International Journal of Pure and Applied Mathematics
Volume 40 No. 1 2007, 89-102

A DISCRIMINANT CONDITION FOR THE TEST OF GREATEST POWER IN THE *MANOVA* MODEL WHEN THE DIMENSION IS LARGE COMPARED TO THE SAMPLE SIZE

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Abstract: The asymptotic non-null distributions of the likelihood ratio, Lawley-Hotelling, and Bartlett-Nanda-Pillai test statistics for the *MANOVA* procedure are obtained when both the sample size and the dimension tend to infinity. These tests are of equal power in the limit. Using the asymptotic distributions of the three test statistics, we compare their asymptotic power. We derive a simple method for selecting the test of greatest power.

AMS Subject Classification: 62F05, 62E20
Key Words: *MANOVA*, likelihood ratio, Lawley-Hotelling, Bartlett-Nanda-Pillai, power comparison, asymptotic distribution

1. Introduction

We consider the multivariate linear model:

$$Y = XQ + \mathcal{E},$$

where Y is the $n_0 \times p$ observation matrix, X is the $n_0 \times k$ design matrix, Q is the $k \times p$ matrix of regression coefficients, and $\mathcal{E}$ is the $n_0 \times p$ error matrix distributed according to $N_{n_0 \times p}(O, I_{n_0} \otimes \Sigma)$. We consider the hypothesis

Received: August 9, 2007

$$H_0 : CQ = O,$$

where C is a $q \times k$ known matrix of full rank q. Among the statistics for testing H_0, (i) the likelihood ratio statistics, (ii) Lawley-Hotelling's generalized T^2 statistics and (iii) Bartlett-Nanda-Pillai test statistics are well known. These three test statistics are defined as

$$\text{(i)}\ \frac{|S_e|}{|S_e + S_h|}, \qquad \text{(ii)}\ \mathrm{tr}(S_h S_e^{-1}), \quad \text{and} \quad \text{(iii)}\ \mathrm{tr}\{S_h(S_e + S_h)^{-1}\},$$

where

$$S_h = \hat{Q}'C'\{C(X'X)^{-1}C'\}^{-1}C\hat{Q} \text{ and } S_e = (Y - X\hat{Q})'(Y - X\hat{Q})$$

with $\hat{Q} = (X'X)^{-1}X'Y$ (Muirhead [2]).

Since the exact distributions of these three statistics are complicated and not easy to deal with, we need some method to approximate the distributions. One technique is to use the asymptotic expansions of the distribution functions when the sample size is large (Anderson [1], Muirhead [2], or Siotani et al [3]). Tonda et al [4] derived an asymptotic expansion of the null distribution function for the LR test under the framework:

$$q : \text{fixed},\ n \to \infty,\ p \to \infty,\ \frac{p}{n} \to c \in (0, 1), \tag{1}$$

where n is the degrees of freedom of the Wishart distribution of S_e. Wakaki et al [5] derived the asymptotic expansions of the null distribution functions and the non-null limiting distributions for the three test statistics under the framework of (1). In this paper, we derive the asymptotic expansions of the non-null distribution functions and the asymptotic powers under the framework of (1). In Section 2, we introduce the Wakaki, Fujikoshi, and Ulyanov result. In Section 3, we derive the asymptotic distributions of the three test statistics. We present a method for selecting the test of greatest power in Section 4.

2. Null Distributions

In this section we present the asymptotic expansion of the null distribution functions and some lemmas (Wakaki et al [5]).

Lemma 1. *Suppose that S_h and S_e are independently distributed according to the noncentral and central Wishart distributions $W_p(q, \Sigma, M'M)$ and $W_p(n, \Sigma)$, respectively, where M is a $q \times k$ matrix. We assume that B and W are independently distributed according to the noncentral and central Wishart distributions $W_q(p, I_q, \Omega)$ and $W_q(m, I_q)$, respectively, where $m = n - p + q$ and* *the noncentrality matrix* Ω *is given by*

$$\Omega = M\Sigma^{-1}M'.$$

The three statistics may then be expressed as

$$\begin{aligned}
&\text{(i)} \quad \frac{|S_e|}{|S_e+S_h|} = \frac{|W|}{|W+B|},\\
&\text{(ii)} \quad \mathrm{tr}(S_hS_e^{-1}) = \mathrm{tr}(BW^{-1}),\\
&\text{(iii)} \quad \mathrm{tr}(S_h(S_e+S_h)^{-1}) = \mathrm{tr}(B(W+B)^{-1}).
\end{aligned}$$

From Lemma 1, we can deduce the following lemmas.

Lemma 2. *Let* T_{LR}, T_{LH} *and* T_{BNP} *be expressed as*

$$\begin{aligned}
T_{LR} &= -\sqrt{p}\left(1+\frac{m}{p}\right)\left\{\log\frac{|S_e|}{|S_e+S_h|} + q\log\left(1+\frac{p}{m}\right)\right\},\\
T_{LH} &= \sqrt{p}\left(\frac{m}{p}\mathrm{tr}(S_hS_e^{-1}) - q\right),\\
T_{BNP} &= \sqrt{p}\left(1+\frac{p}{m}\right)\left[\left(1+\frac{m}{p}\right)\mathrm{tr}\{S_h(S_e+S_h)^{-1}\} - q\right].
\end{aligned}$$

Then the null distribution of T_G *(G=LR, LH and BNP) may be expanded as*

$$\begin{aligned}
Pr\left(\frac{T_G}{\sigma}\le z\right) = \Phi(z) - \phi(z)&\left\{\frac{1}{\sqrt{p}}\left(\frac{b_1}{\sigma}+\frac{b_3}{\sigma^3}h_2(z)\right)\right.\\
&\left.+\frac{1}{p}\left(\frac{b_2}{\sigma^2}h_1(z)+\frac{b_4}{\sigma^4}h_3(z)+\frac{b_6}{\sigma^6}h_5(z)\right)\right\} + O\left(\frac{1}{p\sqrt{p}}\right),
\end{aligned}$$

where $h_j(z)$*'s are the Hermite polynomials given by*

$$\begin{aligned}
&h_1(z) = z,\ h_2(z) = z^2-1,\ h_3(z) = z^3-3z,\\
&h_4(z) = z^4-6z+3,\ h_5(z) = z^5-10z^3+15z,
\end{aligned}$$

$\Phi(z)$ *is the standard normal distribution function and* $\phi(x)$ *is the density function of the standard normal distribution. Here the variance* σ^2 *and the coefficients* b_i *are given by*

$$\begin{aligned}
\sigma^2 &= 2q(1+r),\\
b_1 &= \big(c_1(1+r)+r\big)q(q+1),\\
b_3 &= 4\big(c_1(1+r)+r\big)(1+r)q + \frac{4}{3}(1-r^2)q,\\
b_2 &= 6c_2(1+r)^2q(q+1) + \frac{1}{2}c_1^2(1+r)^2q(q+1)(q^2+q+4)\\
&\quad + c_1(1+r)q(q+1)\big(4+r(q^2+q+12)\big)
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{2}q(q+1)r\big(6+r(q^2+q+8)\big), \\
b_4 &= \frac{4}{3}b_1(1-r^2)q+2(1+r^3)q+8c_2(1+r^3)q \\
& +4c_1^2(1+r)^3q(q^2+q+4) \\
& +8c_1q(1+r)^2\big(2+r(q^2+q+4)\big) \\
& +4(1+r)rq\big(3+r(q^2+q+2)\big), \\
b_6 &= \frac{1}{2}b_3^2,
\end{aligned}
$$

$r = p/m$, and a pair of coefficients (c_1, c_2) is may be defined by

$$
(c_1, c_2) = \begin{cases} \left(-\frac{1}{2}\left(\frac{p}{m+p}\right), \frac{1}{3}\left(\frac{p}{m+p}\right)^2\right) & (G = LR), \\ (0,0) & (G = LH), \\ \left(-\frac{p}{m+p}, \left(\frac{p}{m+p}\right)^2\right) & (G = BNP). \end{cases}
$$

By using Lemma 2, we obtain the following lemma on the Cornish-Fisher expansion.

Lemma 3. *Let z_α be the upper 100α % point of the standard normal distribution, and let*

$$
\begin{aligned}
z_{CF}(\alpha) &= z_\alpha + \frac{1}{\sqrt{p}}\left(\frac{b_1}{\sigma} + (z_\alpha^2 - 1)\frac{b_3}{\sigma^3}\right) + \frac{1}{p}\left(-\frac{1}{2}(\frac{b_1}{\sigma})^2 + z_\alpha\frac{b_2}{\sigma^2}\right. \\
& \left. -z_\alpha(z_\alpha^2-3)\frac{b_1 b_3}{\sigma^4} - z_\alpha(2z_\alpha^2-5)(\frac{b_3}{\sigma^3})^2 + z_\alpha(z_\alpha^2-3)^2\frac{b_4}{\sigma^4}\right).
\end{aligned}
$$

Then

$$
Pr\left(\frac{1}{\sigma}T_G \le z_{CF}(\alpha)\right) = 1 - \alpha + O\left(\frac{1}{p\sqrt{p}}\right).
$$

3. Non-Null Distribution

In this section we derive the asymptotic expansions of the test statistics under the non-null hypothesis under framework (1).

3.1. Stochastic Expansion

Let

$$\begin{aligned}
T^*_{LR} &= -\sqrt{p}\left(1+\frac{m}{p}\right) \\
&\quad \times \left\{\log\frac{|S_e|}{|S_e+S_h|} + \log\left|\left(1+\frac{p}{m}\right)I_q + \frac{1}{m}\Omega\right|\right\} \\
&= -\sqrt{p}\left(1+\frac{m}{p}\right) \\
&\quad \times \left\{\log\frac{|W|}{|W+B|} + \log\left|\left(1+\frac{p}{m}\right)I_q + \frac{1}{m}\Omega\right|\right\}, \\
T^*_{LH} &= \sqrt{p}\left\{\frac{m}{p}\mathrm{tr}(S_h S_e^{-1}) - \mathrm{tr}\left(I_q + \frac{1}{p}\Omega\right)\right\} \\
&= \sqrt{p}\left\{\frac{m}{p}\mathrm{tr}(BW^{-1}) - \mathrm{tr}\left(I_q + \frac{1}{p}\Omega\right)\right\}, \\
T^*_{BNP} &= \sqrt{p}\left(1+\frac{p}{m}\right)\left\{\left(1+\frac{m}{p}\right)\mathrm{tr}[S_h(S_e+S_h)^{-1}]\right. \\
&\quad \left. -\mathrm{tr}\left[\left(I_q + \frac{1}{m+p}\Omega\right)^{-1}\left(I_q + \frac{1}{p}\Omega\right)\right]\right\} \\
&= \sqrt{p}\left(1+\frac{p}{m}\right)\left\{\left(1+\frac{m}{p}\right)\mathrm{tr}[B(W+B)^{-1}]\right. \\
&\quad \left. -\mathrm{tr}\left[\left(I_q + \frac{1}{m+p}\Omega\right)^{-1}\left(I_q + \frac{1}{p}\Omega\right)\right]\right\}.
\end{aligned}$$

We assume that $\Omega = O(p)$. Let U and V be defined by

$$U = \sqrt{p}\left(\frac{1}{p}B - \left(I_q + \frac{1}{p}\Omega\right)\right),\ V = \sqrt{m}\left(\frac{1}{m}W - I_q\right). \tag{2}$$

Then U and V are asymptotically normally distributed. Let

$$D = \sqrt{p}\left(\frac{m}{p}BW^{-1} - \left(I_q + \frac{1}{p}\Omega\right)\right). \tag{3}$$

Then $D = O_p(1)$ when $p \to \infty$, and the three statistics may be expanded in terms of D as follows:

$$T^*_{LR} = -\sqrt{p}\left(1+\frac{m}{p}\right)$$

$$
\begin{aligned}
& \times\left\{-\log |I_q + BW^{-1}| + \log \left|\left(1+\frac{p}{m}\right) I_q + \frac{1}{m}\Omega\right|\right\} \\
= \ & \frac{\sqrt{p}}{r_2} \log \left| I_q + \frac{r_2}{\sqrt{p}} \left(I_q + \frac{r_2}{p}\Omega\right)^{-1} D \right| \\
= \ & \mathrm{tr}[AD] - \frac{r_2}{2\sqrt{p}} \mathrm{tr}[(AD)^2] + O_p\left(\frac{1}{p}\right), \\
T^*_{LH} \ = \ & \sqrt{p}\left\{\frac{m}{p}\mathrm{tr}(BW^{-1}) - \mathrm{tr}\left(I_q + \frac{1}{p}\Omega\right)\right\} \\
= \ & \mathrm{tr}(D), \\
T^*_{BNP} \ = \ & \sqrt{p}\left(1+\frac{p}{m}\right)\left\{\left(1+\frac{m}{p}\right)\mathrm{tr}[BW^{-1}(I_q + BW^{-1})^{-1}] \right. \\
& \left. - \mathrm{tr}\left[\left(I_q + \frac{1}{m+p}\Omega\right)^{-1}\left(I_q + \frac{1}{p}\Omega\right)\right]\right\} \\
= \ & \mathrm{tr}[A^2 D] - \frac{r_2}{\sqrt{p}} \mathrm{tr}[A(AD)^2] + O_p\left(\frac{1}{p}\right),
\end{aligned}
$$

where $r_2 = r/(1+r)$ and $A = \{I_q + (r_2/p)\Omega\}^{-1}$. Then the expansion of T_G ($G = LR$, H, and BNP) is given by

$$
T^*_G = \mathrm{tr}[A^\omega AD] + \frac{c_1}{\sqrt{p}} \mathrm{tr}[A^\omega (AD)^2] + O_p\left(\frac{1}{p}\right), \tag{4}
$$

where a pair of coefficients (c_1, ω) may be defined as

$$
(c_1, \omega) = \begin{cases} \left(-\frac{1}{2}\left(\frac{p}{m+p}\right), 0\right) & (G = LR), \\ (0, -1) & (G = LH), \\ \left(-\frac{p}{m+p}, 1\right) & (G = BNP). \end{cases}
$$

Using (2) and (3), (4) may be expanded as

$$
\begin{aligned}
T^*_G \ = \ & \mathrm{tr}[A^\omega (AU - \sqrt{r}QV)] + \frac{1}{\sqrt{p}}\left\{c_1 \mathrm{tr}[A^\omega \left(AU - \sqrt{r}QV\right)^2] \right. \\
& \left. - \sqrt{r}\mathrm{tr}[A^\omega (AU - \sqrt{r}QV)V]\right\} + O_p\left(\frac{1}{p}\right),
\end{aligned}
$$

where

$$
Q = AK = I_q + \frac{1}{p(1+r)} A\Omega \qquad \text{and} \qquad K = I_q + \frac{1}{p}\Omega.
$$

3.2. The Characteristic Function

The characteristic function $C(t)$ of T_G^* is given by

$$C(t) = \mathrm{E}[\exp(itT_G^*)] = \mathrm{E}[\exp(it\mathrm{tr}[A^\omega AU] - it\sqrt{r}\mathrm{tr}[A^\omega QV])g(U,V)],$$

where

$$\begin{aligned} g(U,V) &= 1 + \frac{it}{\sqrt{p}}\{c_1\mathrm{tr}[A^\omega\big(AU - \sqrt{r}QV\big)^2] \\ &\quad -\sqrt{r}\mathrm{tr}[A^\omega(AU - \sqrt{r}QV)V]\} + O_p\left(\frac{1}{p}\right). \end{aligned}$$

Let Z_1 be a $q \times p$ random matrix distributed according to $N_{q\times p}(O, I_q \otimes I_p)$ and Z_2 be a $q \times m$ random matrix distributed according to $N_{q\times m}(O, I_q \otimes I_m)$. Then

$$\begin{aligned} U &= \frac{1}{\sqrt{p}}Z_1Z_1' - \sqrt{p}I_q + \frac{1}{\sqrt{p}}Z_1\Omega_1' + \frac{1}{\sqrt{p}}\Omega_1Z_1' \quad \text{and} \\ V &= \frac{1}{\sqrt{m}}Z_2Z_2' - \sqrt{m}I_q, \end{aligned} \tag{5}$$

where Ω_1 is a $q \times p$ matrix such that $\Omega_1\Omega_1' = \Omega$. By using (5), we can rewrite the characteristic function as

$$\begin{aligned} C(t) &= (2\pi)^{-q(p+m)/2}\iint \mathrm{etr}\left\{-\frac{1}{2}\left(I_q - \frac{2it}{\sqrt{p}}A^{\omega+1}\right)Z_1Z_1'\right. \\ &\quad \left. + \frac{2it}{\sqrt{p}}A^{\omega+1}\Omega_1Z_1' - \sqrt{p}itA^{\omega+1}\right\} \\ &\quad \times\mathrm{etr}\left\{-\frac{1}{2}\left(I_q + \frac{2it\sqrt{r}}{\sqrt{m}}A^\omega Q\right)Z_2Z_2' + it\sqrt{p}A^\omega Q\right\} \\ &\quad \times g(U(Z_1), V(Z_2))dZ_1dZ_2. \end{aligned}$$

In addition, we make use of the following transformations:

$$\begin{aligned} Z_1 &= \left(I_q - \frac{2it}{\sqrt{p}}A^{\omega+1}\right)^{-1/2} \\ &\quad \times\left(\tilde{Z}_1 + \frac{2it}{\sqrt{p}}\left(I_q - \frac{2it}{\sqrt{p}}A^{\omega+1}\right)^{-1/2}A^{\omega+1}\Omega_1\right), \end{aligned} \tag{6}$$

$$Z_2 = \left(I_q + \frac{2it\sqrt{r}}{\sqrt{m}}A^\omega Q\right)^{-1/2}\tilde{Z}_2. \tag{7}$$

These transformations imply that

$$\begin{aligned} C(t) &= (2\pi)^{-q(p+m)/2}\left|I_q - \frac{2it}{\sqrt{p}}A^{\omega+1}\right|^{-p/2}\left|I_q + \frac{2it\sqrt{r}}{\sqrt{m}}A^{\omega}Q\right|^{-m/2} \\ &\quad \times \mathrm{etr}\left\{(it)(\sqrt{p}A^{\omega}Q - \sqrt{p}A^{\omega+1}) \right. \\ &\quad \left. + \frac{2(it)^2}{p}\left(I_q - \frac{2it}{\sqrt{p}}A^{\omega+1}\right)^{-1}A^{\omega+1}\Omega A^{\omega+1}\right\} \\ &\quad \times \iint \mathrm{etr}\left\{-\frac{1}{2}\tilde{Z}_1\tilde{Z}_1' - \frac{1}{2}\tilde{Z}_2\tilde{Z}_2'\right\} \\ &\quad \times g(U(Z_1(\tilde{Z}_1)), V(Z_2(\tilde{Z}_2)))d\tilde{Z}_1 d\tilde{Z}_2 \\ &\qquad\qquad (\tilde{Z}_1 \sim N_{q\times p}(O, I_{pq}) \text{ and } \tilde{Z}_2 \sim N_{q\times m}(O, I_{mq})) \\ &= \left|I_q - \frac{2it}{\sqrt{p}}A^{\omega+1}\right|^{-p/2}\left|I_q + \frac{2it\sqrt{r}}{\sqrt{m}}A^{\omega}Q\right|^{-m/2} \\ &\quad \times \mathrm{etr}\left\{(it)(\sqrt{p}A^{\omega}Q - \sqrt{p}A^{\omega+1}) \right. \\ &\quad \left. + \frac{2(it)^2}{p}\left(I_q - \frac{2it}{\sqrt{p}}A^{\omega+1}\right)^{-1}A^{\omega+1}\Omega A^{\omega+1}\right\} \\ &\quad \times \mathrm{E}[g(U(Z_1(\tilde{Z}_1)), V(Z_2(\tilde{Z}_2)))]. \end{aligned}$$

Notice that the Jacobian of the transformations (6) and (7) may be expanded as

$$\begin{aligned} &\left|I_q - \frac{2it}{\sqrt{p}}A^{\omega+1}\right|^{-p/2}\left|I_q + \frac{2it\sqrt{r}}{\sqrt{m}}A^{\omega}Q\right|^{-m/2} \\ &= \mathrm{etr}[(it)(\sqrt{p}A^{\omega+1} - \sqrt{p}A^{\omega}Q)]\mathrm{etr}[(it)^2\left(A^{2\omega+2} + r(A^{\omega}Q)^2\right)] \\ &\qquad \times \left\{1 + \frac{4(it)^3}{3\sqrt{p}}\mathrm{tr}\left[A^{3\omega+3} - r^2(A^{\omega}Q)^3\right] + O\left(\frac{1}{p}\right)\right\}, \end{aligned}$$

and that the second term of the characteristic function may be expanded as

$$\begin{aligned} &\mathrm{etr}\left[\frac{2(it)^2}{p}\left(I_q - \frac{2it}{\sqrt{p}}A^{\omega+1}\right)^{-1}A^{\omega+1}\Omega A^{\omega+1}\right] \\ &= \mathrm{etr}\left[2(it)^2A^{2\omega+2}\frac{1}{p}\Omega\right]\left\{1 + \frac{4(it)^3}{\sqrt{p}}\mathrm{tr}\left[A^{3\omega+3}\frac{1}{p}\Omega\right] + O\left(\frac{1}{p}\right)\right\}. \end{aligned}$$

Next, in order to calculate the expectation $\mathrm{E}[g(U(Z_1(\tilde{Z}_1)), V(Z_2(\tilde{Z}_2)))]$, we use the stochastic expansions of U and V thus:

$$U = \frac{1}{\sqrt{p}}\tilde{Z}_1\tilde{Z}_1' - \sqrt{p}I_q + \frac{1}{\sqrt{p}}\tilde{Z}_1\Omega_1' + \frac{1}{\sqrt{p}}\Omega_1\tilde{Z}_1'$$

$$
\begin{aligned}
&+\frac{it}{p}A^{\omega+1}\tilde{Z}_1\tilde{Z}_1' + \frac{it}{p}\tilde{Z}_1\tilde{Z}_1'A^{\omega+1} \\
&+\frac{2it}{p}A^{\omega+1}\Omega + \frac{2it}{p}\Omega A^{\omega+1} + O_p\left(\frac{1}{\sqrt{p}}\right) \\
=\ & \tilde{U} + 2itA^{\omega+1} + \frac{4it}{p}A^{\omega+1}\Omega + O_p\left(\frac{1}{\sqrt{p}}\right),
\end{aligned}
$$

$$
\begin{aligned}
V =\ & \frac{1}{\sqrt{m}}\tilde{Z}_2\tilde{Z}_2' - \sqrt{m}I_q \\
&-\frac{it\sqrt{r}}{m}A^{\omega}Q\tilde{Z}_2\tilde{Z}_2' - \frac{it\sqrt{r}}{m}\tilde{Z}_2\tilde{Z}_2'A^{\omega}Q + O_p\left(\frac{1}{\sqrt{p}}\right) \\
=\ & \tilde{V} - 2it\sqrt{r}A^{\omega}Q + O_p\left(\frac{1}{\sqrt{p}}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{U} &= \frac{1}{\sqrt{p}}\tilde{Z}_1\tilde{Z}_1' - \sqrt{p}I_q + \frac{1}{\sqrt{p}}\tilde{Z}_1\Omega_1' + \frac{1}{\sqrt{p}}\Omega_1\tilde{Z}_1', \\
\tilde{V} &= \frac{1}{\sqrt{m}}\tilde{Z}_2\tilde{Z}_2' - \sqrt{m}I_q.
\end{aligned}
$$

Then we derive the following expansion:

$$
\begin{aligned}
& g(U(Z_1(\tilde{Z}_1)), V(Z_2(\tilde{Z}_2))) \\
&= 1 + \frac{it}{\sqrt{p}}\Bigg\{c_1\mathrm{tr}\Bigg[A^{\omega}A\tilde{U}A\tilde{U} + rA^{\omega}Q\tilde{V}Q\tilde{V} \\
&\quad +4(it)^2A^{3\omega+4}\left(I_q + \frac{2}{p}\Omega + rK^2\right)^2 - \sqrt{r}A^{\omega}A\tilde{U}Q\tilde{V} \\
&\quad -\sqrt{r}A^{\omega}Q\tilde{V}A\tilde{U} - 4it\sqrt{r}A^{2\omega+2}\left(I_q + \frac{2}{p}\Omega + rK^2\right)Q\tilde{V} \\
&\quad +4itA^{2\omega+3}\left(I_q + \frac{2}{p}\Omega + rK^2\right)\tilde{U}\Bigg] \\
&\quad -\sqrt{r}\mathrm{tr}\Bigg[A^{\omega}A\tilde{U}\tilde{V} - \sqrt{r}A^{\omega}Q\tilde{V}^2 \\
&\quad +2itA^{2\omega+2}\left(I_q + \frac{2}{p}\Omega + rK^2\right)\tilde{V} \\
&\quad -2it\sqrt{r}A^{2\omega+1}Q\tilde{U} + 2itrA^{2\omega}Q^2\tilde{V} \\
&\quad -4(it)^2\sqrt{r}A^{3\omega+2}Q\left(I_q + \frac{2}{p}\Omega + rK^2\right)\Bigg]\Bigg\} + O_p\left(\frac{1}{p}\right).
\end{aligned}
$$

We note the following basic formulas for the statistics $\tilde{U}$ and $\tilde{V}$:

$$\begin{aligned} E[\mathrm{tr}(\tilde{U})] &= 0, \\ E[\mathrm{tr}(\tilde{V})] &= 0, \\ E[\mathrm{tr}(\tilde{V}^2)] &= q(q+1), \\ E[\mathrm{tr}(A^\omega A\tilde{U}A\tilde{U})] &= \mathrm{tr}(A^{2+\omega}) + \mathrm{tr}(A)\mathrm{tr}(A^{1+\omega}) \\ &\quad +2\mathrm{tr}\left(A^{2+\omega}\frac{1}{p}\Omega\right) + \mathrm{tr}(A)\mathrm{tr}\left(A^{1+\omega}\frac{1}{p}\Omega\right) \\ &\quad +\mathrm{tr}(A^{1+\omega})\mathrm{tr}\left(A\frac{1}{p}\Omega\right), \\ E[\mathrm{tr}(A^k Q\tilde{V}Q\tilde{V})] &= \mathrm{tr}(A^\omega Q^2) + \mathrm{tr}(Q)\mathrm{tr}(A^\omega Q), \\ E[\mathrm{tr}(A^\omega Q\tilde{V}^2)] &= \mathrm{tr}(A^\omega Q) + q\mathrm{tr}(A^\omega Q). \end{aligned}$$

By using the formulas, we obtain the following expectation:

$$E[g(U(Z_1(\tilde{Z}_1)), V(Z_2(\tilde{Z}_2)))] = 1 + \frac{1}{\sqrt{p}}\{(it)a_1^* + (it)^3 a_3^*\} + O(p^{-1}),$$

where

$$\begin{aligned} a_1^* &= c_1\left\{\mathrm{tr}\left[A^{2+\omega}\left(I_q + \frac{2}{p}\Omega + rK^2\right)\right]\right. \\ &\quad \left. +(1+r)q\mathrm{tr}[A^{1+\omega}K] + \mathrm{tr}[A^{1+\omega}]\mathrm{tr}\left[A\frac{1}{p}\Omega\right]\right\} \\ &\quad +r(1+q)\mathrm{tr}[A^\omega Q], \\ a_3^* &= 4c_1\mathrm{tr}\left[A^{4+3\omega}\left(I_q + \frac{2}{p}\Omega + rK^2\right)^2\right] \\ &\quad +4r\mathrm{tr}\left[A^{2+3\omega}Q\left(I_q + \frac{2}{p}\Omega + rK^2\right)\right]. \end{aligned}$$

Consequently, we can see that the characteristic expansion may be expanded as

$$C(t) = \exp\left\{(it)^2\frac{\sigma_*^2}{2}\right\}\left\{1 + \frac{1}{\sqrt{p}}(itb_1^* + (it)^3 b_3^*)\right\} + O\left(\frac{1}{p}\right),$$

where

$$\begin{aligned} \sigma_* &= \left(2\mathrm{tr}\left[\left(I_q + \frac{r_2}{p}\Omega\right)^{-2(1+\omega)}\left(I_q + \frac{2}{p}\Omega + r\left(I_q + \frac{1}{p}\Omega\right)^2\right)\right]\right)^{1/2}, \\ b_1^* &= a_1^*, \end{aligned}$$

$$b_3^* = a_3^* + \frac{4}{3}\mathrm{tr}\left[\left(I_q + \frac{r_2}{p}\Omega\right)^{-3(1+\omega)}\left(I_q + \frac{3}{p}\Omega - r^2\left(I_q + \frac{1}{p}\Omega\right)^3\right)\right].$$

3.3. Expansions of the Non-Null Distribution

Using the inversion formula of the characteristic function, we obtain the asymptotic expansion of the distribution function of T_G^*.

Theorem 4.

$$Pr\left(\frac{T_G^*}{\sigma_*} \le z\right) = \Phi(z) - \phi(z)\left\{\frac{1}{\sqrt{p}}\left(\frac{b_1^*}{\sigma_*} + \frac{b_3^*}{\sigma_*^3}h_2(z)\right)\right\} + O\left(\frac{1}{p}\right),$$

where σ_* and the coefficients b_j^*'s are as given above.

4. The Selection of the Test of Greatest Power

In this section, we compare the asymptotic power of the three test statistics.

Let

$$\delta_G = T_G - T_G^*,$$

then the power function of T_G may be expressed as

$$\begin{aligned} P_G &= Pr(T_G > \sigma z_{CF}(\alpha)) + O\left(\frac{1}{p}\right) \\ &= Pr\left(\frac{T_G^*}{\sigma_*} > \frac{\sigma z_{CF}(\alpha) - \delta_G}{\sigma_*}\right) + O\left(\frac{1}{p}\right). \end{aligned}$$

If the order of Ω is larger than $\sqrt{p}$, the asymptotic power is 1 since $\delta_G/\sigma_* \to \infty$, while if the order of Ω is smaller than $\sqrt{p}$, the asymptotic power is α since $\delta_G \to 0$ and $\sigma_* \to \sigma$. Therefore we assume $\Omega = O(\sqrt{p})$. Then, coefficients in the asymptotic expansion may be expanded as

$$\frac{\sigma_*^2}{2} = (1+r)q + \frac{1}{p}2(1-r\omega)\mathrm{tr}(\Omega) + O\left(\frac{1}{p}\right),$$

$$A = I_q + O\left(\frac{1}{\sqrt{p}}\right), \quad K = I_q + O\left(\frac{1}{\sqrt{p}}\right), \quad Q = I_q + O\left(\frac{1}{\sqrt{p}}\right),$$

$$a_1^* = a_1 + O\left(\frac{1}{\sqrt{p}}\right), \quad a_3^* = a_3 + O\left(\frac{1}{\sqrt{p}}\right), \quad b_1^* = b_1 + O\left(\frac{1}{\sqrt{p}}\right),$$

$$b_3^* = b_3 + O\left(\frac{1}{\sqrt{p}}\right).$$

These equations yield the asymptotic expansion of the characteristic function as

$$C(t) = \exp\left\{(it)^2\frac{\sigma^2}{2}\right\}\left\{1 + \frac{1}{\sqrt{p}}\big(itb_1 + (it)^2 b_2 + (it)^3 b_3\big)\right\} + O\left(\frac{1}{p}\right),$$

where

$$b_2^* = \frac{2(1-r\omega)}{\sqrt{p}}\mathrm{tr}(\Omega).$$

By the inversion formula of the characteristic function, we obtain

$$Pr\left(\frac{T_G^*}{\sigma_*} \le z\right) = \Phi(z) - \phi(z)\left\{\frac{1}{\sqrt{p}}\left(\frac{b_1}{\sigma} + \frac{b_2^*}{\sigma^2}h_1(z) + \frac{b_3}{\sigma^3}h_2(z)\right)\right\} + O\left(\frac{1}{p}\right).$$

On the other hand, under the assumption $\Omega = O(\sqrt{p})$, δ_G may be expanded as

$$\delta_G = \frac{1}{\sqrt{p}}\mathrm{tr}(\Omega) + \frac{c_1}{p\sqrt{p}}\mathrm{tr}(\Omega^2) + O\left(\frac{1}{p}\right).$$

Consequently, the power function of T_G becomes

$$\begin{aligned}
P_G &= P\left(\frac{T_G^*}{\sigma} > z_\alpha - \frac{1}{\sigma\sqrt{p}}\mathrm{tr}(\Omega) + \frac{1}{\sqrt{p}}\left\{\frac{b_1}{\sigma} + (z_\alpha^2 - 1)\frac{b_3}{\sigma^3}\right.\right. \\
&\quad -\frac{c_1}{\sigma p}\mathrm{tr}(\Omega^2) - \frac{2z_\alpha(1-r\omega)}{\sigma^2\sqrt{p}}\mathrm{tr}(\Omega) \\
&\quad \left.\left.+\frac{2(1-r\omega)}{\sigma^3 p}(\mathrm{tr}(\Omega))^2\right\}\right) + O\left(\frac{1}{p}\right) \\
&= 1 - \Phi\left(z_\alpha - \frac{1}{\sigma\sqrt{p}}\mathrm{tr}(\Omega)\right) + \frac{1}{\sqrt{p}}\phi\left(z_\alpha - \frac{1}{\sigma\sqrt{p}}\mathrm{tr}(\Omega)\right) \\
&\quad \times\left\{\frac{c_1}{\sigma p}\mathrm{tr}(\Omega^2) + \frac{b_2}{\sigma^2}\left\{z_\alpha - \frac{1}{\sigma\sqrt{p}}\mathrm{tr}(\Omega)\right\}\right. \\
&\quad -b_3\left\{\frac{2z_\alpha}{\sigma^4\sqrt{p}}\mathrm{tr}(\Omega) - \frac{1}{\sigma^5 p}\left(\mathrm{tr}(\Omega)\right)^2\right\} \\
&\quad \left.-\frac{2(1-r\omega)}{\sigma^2\sqrt{p}}\mathrm{tr}(\Omega)\left(z_\alpha - \frac{1}{\sigma\sqrt{p}}\mathrm{tr}(\Omega)\right)\right\} + O\left(\frac{1}{p}\right) \\
&= 1 - \Phi\left(z_\alpha - \frac{1}{\sigma\sqrt{p}}\mathrm{tr}(\Omega)\right) + \frac{1}{\sqrt{p}}\phi\left(z_\alpha - \frac{1}{\sigma\sqrt{p}}\mathrm{tr}(\Omega)\right) \\
&\quad \times\left\{\frac{4q}{3\sigma^4\sqrt{p}}(2r+1)(r+1)\mathrm{tr}(\Omega)\left\{2z_\alpha - \frac{1}{\sigma\sqrt{p}}\mathrm{tr}(\Omega)\right\}\right.
\end{aligned}$$

$$+c_1 \left\{ \frac{1}{\sigma p} \mathrm{tr}(\Omega^2) + \frac{1}{qp\sigma} (\mathrm{tr}(\Omega))^2 - \frac{2z_\alpha}{q\sqrt{p}} \mathrm{tr}(\Omega) \right\} \Bigg\} + O\left(\frac{1}{p}\right).$$

The difference of the asymptotic power of LR, LH and BNP becomes

$$\frac{c_1}{\sqrt{p}} \phi \left(z_\alpha - \frac{1}{\sigma\sqrt{p}} \mathrm{tr}(\Omega) \right) \left\{ \frac{1}{\sigma p} \mathrm{tr}(\Omega^2) + \frac{1}{qp\sigma} (\mathrm{tr}(\Omega))^2 - \frac{2z_\alpha}{q\sqrt{p}} \mathrm{tr}(\Omega) \right\}.$$

Hence we obtain the following theorem.

Theorem 5. *Let $g(n,p,q,\Omega)$ be defined as*

$$g(n,p,q,\Omega) = \left(\frac{n-p+q}{2q(n+q)} \right)^{1/2} \left(\frac{1}{p} \mathrm{tr}(\Omega^2) + \frac{1}{qp} (\mathrm{tr}(\Omega))^2 \right) - \frac{2z_\alpha}{q\sqrt{p}} \mathrm{tr}(\Omega).$$

If $g(n,p,q,\Omega) > 0$, then the asymptotic power of T_{LH} is greatest and if $g(n,p,q,\Omega) < 0$, then the asymptotic power of T_{BNP} is greatest.

By the above theorem, the sign of $g(n,p,q,\Omega)$ is important. If $\mathrm{tr}(\Omega)$ is large, the power of T_G becomes large. Therefore, it does not matter which of T_{LR}, T_{LH} and T_{BNP} we choose. If $\mathrm{tr}(\Omega)$ is small, then $g(n,p,q,\Omega) < 0$. In this case, the asymptotic power of T_{BNP} is greatest.

We checked this result by numerical simulation. The values of n, p, Ω were chosen as follows:

$$(n,p) = (40,10), (40,20), (40,30), (80,20), (80,40), (80,60),$$
$$\mathrm{diag}(\Omega) = (3,2), (5,5).$$

In this simulation, we firstly obtained the upper 5 percent points by using 100,000 samples generated by Monte Carlo simulation under the null hypothesis. Next, we calculated the power of T_G from the upper 5 percent point and Monte Carlo simulation under the non-null hypothesis. For this simulation also, T_{BNP} had greatest power.

Acknowledgements

The author is deeply grateful to Professor Y. Fujikoshi, Chuo University; Professor H. Wakaki and Dr. H. Yanagihara, Hiroshima University, for their valuable comments.

diag(Ω)	n	p	T_{LR}	T_{LH}	T_{BNP}
$(3,2)$	40	10	0.154	0.151	0.155
		20	0.100	0.098	0.102
		30	0.074	0.072	0.076
	80	20	0.119	0.119	0.119
		40	0.085	0.084	0.086
		60	0.068	0.066	0.068
$(5,5)$	40	10	0.307	0.307	0.312
		20	0.177	0.165	0.178
		30	0.107	0.097	0.113
	80	20	0.224	0.218	0.226
		40	0.131	0.127	0.134
		60	0.091	0.086	0.091

Table 1: The asymptotic power of T_{LR}, T_{LH} and T_{BNP}

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