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NECESSARY CONDITIONS FOR L^p-CONVERGENCE OF LAGRANGE INTERPOLATION IN FINITE DISC

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Abstract: In this paper we have studied the powerful necessary conditions for convergence of Lagrange interpolation on an arbitrary system of nodes in $L^p(d\alpha)$ with $d\alpha$ belonging to the Szegö class. This provides a partial answer to Problem XI of P. Turàn, *J. Approx. Theory*, **29** (1980), 23-85, in finite disc.

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1. Introduction

Let D_R denote the class of functions regular in the finite disc $U_R = \{z : |z| < R\}$, $0 < R < \infty$, for every $R' \leq R$ but for no $R' > R$. A function is said to be regular in $\overline{U}_{R_o}$, $0 < R_o < R$, the closure of U_{R_o}, if it is regular in $U_{R'}$ for some $R' > R_o$. We denote by $\overline{D}_{R_o}$ the class of functions regular in $\overline{U}_{R_o}$.

Let X denote a triangular matrix of nodes $\{z_{jn}\}$ in U_R, $j = 1, 2, \ldots, n = 1, 2, \ldots$. The Lagrange interpolating polynomial of $f \in D_R$ on X is defined by

$$L_n(X, f) \equiv L_n(X, f, z) = \sum_{k=1}^{n} f(z_{kn})\ell_{kn}(z),\ n = 1, 2 \ldots, \tag{1.1}$$

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$$\ell_{kn}(z)) = \frac{W_n(z)}{(z - z_{kn})W_n'(z_{kn})}, \; k = 1, 2, \ldots, n, \;\; n = 1, 2, \ldots,$$

with $W_n(z) \equiv W_n(X, z) = (z - z_{1n})(z - z_{2n}) \cdots (z - z_{nn})$.

Dealing with mean convergence of $L_n(X, f)$, Turàn [6] proposed:

Problem XI. *Given $p > 1$, what is the necessary and sufficient condition so that*

$$\lim_{n\to\infty} \int_{-1}^{1} |f(x) - L_n(X, f, x)|^p \, dx = 0$$

for every $f \in C[-1, 1]$?

Let $d\alpha(z)$ be a finite Borel positive measure on the finite disc whose support is an infinite set. Let

$$P_n(d\alpha, z) = \gamma_n z^n + \cdots \quad (\gamma_n = \gamma_n(d\alpha) > 0)$$

be the orthonormal polynomials with respect to $d\alpha$. The zeros z_{kn} of $P_n(d\alpha, z)$ are simple and are contained in the smallest set overlapping the support of $d\alpha$. If, in addition $d\alpha$ is an absolutely continuous measure, then $d\alpha(z) = \alpha'(z)dz$, $\alpha'(z)$ is a weight function.

Also, Damelin, Jung and Kwon [1] studied the necessary conditions for mean convergence of Lagrange interpolation for exponential weights but our results are more general then those of Turàn [6] and Damelin, Jung and Kwon [1].

We propose a more general result:

Problem. *Let $d\alpha$ be an absolutely continuous measure supported in U_R along the real line. Given $p > 0$, what is the necessary and sufficient condition so that*

$$\lim_{n\to\infty} \int\int_{\overline{U}_{R_o}} |f(z) - L_n(X, f, z)|^p \, d\alpha(z) = 0 \tag{1.2}$$

for every $f \in \overline{D}_{R_o}$?

Let

$$\phi(z) = \frac{1}{R}\left\{z + \sqrt{z^2 - R^2}\right\}, \quad z \in C \;\; (C \text{ is the complex plane}),$$

denote the conformal map of $C \setminus \overline{U}_{R_0}$ on to the exterior of the finite disc and the branch of square root is taken so that $(1/R)\sqrt{z^2 - R^2}$ behaves like z near infinity.

The analysis of the orthogonal polynomials $\{P_n(d\alpha, .)\}_{n=0}^{\infty}$ associated with general weights $d\alpha$ has been a major theme in classical analysis of the twentieth century. Probably the most elegant part of that theory is due to Szegö: If $d\alpha \in S$ (the Szegö class) which means

$$\iint_{\overline{U}_{R_o}} \log \alpha'(z) \; d\alpha(z) > -\infty, \tag{1.3}$$

then there are very precise asymptotic for $P_n(d\alpha, z), n \to \infty$, for $z \in C \setminus \overline{U}_{R_0}$.

An asymptotic intermediate between the Szegö and the n-th root asymptotic is the ratio asymptotic:

$$\lim_{n\to\infty} \frac{P_{n+1}(d\alpha, z)}{P_n(d\alpha, z)} = \phi(z), \quad \lim_{n\to\infty} P_n(d\alpha, z)^{1/n} = \phi(z), z \in C \setminus \overline{U}_{R_0}. \tag{1.4}$$

This has a close connection to the three term recurrence relation

$$zP_n(d\alpha, z) = A_n P_{n+1}(d\alpha, z) + B_n P_n(d\alpha, z) + A_{n-1} P_{n-1}(d\alpha, z).$$

If (1.2) holds for all $f \in \overline{D}_{R_0}$ for a regular measure $d\alpha$ then (1.4) is essentially equivalent to

$$\lim_{n\to\infty} A_n = R^2/2R_0; \lim_{n\to\infty} B_n = 0,$$

which gives

$$\lim_{n\to\infty} \gamma_n^{1/n}(d\alpha) = 2R_o/R^2,$$

then the asymptotic behaviour of $W_n(X, z)$ behaves like the regular (n-th root) asymptotic behaviour of

$$W_n(d\alpha, z) = \frac{P_n(d\alpha, z)}{\gamma_n(d\alpha)}.$$

In this paper an attempt has been made to give powerful necessary conditions guaranteing (1.2) for all $f \in \overline{D}_{R_o}$ for $d\alpha \in S$.

We need some definitions and notations which have been frequently used. Write for $f \in \overline{D}_{R_o}$ and $0 < p < \infty$,

$$\begin{aligned}
\|f\|_{d\alpha,p} &= \left\{ \iint_{\overline{U}_{R_o}} |f|^p \; d\alpha(z) \right\}^{1/p}, \quad \|f\| = \max_{z\in\overline{U}_{R_o}} |f(z)|, \\
\|L_n(X)\|_{d\alpha,p} &= \sup_{\|f\|\le 1} \|L_n(X, f)\|_{d\alpha,p}, \quad S_n(X, z) = \sum_{k=1}^{n} |(z - z_{kn}) \, \ell_{kn}(z)|, \\
\gamma_n(X) &= \sum_{k=1}^{n} \left| W_n'(z_{k_n}) \right|^{-1}.
\end{aligned}$$

2. Auxiliary Results

In this section we give some auxiliary results which have been used in the sequel.

Let $d\mu(\theta)$ be finite Borel positive and absolutely continuous measure on the interval $[0, 2\pi]$, supported in U_{R_o}, $R_o < R$ along the real line

$$G(d\mu) = \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \log \mu'(\theta)\, d\theta)\right\}.$$

If $d\mu \in S$, i.e. ,

$$\int_0^{2\pi} \log \mu'(\theta)\, d\theta > -\infty\,, \tag{2.1}$$

then the Szegö function is defined by

$$D(t) \equiv D(d\mu, t) = \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \frac{R_o + te^{-i\theta}}{R_o - te^{-i\theta}} \log \mu'(\theta)\, d\theta\right\}$$

which is analytic in $|t| < R_0$, $R_o < R$. For an arbitrary continuous function $F(\theta)$ of period 2π, we have [5, p. 269].

$$\lim_{r \to R_o^-} \int_0^{2\pi} F(\theta) \left|D\left(re^{i\theta}\right)\right|^2 d\theta = \int_0^{2\pi} F(\theta)\mu'(\theta)\, d\theta.$$

Put

$$d_n \equiv d_n(d\mu) = \min_{b_k} \frac{1}{2\pi}\int_0^{2\pi} \left|t^n + b_1 t^{n-1} + \cdots + b_{n-1}t + b_n\right|^p d\mu(\theta),$$
$$t = R_o e^{i\theta} \tag{2.2}$$

If $d\mu \in S$, then (see [5])

$$\lim_{n\to\infty} d_n(d\mu) = G(d\mu). \tag{2.3}$$

Lemma 2.1. *Let $d\mu$ be an absolutely continuous measure supported in $[0, 2\pi]$ and $d\mu \in S$. Let $0 < p < \infty$. If $\rho_n(t) = t^n + a_1 t^{n-1} + \cdots + a_{n-1}t + a_n$ has no zeros in $|t| > R_0$ and satisfies*

$$\int_0^{2\pi} \left|\rho_n\left(R_o e^{i\theta}\right)\right|^p d\mu(\theta) \le \text{const.},$$

then

$$\left|\frac{\rho_n(t)}{t^n}\right| \le \text{const.}, \quad |t| \ge R > R_0.$$

Proof. Consider the function,

$$D(t)[\Psi_n^*(t)]^{p/2} - R_o = \left[D(0)d_n^{-1/2} - R_o\right] + d_{n1}t + d_{n2}t^2 + \cdots ,$$

where

$$\Psi_n^*(t) = t^n \overline{\Psi}_n^* \left(t^{-1}\right) \quad \text{with} \quad \Psi_n(t) = d_n^{-1/p}\rho_n(t).$$

Now, a simple manipulation of the proof of Theorem 12.1 in [5] proves the lemma, so we omit the details.

Lemma 2.2. *Let $d\mu$ be an absolutely continuous measure supported in $\overline{U}_{R_0}$ along real line and $d\alpha \in S$. Let $0 < p < \infty$. If*

$$\iint\limits_{\overline{U}_{R_o}} \left| \left(\frac{2R_0}{R^2}\right)^n W_n\left(X, z\right)\right|^p d\alpha(z) \leq \text{const.} \tag{2.4}$$

$$\left|\frac{2^n R_o^n W_n(X,t)}{R^{2n}[\phi(t)]^n}\right| \leq \text{const.}, \quad t \in \Omega, \tag{2.5}$$

whenever $\Omega \subset C \setminus \overline{U}_{R_o}$ is compact.

Proof. Define $Q_n(z) = \left(\frac{2R_0}{R^2}\right)^n W_n(X,z)$ with $z = (t + t^{-1})/2$. Note that

$$Q_n(z) = \sum_{k=0}^{n} b_{n-k} z^n = \sum_{k=0}^{n} 2^{-k} b_{n-k}\left(t + t^{-1}\right)^k$$
$$= t^{-n} \sum_{k=0}^{2n} a_{2n-k} t^k = t^{-n}\rho_{2n}(t), \tag{2.6}$$

where $a_k = a_{2n-k}$, $k - 0, 1, \ldots, n$, $a_o = a_{2n} = 1$ and $|Q_n(z)| = |\rho_{2n}(t)|$, $t = R_o e^{i\theta}$.

Meanwhile, we observe that $\rho_{2n}(t) \neq 0$ for $|t| \neq R_0$. Further, we define the associated measure $d\mu(\theta)$ on $[0, 2\pi]$ by

$$d\mu(\theta) = |R_0 \sin\theta| d\alpha(R_0 \cos\theta). \tag{2.7}$$

and

$$\int_0^{2\pi} \left|\rho_{2n}\left(R_o e^{i\theta}\right)\right|^p d\mu(\theta) = \int_0^{2\pi} \left|Q_n\left(R_o e^{i\theta}\right)\right|^p |R_0 \sin\theta|\, d\alpha(R_0\cos\theta)$$
$$= 2 \iint\limits_{\overline{U}_{R_o}} |Q_n(z)|^p\, d\alpha(z)\,. \tag{2.8}$$

In view of Lemma 2.1 and (2.4), we get

$$\left|\frac{\rho_{2n}(t)}{t^{2n}}\right| \leq \text{const.}, \quad t \geq R > R_o.$$

By (2.6),

$$\left|\frac{Q_n(z)}{t^n}\right| \leq \text{const.}, \quad t \geq R > R_o.$$

and this establishes (2.5) at once. □

Lemma 2.3. *Let $d\mu$ be an absolutely continuous measure supported in $\overline{U}_{R_0}$ along the real line and $d\mu \in S$ be defined by (2.7). Assume that $0 < p < \infty$ and*

$$e_n \equiv e_n(d\alpha, p)$$
$$= \min_{b_k} \frac{1}{A} \int\int_{\overline{U}_{R_o}} \left| \left(\frac{2R_0}{R^2}\right)^n z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + b_n \right|^p d\alpha(z),$$

where A is the area of the open disc domain U_{R_o}. Then

$$e_n(d\alpha) \geq d_{2n}(d\mu) \geq G(d\mu). \tag{2.9}$$

Meanwhile,

$$\lim_{n\to\infty} e_n(d\alpha) > 0 \iff d\alpha \in S. \tag{2.10}$$

Proof. By definition, we have $d_n(d\mu) \geq d_{n+1}(d\mu)$, This, with (2.3) implies $d_{2n}(d\mu) \geq G(d\mu)$. Further, (2.8) yields $e_n(d\alpha) \geq d_{2n}(d\mu)$.

To prove the last assertion we note that for $d\alpha \in S$, (2.9) implies (2.10).

Conversely, let $p \geq 1$ and let $\phi_n(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ be solution of (2.2). Since $\mu(\theta)$ is even, it follows that

$$\overline{\phi}(t^{-1}) = \overline{\phi}(t) = t^n + \overline{a}_1 t^{n-1} + \cdots + \overline{a}_n$$

is also a solution of (2.2). Thus, $\psi_n(t) = (\phi_n(t) + \overline{\phi}(t))/2$ is a solution of (2.2). Let $t = R_o e^{i\theta}$ and

$$\psi_n(t) = \sum_{k=0}^{n} b_{n-k} t^k = \sum_{k=0}^{n} b_{n-k} R_o^k (\cos k\theta + i \sin k\theta), \ b_o = 1.$$

If $T_k(z)$ denote the k-th Chebyshev polynomial of first kind, then

$$\begin{aligned} d_n(d\mu) &= \frac{1}{2\pi}\int_0^{2\pi} \left|\psi_n\left(R_o e^{i\theta}\right)\right|^p d\mu(\theta) \\ &\geq \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^n b_{n-k}R_0^k\cos k\theta\right|^p d\mu(\theta) = \frac{1}{A}\iint\limits_{\overline{U}_{R_o}}\left|\sum_{k=0}^n b_{n-k}T_k(z)\right|^p d\alpha(z) \\ &= \frac{1}{A2^p}\iint\limits_{\overline{U}_{R_o}}\left|\left(\frac{2R_0}{R^2}\right)^n z^n+\cdots\right|^p d\alpha(z) \geq 2^{-p}e_n(d\alpha). \end{aligned}$$

Thus, for $p \geq 1$, $d\alpha \in S$, follows at once from (2.3), (2.10) and the above inequality. Using the fact under the assumption (2.7), (1.3) is equivalent to (2.1). If $0 < p < 1$, then by Hölder's inequality,

$$e_n(d\alpha, p) \leq \text{const. } e_n(d\alpha, 1).$$

This means that $\lim_{n\to\infty} e_n(d\alpha) > 0$ and hence $d\alpha \in S$. □

Lemma 2.4. *Let X be defined by (1.1). Then*

$$0 < c_1 \leq \left|\frac{2^n R_o^n W_n(X,t)}{R^{2n}[\phi(t)]^n}\right| \leq c_2, \quad t \in \Omega \tag{2.11}$$

holds for every compact set Ω in $C\backslash\overline{U}_{R_o}$, where c_1, c_2 are constants independent of n, is equivalent to

$$\left|\sum_{k=1}^n f\left(z_{n_k}\right) - \frac{n}{A}\iint\limits_{\overline{U}_{R_o}} f(z)\, d\alpha(z)\right| \leq \text{const. } \|f\|_\Delta, \tag{2.12}$$

where f is analytic in an open disc $\Delta \supset \overline{U}_{R_o}$ and $\|f\|_\Delta = \sup_{z\in\Delta}|f(z)|$.

Proof. Let $\Delta \supset \overline{U}_{R_o}$ be an open disc. Choose a closed curve Γ in Δ enclosing $\overline{U}_{R_o}$ so that distance of Γ from $\overline{U}_{R_o}$ is positive, and put $\Omega = \Gamma$. Then, for an arbitrary sequence $\alpha(n)$ satisfying $\alpha(n) \to 0$ as $n \to \infty$, it follows from (2.11) that

$$\lim_{n\to\infty}\frac{\alpha(n)2^n R_o^n W_n(t)}{R^{2n}\phi(t)^n} = 0$$

holds uniformly on Ω. Thus, by differentiation, we get

$$\lim_{n\to\infty}\frac{\alpha(n)2^n R_o^n W_n(t)}{R^{2n}\phi(t)^n}\left\{\frac{W_n'(t)}{W_n(t)} - \frac{n}{t-z}\right\} = 0$$

also holds uniformly on Ω. Since $\alpha(n)$ is arbitrary, by (2.11), it follows that

$$R_n(t) = \left\{ \frac{W_n'(t)}{W_n(t)} - \frac{n}{t-z} \right\} \tag{2.13}$$

satisfies $|R_n(t)| \leq$ const., $t \in \Omega$. From (2.13), we get for every f analytic in Δ

$$\frac{1}{2\pi i} \int_\Gamma f(t) \left\{ \frac{W_n'(t)}{W_n(t)} - \frac{n}{t-z} \right\} dt = \frac{1}{2\pi i} \int_\Gamma f(t) R_n(t)\, dt.$$

Applying residue theorem, we have

$$\frac{1}{2\pi i} \int_\Gamma f(t) \frac{W_n'(t)}{W_n(t)}\, dt = \sum_{k=0}^{n} f\left(z_{kn}\right).$$

Meanwhile,

$$\frac{n}{2\pi i} \int_\Gamma \frac{f(t)}{t-z}\, dt = \frac{n}{A} \int\!\!\int_{\overline{U}_{R_o}} f(z)\, d\alpha(z)$$

and

$$\left| \frac{1}{2\pi i} \int_\Gamma f(t) R_n(t)\, dt \right| \leq \text{const. } \|f\|_\Delta.$$

Then, (2.12) is immediate.

($\Leftarrow$) Let $\Omega \subset C \setminus \overline{U}_{R_o}$ be an arbitrary compact set. Choose an open set Δ so that $\overline{U}_{R_o} \subset \Delta$ and distance between Ω and Δ is positive, since Ω is compact and Ω and $\overline{U}_{R_o}$ are disjoint. Then, $f(z) = \log(t-z)$ with $t \in \Omega$ is analytic in Δ. For this function (2.12) gives

$$\log W_n(t) - \frac{n}{A} \int\!\!\int_{\overline{U}_{R_o}} \log(t-z)\, d\alpha(z) \leq \text{const. } \|f\|_\Delta \leq c_o,\ t \in \Omega. \tag{2.14}$$

The relation

$$\frac{1}{A} \int\!\!\int_{\overline{U}_{R_o}} \log(t-z)\, d\alpha(z) = \log\left(\frac{R^2 \phi(t)}{2R_o} \right), \quad t \in C \setminus \overline{U}_{R_o},$$

is to be used. This may be proved by argument used in [2], Lemma 2.2. It follows from (2.14) that

$$\left| \log \frac{2^n R_o^n W_n(X,t)}{R^{2n} [\phi(t)]^n} \right| \leq c_o, \quad t \in \Omega$$

which gives (2.11) provided we set $c_1 = e^{-c_o}$ and $c_2 = e^{c_o}$. □

Lemma 2.5. *Let X be given by (1.1). Then*

$$\left|\frac{S_n(X,t)}{[\phi(t)]^n}\right| \geq c > 0, \quad t \in \Omega \tag{2.15}$$

holds for every compact set Ω in $C \setminus \overline{U}_{R_o}$ and if

$$\frac{\gamma_n(X)R^{2n}}{2^n R_o^n} \leq \text{const.}, \tag{2.16}$$

holds, then

$$\left|\frac{2^n R_o^n W_n(X,t)}{R^{2n}[\phi(t)]^n}\right| \geq c_1, \quad t \in \Omega \tag{2.17}$$

holds for every compact set Ω in $C \setminus \overline{U}_{R_o}$. Here c_1 and c_2 are constants independent of n.

Proof. Using the Lagrange interpolation formula on the nodes $z_1, z_2, \ldots, z_n$, the $(n-1)$-th Chebyshev polynomial of the first kind $T_{n-1}(t)$, $t \in \Omega$ can be expressed as

$$T_{n-1}(t) = \sum_{k=1}^{n} T_{n-1}(z_k)\,\ell_k(t) = \sum_{k=1}^{n} T_{n-1}(z_k)\,\frac{W_n(t)}{(t-z_k)W_n'(z_k)}.$$

Then,

$$|T_{n-1}(t)| \leq |S_n(t)| \max_{1\leq k\leq n} \frac{1}{|t-z_k|} \leq |S_n(t)|\, d(t)\,, \tag{2.18}$$

where $d(t) = 1/\text{dist}(t, \overline{U}_{R_o})$. Taking logarithms on both sides in (2.18) yields

$$\log|T_{n-1}(t)| \leq \log S_n(t) + \log d(t) \leq \log|S_n(t)| + c_3\,,$$

or in other form

$$\left|\frac{S_n(t)}{T_{n-1}(t)}\right| \geq c_4 > 0, \quad t \in \Omega.$$

Using the formula (see [4], p. 239),

$$\lim_{n\to\infty} \frac{T_n(t)}{[\phi(t)]^n} = \frac{(2R_o)^n}{2R^{2n}}, \quad t \in C \setminus \overline{U}_{R_o}$$

and (2.15) is immediate. Meanwhile, by definition we get

$$S_n(X,t) = \gamma_n(X) W_n(X,t). \tag{2.19}$$

Thus, (2.17) is an immediate consequence of (2.15) and (2.16). □

Lemma 2.6. *The statement*

$$\left|\frac{S_n(t)}{[\phi(t)]^n}\right| \leq \text{const.}, \quad t \in \Omega \tag{2.20}$$

holds for every compact set Ω *in* $C \setminus \overline{U}_{R_o}$ *if and only if (2.16) and (2.11) hold.*

Proof. By (2.19), (2.16) and (2.11) imply (2.20). □

Conversely, by the same arguments as in Lemma 2.4 it follows from (2.20) and (2.15) that (2.12) holds. By Lemma 2.4, (2.11) is true. Furthermore, by (2.19), (2.20) and (2.11) yield (2.16).

Lemma 2.7. *Let* $d\mu$ *be an arbitrary measure supported in* U_R *supported along real line and* $0 < p_0 \leq p \leq \infty$. *Then, for arbitrary system* X *of nodes,*

$$\|S_n(X)\|_{d\alpha,p} \leq c(p_o)\,\|L_n(X)\|_{d\alpha,p}\,, \quad n \geq 1.$$

The proof is left to the reader.

3. Main Result

Our result is the following theorem.

Theorem 3.1. *Let* $d\mu$ *be an absolutely continuous measure supported in* U_{R_o} *along the real line, and* $d\alpha \in S$. *Let* $0 < p < \infty$. *If (1.2) holds for every* $f \in \overline{D}_{R_o}$, *then:*

(a) $\frac{R^{2n}\gamma_n(X)}{2^n R_o^n} \leq \text{const.}$;

(b) $\frac{S_n(X,t)}{[\phi(t)]^n} \leq \text{const.}$, $t \in \Omega$ holds for every compact set Ω in $C \setminus \overline{U}_{R_o}$;

(c) $0 < c_1 \leq \frac{2^n R_o^n W_n(X,t)}{R^{2n}[\phi(t)]^n} \leq c_2$, $t \in \Omega$ holds for every compact set Ω in $C \setminus \overline{U}_{R_o}$;

(d) $\left|\sum_{k=1}^{n} f(z_{n_k}) - \frac{n}{A}\int\int_{\overline{U}_{R_o}} f(z)d\alpha(z)\right| \leq \text{const.}\ \|f\|_\Delta$.

Here c_1 *and* c_2 *are constants independent of* n.

Proof. If (1.2) holds for every $f \in \overline{D}_{R_o}$, then by the Banach Theorem

$$\|L_n(X)\|_{d\alpha,p} \leq \text{const.}$$

So, by Lemma 2.7,

$$\|S_n(X)\|_{d\alpha,p} \leq \text{const.},$$

or, equivalently

$$\gamma_n(X)\,\|W_n(X)\|_{d\alpha,p} \le \text{const.} \tag{3.1}$$

Since, $d\alpha \in S$, by (2.17),

$$\left(\frac{2R_0}{R^2}\right)^n \|W_n(X)\|_{d\alpha,p} \ge [AG(d\mu)]^{1/p} > 0$$

which by (3.1) implies statement (a).

On the other hand it is well known that for an arbitrary system X of nodes $\gamma_n \ge (2R_o)^{n-1}/(R^2)^{(n-1)}$ from which we conclude that

$$\left(\frac{2R_0}{R^2}\right)^n \|W_n(X)\|_{d\alpha,p} \le \text{const.}$$

Then, by applying Lemma 2.2 and Lemma 2.5, we get statement (c).

Statement (b) follows from statements (a) and (c) in view of Lemma 2.6.

Meanwhile, statement (d) follows from (c) by Lemma 2.4. □

Remarks. 1. Note that $(b) \iff (a)$, cf. Lemma 2.6, and $(c) \iff (d)$, cf. Lemma 2.4.

2. For $d\alpha \in S$, the orthogonal polynomials $W_n(d\alpha, z) = P_n(d\alpha, z)/\gamma_n(d\alpha)$ must satisfy the conditions $(a) - (d)$. In particular, statement (d) is a representation of the asymptotic distribution of nodes. Thus, our theorem shows that under the assumptions of the theorem the asymptotic behaviour of $W_n(X, z)$ behaves like the asymptotic behaviour of $W_n(d\alpha, z)$.

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