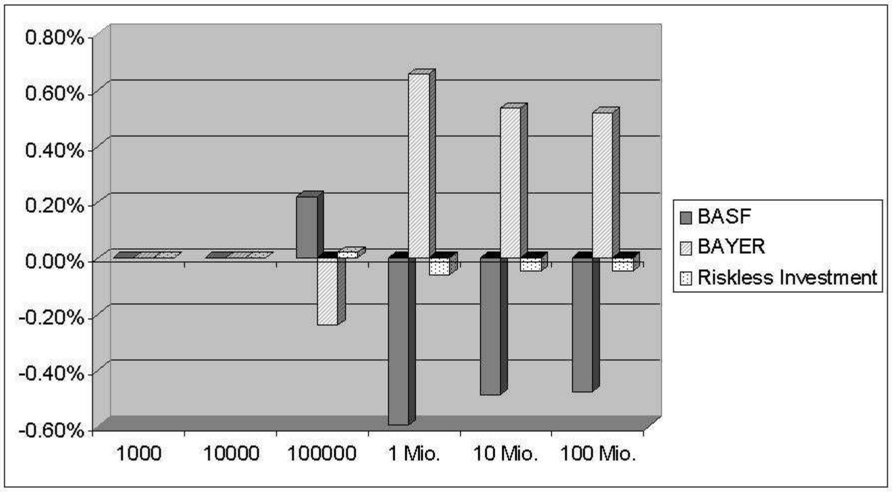

Figure 1: Change of the optimal portfolio under liquidity cost relative to the optimal portfolio without liquidity cost with increasing budget

little as now all additional shares are under liquidity cost. The optimal portfolio weights relative to the optimal portfolio under no transaction cost are shown in Figure 1.

References

[1] R. Almgren, C. Thum, E. Hauptmann, H. Li, Direct estimation of equity market impact, *Working Paper*, University of Toronto, Toronto (2005).

[2] BARRA, The market impact model $^{\text{TM}}$, *BARRA Newsletter* (1998), 165-168.

[3] M.J. Best, J. Hlouskova, Portfolio selection and transaction costs, *Computational Optimization and Applications*, **24**. No. 1 (2003), 95-116.

[4] J.A. Bikker, L. Spierdijk, P.J. van der Sluis, Market impact costs of institutional equity traders, *DNB Working Papers*, Netherlands Central Bank, Netherlands, **001** (2004).

[5] M.J. Brennan, The optimal number of securities in a risky asset portfolio when there are fixed costs of transacting: theory and some empirical results, *Journal of Financial and Quantitative Analysis*, **10**, No. 3 (1975), 483-496.

[6] L.K.C. Chan, J. Lakonishok, Institutional trades and intraday stock price behavior, *Journal of Financial Economics*, **33**, No. 2 (1993), 173-199.

[7] L.K.C. Chan, J. Lakonishok, The behavior of stock prices around institutional trades, *Journal of Finance*, **50**, No. 4 (1995), 1147-1174.

[8] G. Cornuejols, R. Tütüncü, *Optimization Methods in Finane*, Cambridge University Press, Cambridge (2007).

[9] D.R. Gallagher, A. Looi, An examination of the market impact cost of active Australian equity managers, *SIRCA Research Report*, Sydney (2003).

[10] R. Hafner, The RiskLab transaction cost model (TraC'M), *Solutions*, **5**, No. 3/4 (2001), 35-49.

[11] S. Isaenko, Portfolio choice under convex transaction costs, *Working paper*, Concordia University, Montreal (2005).

[12] S. Isaenko, Diversification in illiquid markets, *Working paper*, Concordia University, Montreal (2006).

[13] D.B. Keim, A. Madhavan, Transaction costs and investment style: an interexchange analysis of institutional equity trades, *Journal of Financial Economics*, **46** (1997), 265-292.

[14] D.B. Keim, A. Madhavan, The cost of institutional equity trades, *Financial Analysts Journal*, **54**, No. 4 (1998), 50-69.

[15] H. Konno, A. Wijayanayake, Portfolio optimization problems under concave transaction costs and minimal transaction unit constraints, *Mathematical Programming*, **89** (2001), 233-250.

[16] H. Konno, A. Wijayanayake, Portfolio optimization under D.C. transaction costs and minimal transaction unit constraints, *Journal of Global Optimization*, **22**, No. 1-4 (2004), 137-154.

[17] H. Konno, A. Wijayanayake, Optimization of a long-short portfolio under nonconvex transaction costs, *Computational Optimization and Applications*, **32**, NNo. 1-2 (2005), 115-132.

[18] P. Krokhmal, J. Palmquist, S. Uryasev, Portfolio optimization with conditional value-at-risk objective and constraints, *Journal of Risk*, **4**, No. 2 (2002), 43-68.

[19] M.S. Lobo, M. Fazel, S. Boyd, Portfolio optimization with linear and fixed transaction costs, *Working paper*, California Institute of Technology, Pasadena (2005).

[20] D. Maringer, Portfolioselektion bei Transaktionskosten und Ganzzahligkeitsbeschränkungen, *Zeitschrift für Betriebswirtschaft*, **72**, No. 11 (2002), 1155-1176.

[21] D. Maringer, *Portfolio Management with Heuristic Optimization*, Springer, Berlin (2005).

[22] H.M. Markowitz, Portfolio selection, *The Journal of Finance*, **7** (1952), 77-91.

[23] J.E. Mitchell, S. Braun, Rebalancing an investment portfolio in the presence of convex transaction costs, *Technical Report*, Rensselaer Polytechnic Institute, Troy (2004).

[24] G.A. Pogue, An extension of the Markowitz portfolio selection model to include variable transactions' costs, short sales, leverage policies and taxes, *The Journal of Finance*, **25**, No. 5 (1970), 1005-1027.

[25] B. Scherer, R.D. Martin, *Modern Portfolio Optimization*, Springer, Berlin (2005).

[26] W. Sharpe, Capital asset prices: A theory of market equilibrium under conditions of risk, *The Journal of Finance*, **29** (1964), 425-442.