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WAVE PROPAGATION ACCORDING TO THE LINEARIZED FINITE THEORY OF ELASTICITY

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Abstract: This paper completely solves the problem of wave propagation in constrained linear elastic materials within the framework of the linearized finite theory of elasticity proposed by Hoger et al [2], [3]. By means of a procedure of linearization appropriate for such a theory, in [4] we have derived the amplitude condition. In this paper we obtain the acoustic tensor and the propagation condition solving an eigenvalue problem related to this tensor. Moreover, we solve with the right degree of accuracy the characteristic equation. In general, our results differ by terms which are first order in the displacement gradient from the corresponding results obtained in the classical linear elasticity, as explicitly shown by the study of the constraints of incompressibility and inextensibility.

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1. Introduction

In 1995 Hoger et al [2], [3] formulated the so-called linearized finite theory of elasticity (in the following denoted for brevity by LFTE), appropriate for linear elastic materials subject to constraints. In such a theory, the linear constitutive equations, derived by linearization of the corresponding finite constitutive equations with respect to the displacement gradient, differ from the equations of the classical linear elasticity by terms that are first order in the strain.

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Even if the constrained linear elastic materials are usually studied by using the classical linear elasticity, many reasons, extensively exposed in [2], [3], allow us to prefer the use of LFTE, in order to have the accuracy required by a linear theory.

In [4] we have applied the LFTE to the study of propagation of acceleration waves in constrained linear elastic materials; in particular, following the same procedure of linearization used for the constitutive equations, we have derived the Scott-Hayes constraint vector and the amplitude condition (see [4], Section 3).

As shown in [4], Section 4, the amplitude condition obtained in LFTE differs from the amplitude condition provided by the classical linear elasticity by terms that are first order in the strain, even if for particular constraints, as incompressibility, the two conditions coincide. The results of [4] represent a first step in the study of wave propagation according to LFTE; in fact, in [4] we have derived the propagation condition only for particular constraints.

The aim of this paper is to obtain the propagation condition for a general constraint, in order to complete the results of [4]. By means of a suitable procedure of linearization we obtain the explicit expression for the acoustic tensor and then the propagation condition solving an eigenvalue problem for such a tensor; moreover, we solve the characteristic equation with the degree of accuracy appropriate for LFTE. Finally, we apply the previous results both to incompressible materials and inextensible materials.

In Section 2, referring to [2], [3], we briefly recall the field equations used in LFTE.

In Section 3 we summarize the main results obtained in [4] for acceleration waves propagating into materials subject to a constraint; in particular, we recall the expression of the Scott-Hayes constraint vector and the amplitude condition appropriate for LFTE; we show that the amplitude condition in LFTE differs from the corresponding condition in classical linear elasticity by terms at most linear in the displacement gradient.

In Section 4, by means of the usual method of the singular surfaces, we obtain a first form for the propagation condition of acceleration waves.

In Section 5, following a suitable procedure of linearization, we transform the propagation condition of Section 4 into an eigenvalue problem and then we identify the acoustic tensor.

In Section 6 we solve the characteristic equation with the degree of accuracy required by a linear theory. According to the results derived for the constraint vector, we obtain the squares of the speeds of propagation as functions at most linear in the displacement gradient, while in the classical linear elasticity they are constant. Finally, we derive conditions which guarantee real speeds of propagation.

In Sections 7 and 8 we apply the results obtained in the previous sections to the constraints of incompressibility and inextensibility, respectively. In both cases we derive the acoustic tensor; in particular, for incompressible materials, according to [4] we solve the propagation condition and we obtain not constant speeds of propagation.

Finally, in Section 9 we indicate future developments and applications.

2. Field Equations for the Linearized Finite Theory of Elasticity

In this section we briefly recall the field equations for LFTE appropriate for constrained linear hyperelastic materials; we refer to [2], [3] for an exhaustive exposition of such a theory.

Denote by $\mathcal{B}_0$ and $\mathcal{B} = \mathbf{f}(\mathcal{B}_0)$ the reference configuration and the deformed configuration of a body, respectively, where $\mathbf{f}$ is a deformation function depending on the time t; $\mathbf{f}$ carries point $\mathbf{X} \in \mathcal{B}_0$ into point $\mathbf{x} = \mathbf{f}(\mathbf{X}, t) \in \mathcal{B}$. Then, we define the displacement $\mathbf{u}$, the deformation gradient $\mathbf{F}$ and the displacement gradient $\mathbf{H}$ as

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{f}(\mathbf{X}, t) - \mathbf{X}\,, \tag{1}$$

$$\mathbf{F} = \mathrm{Grad}\mathbf{f}\,, \tag{2}$$

$$\mathbf{H} = \mathrm{Grad}\mathbf{u} = \mathbf{F} - \mathbf{I}\,, \tag{3}$$

where Grad is the gradient operation taken with respect to $\mathbf{X}$ and $\mathbf{I}$ is the identity tensor. In the following $\mathbf{H}$ is assumed to be small and everywhere only terms that are at most linear in $\mathbf{H}$ are retained.

Denoting by $\mathbf{E}_G = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)$ the finite Green strain tensor and using (3), it follows that the infinitesimal strain tensor $\mathbf{E}$ is

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right), \tag{4}$$

since we linearize $\mathbf{E}_G$ about the zero strain state.

For an elastic material subject to a single constraint, the finite constraint equation is

$$\hat{c}(\mathbf{E}_G) = 0. \tag{5}$$

By introducing the linear constraint function $\tilde{c}(\mathbf{E})$ defined as $\tilde{c}(\mathbf{E}) = \dfrac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) \cdot \mathbf{E}$, where $\mathbf{O}$ denotes the zero tensor, the linearized constraint equation is

$$\tilde{c}(\mathbf{E}) = 0 \tag{6}$$

(see [4], Section 2, for more details and remarks concerning the right procedure of linearization).

For hyperelastic constrained materials with strain energy function $W = \hat{W}(\mathbf{E}_G)$, the finite constitutive equation for the Cauchy stress $\mathbf{T}$ is

$$\mathbf{T} = \frac{1}{\det \mathbf{F}} \mathbf{F} \frac{\partial \hat{W}}{\partial \mathbf{E}_G}(\mathbf{E}_G)\mathbf{F}^T + q\mathbf{F}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{E}_G)\mathbf{F}^T, \tag{7}$$

where q is a Lagrange multiplier.

According to LFTE, the linear constitutive equation is obtained by linearization of (7) with respect to the displacement gradient $\mathbf{H}$; the accurate procedure of linearization formulated in [3], Section 3, in which the linearization of the derivative of the strain energy density parallels that of the derivative of the constraint function, yields for $\mathbf{T}$ the expression (see [3], formula (3.22))

$$\begin{aligned}\mathbf{T} \;=\; & \frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\tilde{c}} \mathbf{E} + q\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + q\left\{\mathbf{H}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) \right.\\ & \left. + \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O})\mathbf{H}^T\right\} + q\,\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\tilde{c}} \mathbf{E}.\end{aligned} \tag{8}$$

In formula (8), $\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\Big|_{\tilde{c}} \mathbf{E}$ and $\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\Big|_{\tilde{c}} \mathbf{E}$ indicate the evaluation of $\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\mathbf{E}$ and $\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\mathbf{E}$ on the linearized constraint equation (6); moreover, in (8) and in the following the terms of order $o(\mathbf{H})$ are discarded.

As shown in [3], Section 5, equation (8) has the accuracy required by a linear theory, since this equation contains some terms that are first order in the strain, usually dropped for different reasons in classical linear elasticity for constrained materials.

In terms of the Cauchy stress $\mathbf{T}$, the field equations for LFTE in the deformed region $\mathcal{B}$ are given by (3), (4), (6), (8) and the equation of motion

$$\mathrm{div}\mathbf{T} + \rho\mathbf{b} = \rho\ddot{\mathbf{u}}, \tag{9}$$

where div is the divergence operator taken with respect to $\mathbf{x}$, ρ and $\mathbf{b}$ are the mass density and the body force density of $\mathcal{B}$, respectively, while the superposed dot denotes time differentiation.

3. Acceleration Waves: Compatibility Conditions, Scott-Hayes Constraint Vector, Amplitude Condition

In [4] the method of the discontinuity surfaces is adapted to LFTE and used to study the propagation of acceleration waves in constrained materials according to such a theory, even if the explicit expression for the acoustic tensor is found only for particular constraints. However in [4] we have derived for a general constraint the amplitude condition, that is the condition imposed by the constraint on the amplitudes, which are the eigenvectors of the acoustic tensor. In this section we briefly recall the amplitude condition obtained in [4], Section 3.

An acceleration wave is a surface propagating through the body across which at least one of the second derivatives of $\mathbf{x}$ suffers a jump discontinuity, while the first derivatives of $\mathbf{x}$ are continuous across this surface (see [5], Section 71).

For the singular surface in the deformed configuration the geometrical and kinematical compatibility conditions for acceleration waves, appropriate for LFTE, are

$$\begin{aligned} [\![\mathrm{grad}\mathbf{H}]\!] &= \mathbf{s} \otimes \left(\mathbf{I} + \mathbf{H}^T\right) \mathbf{n} \otimes \mathbf{n}\,, \\ [\![\dot{\mathbf{H}}]\!] &= -U\mathbf{s} \otimes \left(\mathbf{I} + \mathbf{H}^T\right) \mathbf{n}\,, \\ [\![\ddot{\mathbf{x}}]\!] &= [\![\ddot{\mathbf{u}}]\!] = U^2\mathbf{s} \end{aligned} \tag{10}$$

(see [4], formulas (3.2)); in (10) the bracket $[\![\ ]\!]$ denotes the jump, $\mathbf{n}$ is the unit normal to the singular surface, U is the wave speed and $\mathbf{s}$ is the amplitude vector.

In finite elasticity, for a material subject to a constraint the amplitude $\mathbf{s}$ must satisfy the so-called amplitude condition. In fact, the constraint equation (5) written in terms of $\mathbf{F}$

$$\check{c}(\mathbf{F}) = 0 \tag{11}$$

allows us to define a vector $\boldsymbol{\nu}$, called Scott-Hayes constraint vector,

$$\boldsymbol{\nu} = \frac{\partial \check{c}}{\partial \mathbf{F}}(\mathbf{F})\mathbf{F}^T\mathbf{n}, \tag{12}$$

which is orthogonal to the amplitude vector (amplitude condition)

$$\boldsymbol{\nu} \cdot \mathbf{s} = 0 \tag{13}$$

(see [4], Section 3, and references herein quoted for more details).

In LFTE, by applying on (12) the same procedure of linearization adopted for the constitutive equations we obtain for $\boldsymbol{\nu}$ the expression

$$\boldsymbol{\nu} = \left\{ \frac{\partial \check{c}}{\partial \mathbf{F}}(\mathbf{I})(\mathbf{I} + \mathbf{H}^T) + \frac{\partial^2 \check{c}}{\partial \mathbf{F} \partial \mathbf{F}}(\mathbf{I})\mathbf{H} \right\} \mathbf{n}, \tag{14}$$

so that the amplitude condition (13) becomes

$$\left\{ \frac{\partial \check{c}}{\partial \mathbf{F}}(\mathbf{I})(\mathbf{I} + \mathbf{H}^T) + \frac{\partial^2 \check{c}}{\partial \mathbf{F} \partial \mathbf{F}}(\mathbf{I})\mathbf{H} \right\} \mathbf{n} \cdot \mathbf{s} = 0 \tag{15}$$

(see [4], formulas (3.10), (3.11)).

In order to completely formulate the problem of wave propagation in LFTE, the next step is now to obtain for a general constraint an explicit expression for the acoustic tensor in agreement with such a theory.

4. Acceleration Waves: The Propagation Condition

In [4] the propagation condition has been obtained and solved only in two particular cases: acceleration waves propagating into incompressible materials ([4], Section 5) and plane acceleration waves propagating into homogeneously strained inextensible materials ([4], Section 6).

The aim of this section is to obtain the expression of the propagation condition for acceleration waves propagating into a material subject to a general constraint in LFTE.

In the following, the usual method of the singular surfaces is applied. First, we substitute the constitutive equation (8), with the use of (3), (4), into the equation of motion (9). Then, we take the jumps of such an equation across a singular surface of order 2, assuming ρ and $\mathbf{b}$ continuous across the wave front and noting that $[\![q]\!] = 0$, as shown in [5], formula (78.2). Finally, we use the compatibility conditions $(10)_{1,3}$ and the Maxwell's Theorem (see [5], formula (78.3))

$$[\![\mathrm{grad} q]\!] = M\mathbf{n}, \tag{16}$$

where $M = [\![\mathrm{grad} q \cdot \mathbf{n}]\!]$.

This procedure leads to the following equation in component form with respect to a rectangular cartesian coordinate system

$$\begin{aligned}
&\frac{1}{2}\hat{W}_{ijhk}\left(s_h n_j n_k + s_k n_j n_h + H_{lk} s_h n_j n_l + H_{lh} s_k n_j n_l\right)\\
&+ q\Big\{\left(s_i n_j n_h + H_{lh} s_i n_j n_l\right)\hat{\hat{C}}_{hj} + \hat{\hat{C}}_{ih}\left(s_j n_j n_h + H_{lh} s_j n_j n_l\right)\\
&+ \frac{1}{2}\hat{C}_{ijhk}\left(s_h n_j n_k + s_k n_j n_h + H_{lk} s_h n_j n_l + H_{lk} s_k n_j n_l\right)\Big\}\\
&+ M n_j\left\{\hat{\hat{C}}_{ij} + H_{ih}\hat{\hat{C}}_{hj} + \hat{\hat{C}}_{ih} H_{jh} + \frac{1}{2}\hat{C}_{ijhk}(H_{hk} + H_{kh})\right\} = \rho U^2 s_i.
\end{aligned} \tag{17}$$

In (17) the indices $i, j, \ldots$ take the values $1, 2, 3$; here and in the following summation over repeated indices is implied; moreover $\hat{\mathbf{W}}$ and $\hat{\mathbf{C}}$ are two fourth-order tensors, $\hat{\hat{\mathbf{C}}}$ a second-order tensor, defined respectively as follows

$$\begin{aligned}\hat{\mathbf{W}} &= \frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\Big|_{\tilde{c}}, \\ \hat{\mathbf{C}} &= \frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\Big|_{\tilde{c}}, \\ \hat{\hat{\mathbf{C}}} &= \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O})\end{aligned} \tag{18}$$

(see formula (8)).

Our aim is now to obtain for (17) a coordinate-free and simplified form. First we note that the symmetry of $\mathbf{E}_G$ and the assumed differentiability properties both of the strain energy function $\hat{W}$ and the constraint function $\hat{c}$ lead to the following symmetries for the tensors $\hat{\mathbf{W}}$, $\hat{\mathbf{C}}$, $\hat{\hat{\mathbf{C}}}$ defined in (18)

$$\begin{aligned}&\hat{\mathbf{W}}\mathbf{A} = \frac{1}{2}\left(\hat{\mathbf{W}}\mathbf{A} + (\hat{\mathbf{W}}\mathbf{A})^T\right), \ \hat{\mathbf{W}}\mathbf{A} = \hat{\mathbf{W}}\left(\frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right)\right), \\ &\hat{\mathbf{W}} = \hat{\mathbf{W}}^T, \quad \hat{\mathbf{C}}\mathbf{A} = \frac{1}{2}\left(\hat{\mathbf{C}}\mathbf{A} + (\hat{\mathbf{C}}\mathbf{A})^T\right), \\ &\hat{\mathbf{C}}\mathbf{A} = \hat{\mathbf{C}}\left(\frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right)\right), \quad \hat{\mathbf{C}} = \hat{\mathbf{C}}^T, \quad \hat{\hat{\mathbf{C}}} = \hat{\hat{\mathbf{C}}}^T,\end{aligned} \tag{19}$$

for any second-order tensor $\mathbf{A}$.

The second step is to obtain for the constraint vector $\boldsymbol{\nu}$, which in LFTE is given by (14), an expression derived by (5) rather than by (11), that is an expression involving the tensors $\hat{\mathbf{C}}$ and $\hat{\hat{\mathbf{C}}}$. To this aim, first we note that

$$\frac{\partial \check{c}}{\partial \mathbf{F}}(\mathbf{I}) = \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O})\frac{\partial \mathbf{E}_G}{\partial \mathbf{F}}(\mathbf{I}); \tag{20}$$

by using the definition of $\mathbf{E}_G$, $(18)_3$ and the symmetry $(19)_7$, formula (20) becomes

$$\frac{\partial \check{c}}{\partial \mathbf{F}}(\mathbf{I}) = \hat{\hat{\mathbf{C}}}. \tag{21}$$

Now we turn to the term $\frac{\partial^2 \check{c}}{\partial \mathbf{F}\partial \mathbf{F}}(\mathbf{I})$ in (14); by using again the chain rule for the second-order derivative, the symmetry $(19)_7$ for $\hat{\hat{\mathbf{C}}}$, as well as the symmetries $(19)_{4,5}$ for $\hat{\mathbf{C}}$, we find

$$\frac{\partial^2 \check{c}}{\partial \mathbf{F}\partial \mathbf{F}}(\mathbf{I}) = \mathbf{I} \boxtimes \hat{\hat{\mathbf{C}}} + \hat{\mathbf{C}}. \tag{22}$$

For a careful definition of the tensor product $\boxtimes$ appearing in (22) we refer to [1], § 52; here we only recall that the tensor product $\boxtimes$ of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ can be defined as the fourth-order tensor $\mathbf{A}\boxtimes\mathbf{B}$ characterized by

$$(\mathbf{A}\boxtimes\mathbf{B})\mathbf{L} = \mathbf{A}\mathbf{L}\mathbf{B}^T, \tag{23}$$

for any second-order tensor $\mathbf{L}$.

Finally, incorporation of (21), (22), (23), $(19)_7$ into (14) provides the following expression for the constraint vector $\boldsymbol{\nu}$

$$\boldsymbol{\nu} = \left\{\hat{\hat{\mathbf{C}}} + \hat{\hat{\mathbf{C}}}\mathbf{H}^T + \mathbf{H}\hat{\hat{\mathbf{C}}} + \hat{\mathbf{C}}\mathbf{H}\right\}\mathbf{n}. \tag{24}$$

Now we devote our attention to equation (17); taking into account the symmetry properties (19) for $\hat{\mathbf{W}}$, $\hat{\mathbf{C}}$, $\hat{\hat{\mathbf{C}}}$ and the expression (24) for $\boldsymbol{\nu}$, we can rewrite (17) in the coordinate-free form

$$(\mathbf{Q}_1 + q\mathbf{Q}_2)\mathbf{s} + M\boldsymbol{\nu} = \rho U^2\mathbf{s}, \tag{25}$$

where the tensors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are defined respectively as

$$\begin{aligned}
\mathbf{Q}_1 &= \left(\mathbf{n}\otimes\hat{\mathbf{W}}\mathbf{n}\right)^T\mathbf{I} + \left[\left(\mathbf{H}^T\mathbf{n}\otimes\hat{\mathbf{W}}\mathbf{n}\right)^T\mathbf{I}\right]^T,\\
\mathbf{Q}_2 &= \left(\mathbf{n}\otimes\hat{\mathbf{C}}\mathbf{n}\right)^T\mathbf{I} + \left[\left(\mathbf{H}^T\mathbf{n}\otimes\hat{\mathbf{C}}\mathbf{n}\right)^T\mathbf{I}\right]^T + \left(\hat{\hat{\mathbf{C}}}\mathbf{H}^T\mathbf{n}\right)\cdot\mathbf{n}\mathbf{I}\\
&\quad + \left(\hat{\hat{\mathbf{C}}}\mathbf{H}^T\mathbf{n}\right)\otimes\mathbf{n} + \left(\hat{\hat{\mathbf{C}}}\mathbf{n}\cdot\mathbf{n}\right)\mathbf{I} + \hat{\hat{\mathbf{C}}}\mathbf{n}\otimes\mathbf{n}.
\end{aligned} \tag{26}$$

Equation (25), together with (26), governs the propagation of acceleration waves into a material subject to a general constraint, according to LFTE; particular cases of (25) have been obtained and discussed in [4] (see [4], Section 5, for the constraint of incompressibility and [4], Section 6, for the constraint of inextensibility).

5. Final Form for the Propagation Condition

In this section by means of a suitable procedure of linearization we transform (25) into an eigenvalue problem and we identify the final form of the acoustic tensor governing the wave propagation.

As usual in the study of wave propagation in constrained materials, the first step is to obtain M from (25). To this aim, we take the scalar product of (25) by $\boldsymbol{\nu}$, recalling that the amplitude condition requires $\boldsymbol{\nu}\cdot\mathbf{s} = 0$. Moreover, we introduce for the constraint vector $\boldsymbol{\nu}$ in (24) the following decomposition

$$\boldsymbol{\nu} = \boldsymbol{\nu}^{(0)} + \boldsymbol{\nu}^{(1)}, \tag{27}$$

where

$$\boldsymbol{\nu}^{(0)} = \hat{\hat{\mathbf{C}}}\mathbf{n}\,,\qquad \boldsymbol{\nu}^{(1)} = \hat{\hat{\mathbf{C}}}\mathbf{H}^T\mathbf{n} + \mathbf{H}\hat{\hat{\mathbf{C}}}\mathbf{n} + \hat{\mathbf{C}}\mathbf{H}\mathbf{n}. \tag{28}$$

It follows that the assumption of linearity of the theory and the binomial series expansion provide

$$(\boldsymbol{\nu}\cdot\boldsymbol{\nu})^{-1} = \epsilon^{(0)} + \epsilon^{(1)}, \tag{29}$$

where

$$\epsilon^{(0)} = (\boldsymbol{\nu}^{(0)}\cdot\boldsymbol{\nu}^{(0)})^{-1}\,, \quad \epsilon^{(1)} = -2\frac{\boldsymbol{\nu}^{(0)}\cdot\boldsymbol{\nu}^{(1)}}{(\boldsymbol{\nu}^{(0)}\cdot\boldsymbol{\nu}^{(0)})^2}. \tag{30}$$

Finally, we also introduce for $\mathbf{Q}_1$ and $\mathbf{Q}_2$ in (26) the following decompositions

$$\mathbf{Q}_1 = \mathbf{Q}_1^{(0)} + \mathbf{Q}_1^{(1)}\,, \tag{31}$$

$$\mathbf{Q}_2 = \mathbf{Q}_2^{(0)} + \mathbf{Q}_2^{(1)}\,, \tag{32}$$

where

$$\mathbf{Q}_1^{(0)} = (\mathbf{n}\otimes\hat{\mathbf{W}}\mathbf{n})^T\mathbf{I}\,,\qquad \mathbf{Q}_1^{(1)} = \left[(\mathbf{H}^T\mathbf{n}\otimes\hat{\mathbf{W}}\mathbf{n})^T\mathbf{I}\right]^T\,, \tag{33}$$

and

$$\mathbf{Q}_2^{(0)} = (\mathbf{n}\otimes\hat{\mathbf{C}}\mathbf{n})^T\mathbf{I} + \left(\hat{\hat{\mathbf{C}}}\mathbf{n}\cdot\mathbf{n}\right)\mathbf{I} + \hat{\hat{\mathbf{C}}}\mathbf{n}\otimes\mathbf{n}\,,\qquad \mathbf{Q}_2^{(1)} = \left[(\mathbf{H}^T\mathbf{n}\otimes\hat{\mathbf{C}}\mathbf{n})^T\mathbf{I}\right]^T + \left(\hat{\hat{\mathbf{C}}}\mathbf{H}^T\mathbf{n}\right)\cdot\mathbf{n}\mathbf{I} + \left(\hat{\hat{\mathbf{C}}}\mathbf{H}^T\mathbf{n}\right)\otimes\mathbf{n}. \tag{34}$$

By using (27)-(34), the scalar product of (25) by $\boldsymbol{\nu}$ provides for M the expression

$$\begin{aligned} M = & -\left\{\left[\left(\epsilon^{(0)}+\epsilon^{(1)}\right)\mathbf{Q}_1^{(0)}\mathbf{s} + \epsilon^{(0)}\mathbf{Q}_1^{(1)}\mathbf{s}\right]\cdot\boldsymbol{\nu}^{(0)} \right.\\ & \left. + \epsilon^{(0)}\mathbf{Q}_1^{(0)}\mathbf{s}\cdot\boldsymbol{\nu}^{(1)}\right\} - q\left\{\left[\left(\epsilon^{(0)}+\epsilon^{(1)}\right)\mathbf{Q}_2^{(0)}\mathbf{s} \right.\right.\\ & \left.\left. + \epsilon^{(0)}\mathbf{Q}_2^{(1)}\mathbf{s}\right]\cdot\boldsymbol{\nu}^{(0)} + \epsilon^{(0)}\mathbf{Q}_2^{(0)}\mathbf{s}\cdot\boldsymbol{\nu}^{(1)}\right\}. \end{aligned} \tag{35}$$

Moreover, we also note that (35) has been obtained retaining only terms which are at most linear in $\mathbf{H}$, in agreement with the assumption of linearity of the theory. Now we substitute in (25) the expressions (31)-(34) for $\mathbf{Q}_1$ and $\mathbf{Q}_2$, the expressions (27), (28) for $\boldsymbol{\nu}$ and (35) for M; moreover, in the product $M\boldsymbol{\nu}$ we neglect the terms quadratic in the displacement gradient $\mathbf{H}$.

Such a procedure transforms (25) into the following equation

$$\left(\tilde{\mathbf{Q}} - \rho U^2 \mathbf{I}\right) \mathbf{s} = \mathbf{o}, \tag{36}$$

where we have set

$$\begin{aligned}\tilde{\mathbf{Q}} = & \left[\mathbf{I} - \epsilon^{(0)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(0)} - \epsilon^{(0)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(1)} - \epsilon^{(0)}\boldsymbol{\nu}^{(1)} \otimes \boldsymbol{\nu}^{(0)} \right.\\ & \left. - \epsilon^{(1)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(0)}\right] \mathbf{Q}_1^{(0)} + \left[\mathbf{I} - \epsilon^{(0)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(0)}\right] \mathbf{Q}_1^{(1)} \\ & + q\left[\mathbf{I} - \epsilon^{(0)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(0)} - \epsilon^{(0)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(1)} - \epsilon^{(0)}\boldsymbol{\nu}^{(1)} \otimes \boldsymbol{\nu}^{(0)} \right.\\ & \left. - \epsilon^{(1)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(0)}\right] \mathbf{Q}_2^{(0)} + \left[\mathbf{I} - \epsilon^{(0)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(0)}\right] \mathbf{Q}_2^{(1)};\end{aligned} \tag{37}$$

in (37) $\epsilon^{(0)}$, $\epsilon^{(1)}$, $\boldsymbol{\nu}^{(0)}$, $\boldsymbol{\nu}^{(1)}$, $\mathbf{Q}_1^{(0)}$, $\mathbf{Q}_1^{(1)}$, $\mathbf{Q}_2^{(0)}$, $\mathbf{Q}_2^{(1)}$ are given by (30), (28), (33), (34) respectively.

Now we directly impose on the left-hand side of (36) the amplitude condition $\boldsymbol{\nu}\cdot\mathbf{s} = 0$, in order to identify the acoustic tensor governing the wave propagation and to write the propagation condition as an eigenvalue problem with respect to such a tensor. To this aim, we introduce a suitable basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, defined as follows: $\mathbf{v}_3$ is the constraint vector $\boldsymbol{\nu}$, while $\mathbf{v}_\alpha$, $\alpha = 1, 2$, is any pair of unit vectors, mutually orthogonal, in the plane orthogonal to $\boldsymbol{\nu}$. Note that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is an orthogonal basis, but in general it is not an orthonormal basis, since the constraint vector $\boldsymbol{\nu}$ is not a unit vector. By (36) and the amplitude condition $\boldsymbol{\nu}\cdot\mathbf{s} = 0$, we obtain

$$\left(\tilde{\tilde{\mathbf{Q}}} - \rho U^2 \mathbf{I}\right) \mathbf{s} = \mathbf{o}, \tag{38}$$

where we have set $\tilde{\tilde{\mathbf{Q}}} = \tilde{Q}_{\alpha\beta}\mathbf{v}_\alpha \otimes \mathbf{v}_\beta$, $\tilde{Q}_{\alpha\beta} = \tilde{\mathbf{Q}}\mathbf{v}_\beta \cdot \mathbf{v}_\alpha$, and $\mathbf{I}$ is now the 2×2 identity tensor.

The 2×2 tensor $\tilde{\tilde{\mathbf{Q}}}$ is the acoustic tensor and (38) is the final form of the propagation condition written as an eigenvalue problem with respect to such a tensor.

6. The Characteristic Equation

The aim of this section is to solve the characteristic equation of (38), that is

$$\left(\rho U^2\right)^2 - \left(\mathrm{tr}\tilde{\tilde{\mathbf{Q}}}\right) \rho U^2 + \det \tilde{\tilde{\mathbf{Q}}} = 0, \tag{39}$$

with the degree of accuracy appropriate for a linear theory. In fact, also the solutions of (39) are required to be expressions at most linear in $\mathbf{H}$.

To this aim, we introduce for $\tilde{\mathbf{Q}}$ in (37) the following decomposition

$$\tilde{\mathbf{Q}} = \tilde{\mathbf{Q}}^{(0)} + \tilde{\mathbf{Q}}^{(1)}, \tag{40}$$

where

$$\begin{aligned}
\tilde{\mathbf{Q}}^{(0)} &= \left[\mathbf{I} - \epsilon^{(0)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(0)}\right] \mathbf{Q}_1^{(0)} + q\left[\mathbf{I} - \epsilon^{(0)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(0)}\right] \mathbf{Q}_2^{(0)} \\
\tilde{\mathbf{Q}}^{(1)} &= \left[- \epsilon^{(0)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(1)} - \epsilon^{(0)}\boldsymbol{\nu}^{(1)} \otimes \boldsymbol{\nu}^{(0)}\right. \\
&\quad \left.-\epsilon^{(1)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(0)}\right] \mathbf{Q}_1^{(0)} + \left[\mathbf{I} - \epsilon^{(0)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(0)}\right] \mathbf{Q}_1^{(1)} \\
&\quad +q\left[-\epsilon^{(0)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(1)} - \epsilon^{(0)}\boldsymbol{\nu}^{(1)} \otimes \boldsymbol{\nu}^{(0)}\right. \\
&\quad \left.-\epsilon^{(1)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(0)}\right] \mathbf{Q}_2^{(0)} + \left[\mathbf{I} - \epsilon^{(0)}\boldsymbol{\nu}^{(0)} \otimes \boldsymbol{\nu}^{(0)}\right] \mathbf{Q}_2^{(1)}.
\end{aligned} \tag{41}$$

Decomposition (40), (41) gives rise to a similar decomposition for the components $\tilde{Q}_{\alpha\beta}$ of $\tilde{\tilde{\mathbf{Q}}}$, so that, with obvious meaning of the symbols, we can write

$$\tilde{Q}_{\alpha\beta} = \tilde{Q}_{\alpha\beta}^{(0)} + \tilde{Q}_{\alpha\beta}^{(1)}. \tag{42}$$

If $\left[\left(\mathrm{tr}\tilde{\tilde{\mathbf{Q}}}\right)^2 - 4\det\tilde{\tilde{\mathbf{Q}}}\right]^{\frac{1}{2}}$ is at most linear in $\mathbf{H}$, equation (39) directly provides roots at most linear in $\mathbf{H}$. This happens, for instance, in two cases discussed in [4]: incompressible materials and plane waves propagating into homogeneously strained inextensible materials (see [4], Sections 5 and 6, respectively).

In the other cases, a suitable procedure of linearization must be applied; first, we use the following approximation

$$\left(\mathrm{tr}\tilde{\tilde{\mathbf{Q}}}\right)^2 - 4\det\tilde{\tilde{\mathbf{Q}}} = \Delta^{(0)} + \Delta^{(1)}, \tag{43}$$

where

$$\begin{aligned}
&\Delta^{(0)} = \left(\tilde{Q}_{11}^{(0)} - \tilde{Q}_{22}^{(0)}\right)^2 + 4\tilde{Q}_{12}^{(0)}\tilde{Q}_{21}^{(0)}, \\
&\Delta^{(1)} = 2\left(\tilde{Q}_{11}^{(0)}\tilde{Q}_{11}^{(1)} + \tilde{Q}_{22}^{(0)}\tilde{Q}_{22}^{(1)}\right) - 2\left(\tilde{Q}_{11}^{(0)}\tilde{Q}_{22}^{(1)} + \tilde{Q}_{22}^{(0)}\tilde{Q}_{11}^{(1)}\right) \\
&+ 4\left(\tilde{Q}_{12}^{(0)}\tilde{Q}_{21}^{(1)} + \tilde{Q}_{21}^{(0)}\tilde{Q}_{12}^{(1)}\right).
\end{aligned} \tag{44}$$

By (43), (44) and the binomial series expansion it follows that

$$\left[\left(\mathrm{tr}\tilde{\tilde{\mathbf{Q}}}\right)^2 - 4\det\tilde{\tilde{\mathbf{Q}}}\right]^{\frac{1}{2}} = \left[\Delta^{(0)}\right]^{\frac{1}{2}}\left(1 + \frac{\Delta^{(1)}}{2\Delta^{(0)}}\right), \tag{45}$$

if $\Delta^{(0)} > 0$.

Then the final expression for the squares of the speeds of propagation is

$$U^2 = \frac{1}{2\rho}\left\{\tilde{Q}_{11}^{(0)} + \tilde{Q}_{22}^{(0)} + \tilde{Q}_{11}^{(1)} + \tilde{Q}_{22}^{(1)} \pm \left[\Delta^{(0)}\right]^{\frac{1}{2}}\left(1 + \frac{\Delta^{(1)}}{2\Delta^{(0)}}\right)\right\}, \qquad (46)$$

where $\Delta^{(0)}$ and $\Delta^{(1)}$ are defined in (44).

Finally we set

$$\begin{aligned} U^{(0)} &= \frac{1}{2\rho}\left\{\tilde{Q}_{11}^{(0)} + \tilde{Q}_{22}^{(0)} \pm \left[\Delta^{(0)}\right]^{\frac{1}{2}}\right\}, \\ U^{(1)} &= \frac{1}{2\rho}\left\{\tilde{Q}_{11}^{(1)} + \tilde{Q}_{22}^{(1)} \pm \frac{1}{2}\left[\Delta^{(0)}\right]^{-\frac{1}{2}}\Delta^{(1)}\right\}, \end{aligned} \qquad (47)$$

so that (46) becomes

$$U^2 = U^{(0)} + U^{(1)}; \qquad (48)$$

as occurs for the constraint vector $\boldsymbol{\nu}$ in (24) and the components $\tilde{Q}_{\alpha\beta}$ of the acoustic tensor in (42), also U^2 is written as the sum of two addenda, the former independent of $\mathbf{H}$, the latter linear in $\mathbf{H}$, in agreement with the degree of accuracy required by LFTE.

Of course, only real speeds of propagation U can correspond to real acceleration waves. Then, in order to satisfy such a condition, the following two inequalities must be imposed on the right-hand sides of (43) and (46), respectively

$$\Delta^{(0)} + \Delta^{(1)} \geq 0\,, \qquad (49)$$

$$\tilde{Q}_{11}^{(0)} + \tilde{Q}_{22}^{(0)} + \tilde{Q}_{11}^{(1)} + \tilde{Q}_{22}^{(1)} \pm \left[\Delta^{(0)}\right]^{\frac{1}{2}}\left(1 + \frac{\Delta^{(1)}}{2\Delta^{(0)}}\right) > 0; \qquad (50)$$

condition (49) guarantees real roots for (39), while condition (50) ensures that such roots are positive.

7. Incompressible Materials

The general results obtained in the previous sections are now applied to the constraint of incompressibility. In the following, we refer to [3], Section 4.1, for the details related to the derivation of the constitutive equation for the Cauchy stress $\mathbf{T}$ in incompressible materials according to LFTE. As regards the propagation of acceleration waves in such materials, the method exposed in Sections 5 and 6 allows us to confirm all results obtained in [4], Sections 3.3 and 5.

For an incompressible isotropic elastic material the constraint equation (5) is

$$\det(2\mathbf{E}_G + \mathbf{I}) - 1 = 0 \tag{51}$$

and the linearized constraint equation (6) is

$$\mathrm{tr}\mathbf{E} = 0 \tag{52}$$

(see [3], formulas (4.2) and (4.6), respectively). So, with the notations of $(18)_{2,3}$, direct computation gives

$$\begin{aligned} \hat{\mathbf{C}} &= -(\mathbf{I} \boxtimes \mathbf{I} + \mathbf{I} \diamondplus \mathbf{I})\,, \\ \hat{\hat{\mathbf{C}}} &= \mathbf{I}; \end{aligned} \tag{53}$$

the tensor product $\diamondplus$ of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ can be defined as the fourth-order tensor $\mathbf{A} \diamondplus \mathbf{B}$ characterized by

$$(\mathbf{A} \diamondplus \mathbf{B})\mathbf{L} = \mathbf{B}\mathbf{L}^T\mathbf{A}^T, \tag{54}$$

for any second-order tensor $\mathbf{L}$.

Moreover, if we turn to the strain energy function $\hat{W}$, for isotropic materials $(18)_1$ reduces to

$$\hat{\mathbf{W}} = \mu\mathbf{I} \boxtimes \mathbf{I} + \mu\mathbf{I} \diamondplus \mathbf{I}, \tag{55}$$

where μ is the shear modulus.

Note that all symmetries listed in (19) are satisfied by (53), (55).

Direct substitution of (53), (55) into (8) gives the following expression for the Cauchy stress

$$\mathbf{T} = 2\mu\mathbf{E} + 2q\mathbf{I} \tag{56}$$

(see [3], formula (4.15)).

Substitution of (53) into (24) provides the constraint vector

$$\boldsymbol{\nu} = \mathbf{n}, \tag{57}$$

so that the amplitude condition is

$$\mathbf{n} \cdot \mathbf{s} = 0. \tag{58}$$

Moreover, if we substitute (55) into $(26)_1$ and (53) into $(26)_2$, we obtain

$$\begin{aligned} \mathbf{Q}_1 = & \ \mu(\mathbf{I}+\mathbf{n}\otimes\mathbf{n}) + \mu\left[(\mathbf{Hn}\cdot\mathbf{n})\mathbf{I} + (\mathbf{H}^T\mathbf{n})\otimes\mathbf{n}\right] , \\ \mathbf{Q}_2 = & \ \mathbf{O} , \end{aligned} \tag{59}$$

then, equation (25) reduces to

$$\left\{\mu(\mathbf{I}+\mathbf{n}\otimes\mathbf{n}) + \mu\left[(\mathbf{Hn}\cdot\mathbf{n})\mathbf{I} + (\mathbf{H}^T\mathbf{n})\otimes\mathbf{n}\right]\right\}\mathbf{s} + M\mathbf{n} = \rho U^2\mathbf{s}, \tag{60}$$

according to [4], formula (5.3).

Taking the scalar product of (60) with $\mathbf{n}$ and using (58), we obtain $M = 0$, according to (35). Of course, the procedure exposed in Section 5 in order to transform (25) into (36) and then (36) into (38) in this case is greatly simplified. In fact we have

$$\begin{aligned} &\epsilon^{(0)} = 1, \quad \epsilon^{(1)} = 0 , \\ &\mathbf{Q}_1^{(0)} = \mu(\mathbf{I}+\mathbf{n}\otimes\mathbf{n}), \quad \mathbf{Q}_1^{(1)} = \mu\left[(\mathbf{Hn}\cdot\mathbf{n})\mathbf{I} + (\mathbf{H}^T\mathbf{n})\otimes\mathbf{n}\right] , \\ &\mathbf{Q}_2^{(0)} = \mathbf{O}, \quad \mathbf{Q}_2^{(1)} = \mathbf{O}, \end{aligned}$$

so that

$$\tilde{\mathbf{Q}} = \mu\mathbf{I} + \mu(\mathbf{Hn}\cdot\mathbf{n})\mathbf{I} - \mu\mathbf{n}\otimes\mathbf{n} + \mu(\mathbf{H}^T\mathbf{n})\otimes\mathbf{n} - 2\mu(\mathbf{Hn}\cdot\mathbf{n})\mathbf{n}\otimes\mathbf{n}.$$

It follows that $\tilde{\tilde{\mathbf{Q}}} = \mu\mathbf{I} + \mu(\mathbf{Hn}\cdot\mathbf{n})\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity tensor; then

$$\left[\left(\mathrm{tr}\tilde{\tilde{\mathbf{Q}}}\right)^2 - 4\det\tilde{\tilde{\mathbf{Q}}}\right]^{\frac{1}{2}} = 0$$

and without any linearization (39) provides the double root

$$\rho U^2 = \mu + \mu\mathbf{Hn}\cdot\mathbf{n} \tag{61}$$

(see [4], formula (5.9)).

For a detailed discussion concerning the positiveness of the roots (61) and related comments we refer to [4], p. 32.

8. Inextensible Materials

In this section we briefly recall the results concerning wave propagation in inextensible materials obtained in [4]. Moreover, by means of the method exposed in Section 5 we obtain the explicit expression for the acoustic tensor. We refer to [3], Section 4.2, for the derivation of the constitutive equation for the Cauchy stress $\mathbf{T}$ in inextensible materials within the framework of LFTE. For the detailed derivation of the constraint condition and the propagation condition we refer to [4], Sections 3 and 6.

For an inextensible transversely isotropic elastic material whose axis of inextensibility coincides with the axis of symmetry $\mathbf{k}$, the constraint equation (5) and the linearized constraint equation (6) are respectively

$$(\mathbf{k} \otimes \mathbf{k}) \cdot \mathbf{E}_G = 0 \tag{62}$$

and

$$(\mathbf{k} \otimes \mathbf{k}) \cdot \mathbf{E} = 0 \tag{63}$$

(see [3], formulas (4.18), (4.25)).

It follows that $(18)_{2,3}$ become

$$\hat{\mathbf{C}} = \mathbf{O}\,, \quad \hat{\hat{\mathbf{C}}} = \mathbf{k} \otimes \mathbf{k}. \tag{64}$$

For a transversely isotropic material with axis of symmetry $\mathbf{k}$ the tensor $\hat{\mathbf{W}}$ in $(18)_1$ takes the form

$$\begin{aligned} \hat{\mathbf{W}} = {} & \alpha_1 \mathbf{I} \otimes \mathbf{I} + \alpha_3 (\mathbf{k} \otimes \mathbf{k}) \otimes \mathbf{I} + \alpha_5 (\mathbf{I} \boxtimes \mathbf{I} + \mathbf{I} \diamondplus \mathbf{I}) \\ & + \frac{1}{2} \alpha_6 \left[(\mathbf{k} \otimes \mathbf{k}) \boxtimes \mathbf{I} + \mathbf{I} \boxtimes (\mathbf{k} \otimes \mathbf{k}) + (\mathbf{k} \otimes \mathbf{k}) \diamondplus \mathbf{I} + \mathbf{I} \diamondplus (\mathbf{k} \otimes \mathbf{k}) \right], \end{aligned} \tag{65}$$

where $\alpha_1, \alpha_3, \alpha_5, \alpha_6$ are elastic moduli.

The tensors (64), (65) satisfy the symmetry properties listed in (19). Substitution of (64), (65) into (8) provides the following expression for the Cauchy stress

$$\begin{aligned} \mathbf{T} = {} & \alpha_1 (\mathrm{tr}\mathbf{E}) \mathbf{I} + \alpha_3 (\mathrm{tr}\mathbf{E}) \mathbf{k} \otimes \mathbf{k} + 2\alpha_5 \mathbf{E} + \alpha_6 \{ \mathbf{E} (\mathbf{k} \otimes \mathbf{k}) \\ & + (\mathbf{k} \otimes \mathbf{k}) \mathbf{E} \} + q \mathbf{k} \otimes \mathbf{k} + q \left\{ \mathbf{H} (\mathbf{k} \otimes \mathbf{k}) + (\mathbf{k} \otimes \mathbf{k}) \mathbf{H}^T \right\} \end{aligned} \tag{66}$$

(see [3], formula (4.27)).

Now we turn our attention to equation (25). First we note that in virtue of (64) the constraint vector becomes

$$\boldsymbol{\nu} = \left\{ \mathbf{k} \otimes \mathbf{k} + \mathbf{H} (\mathbf{k} \otimes \mathbf{k}) + (\mathbf{k} \otimes \mathbf{k}) \mathbf{H}^T \right\} \mathbf{n}, \tag{67}$$

so that the amplitude condition is

$$\left\{ \mathbf{k} \otimes \mathbf{k} + \mathbf{H} (\mathbf{k} \otimes \mathbf{k}) + (\mathbf{k} \otimes \mathbf{k}) \mathbf{H}^T \right\} \mathbf{n} \cdot \mathbf{s} = 0; \tag{68}$$

moreover, by (64), (65) the tensors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ defined in (26) take the form

$$\begin{aligned}\mathbf{Q}_1 = {}& \alpha_1 \mathbf{n}\otimes\mathbf{n} + \alpha_3 \cos\phi \mathbf{k}\otimes\mathbf{n} + \alpha_5(\mathbf{I} + \mathbf{n}\otimes\mathbf{n}) \\ & + \frac{1}{2}\alpha_6 \left[\cos^2\phi \mathbf{I} + \cos\phi \mathbf{n}\otimes\mathbf{k} + \cos\phi\mathbf{k}\otimes\mathbf{n} + \mathbf{k}\otimes\mathbf{k}\right] \\ & + \alpha_1 \mathbf{n}\otimes\mathbf{H}^T\mathbf{n} + \alpha_3\cos\phi\mathbf{k}\otimes\mathbf{H}^T\mathbf{n} \\ & + \alpha_5\left[(\mathbf{Hn}\cdot\mathbf{n})\mathbf{I} + \mathbf{H}^T\mathbf{n}\otimes\mathbf{n}\right] + \frac{1}{2}\alpha_6\left[\cos\phi(\mathbf{H}^T\mathbf{n}\cdot\mathbf{k})\mathbf{I}\right. \\ & \left. + \cos\phi\mathbf{H}^T\mathbf{n}\otimes\mathbf{k} + (\mathbf{Hn}\cdot\mathbf{n})\mathbf{k}\otimes\mathbf{k} + (\mathbf{H}^T\mathbf{n}\cdot\mathbf{k})\mathbf{k}\otimes\mathbf{n}\right], \\ \mathbf{Q}_2 = {}& \cos^2\phi\mathbf{I} + \cos\phi\mathbf{k}\otimes\mathbf{n} + \cos\phi(\mathbf{Hk}\cdot\mathbf{n})\mathbf{I} + (\mathbf{Hk}\cdot\mathbf{n})\mathbf{k}\otimes\mathbf{n},\end{aligned} \tag{69}$$

according to [4], formulas (6.3), (6.4); in (69) ϕ denotes the angle between $\mathbf{n}$ and $\mathbf{k}$. Equation (25) with $\boldsymbol{\nu}, \mathbf{Q}_1, \mathbf{Q}_2$ given by (67), $(69)_{1,2}$ respectively has been solved in [4] only for plane waves propagating into a homogeneously strained material.

Now, in virtue of the method exposed in Section 5, our analysis can be improved. In fact, we can write the propagation condition in the final form (38), taking into account that the explicit expression for the tensor $\tilde{\mathbf{Q}}$ is given by (37), where the following identifications hold

$$\begin{aligned}\epsilon^{(0)} &= (\cos\phi)^{-2}, \\ \epsilon^{(1)} &= -2(\cos\phi)^{-3}(\mathbf{Hk}\cdot\mathbf{k}\cos\phi + \mathbf{Hk}\cdot\mathbf{n}), \\ \boldsymbol{\nu}^{(0)} &= \cos\phi\mathbf{k}, \\ \boldsymbol{\nu}^{(1)} &= \cos\phi\mathbf{Hk} + (\mathbf{Hk}\cdot\mathbf{n})\mathbf{k},\end{aligned} \tag{70}$$

while $\mathbf{Q}_1^{(0)}$, $\mathbf{Q}_1^{(1)}$, $\mathbf{Q}_2^{(0)}$, $\mathbf{Q}_2^{(1)}$ immediately follow by (69).

Unfortunately the explicit expression for the eigenvalue problem (38) is not simple, but this fact is a consequence of the complexity of the constitutive equations appropriate for LFTE. Even if the speeds of propagation cannot be explicitly obtained, it is worth noting that the method exposed in Sections 5 and 6 completely solves the wave propagation problem in inextensible materials.

9. Conclusions

This paper represents the natural continuation and completion of [4]. In this paper we solve the problem of wave propagation in constrained linear elastic materials according to LFTE by means of a procedure of linearization appropriate for such a theory. In general our results for speeds, amplitudes and acoustic tensor differ by terms which are first order in the displacement gradient from the corresponding results obtained by the classical linear elasticity, as extensively discussed in [4] for the constraints of incompressibility and inextensibility.

It is worth noting that this paper completes [4], but it does not exhaust all problems related to the study of wave propagation in LFTE. We leave such problems for future study. For instance, we can expect interesting results when two constraints are imposed on the material at the same time. Moreover, we leave for future work also the problem of deriving the equation governing the growth or decay of the amplitude; here we only remark that in LFTE such a problem is not trivial since the speeds of propagation are not constant, as occurs in classical linear elasticity.

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