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WELL POSEDNESS AND REGULARITY OF THE CONTROLLED MINDLIN-TIMOSHENKO PLATE MODEL

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Abstract: The Mindlin-Timoshenko model describes the elastic motion of a homogeneous and isotropic thin plate, and Ciarlet, [3], has shown that this and other models are "correct", first order approximations of the full, nonlinear models. We consider here the M-T model with boundary control in a variational formulation and establish the well-posedness and regularity properties of the model, in particular in the case where the spatial domain has corners. These issues are vital in the analysis of the controllability problems connected to the plate. The work is closely related to the previous work of Lions and Lagnese, [6], [7], and Pedersen, [12], [14].

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1. Introduction and Notation

The Mindlin-Timoshenko model describes the elastic motion of a homogeneous and isotropic thin plate, the motion is assumed to be elastic, in the sense that no permanent deformation of the plate occurs.

We consider the plate in rectangular coordinates $(x, y, z) \in R^3$, when the plate is in equilibrium it occupies the volume denoted $\Omega\times]\frac{-h}{2}, \frac{h}{2}[$, where h is the thickness of the plate and Ω is called the middle surface. We assume Ω is a bounded subset of R^2 with a sufficiently smooth boundary $\Gamma = \partial\Omega$.

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Introducing the usual notation $Q = \Omega\times]0,T[$, $\Sigma_0 = \Gamma_0\times]0,T[$ and $\Sigma_1 = \Gamma_1\times]0,T[$ the Mindlin-Timoshenko plate model is the system of equations:

$$\begin{cases} \frac{\rho h^3}{12}\psi'' - D(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy} + \frac{1+\mu}{2}\varphi_{xy}) + K(\psi + w_x) & = 0 \quad \text{in} \quad Q, \\ \frac{\rho h^3}{12}\varphi'' - D(\varphi_{yy} + \frac{1-\mu}{2}\varphi_{xx} + \frac{1+\mu}{2}\psi_{xy}) + K(\varphi + w_y) & = 0 \quad \text{in} \quad Q, \\ \rho h w'' - K\left[(\psi + w_x)_x + (\varphi + w_y)_y\right] & = 0 \quad \text{in} \quad Q, \end{cases} \tag{1}$$

with the boundary conditions :

$$\begin{cases} u = (\psi, \varphi, w) & = 0 \quad \text{on} \quad \Sigma_1, \\ D(n_1\psi_x + \mu n_1\varphi_y + \frac{1-\mu}{2}n_2(\psi_y + \varphi_x)) & = \kappa_1 \quad \text{on} \quad \Sigma_0, \\ D(n_2\varphi_y + \mu n_2\psi_x + \frac{1-\mu}{2}n_1(\psi_y + \varphi_x)) & = \kappa_2 \quad \text{on} \quad \Sigma_0, \\ K(\partial_n w + n_1\psi + n_2\varphi) & = \kappa_3 \quad \text{on} \quad \Sigma_0, \end{cases} \tag{2}$$

and with appropriate initial conditions $u(0) = u^0$ and $u'(0) = u^1$ in Ω. In order to keep a simple notation we will write this as

$$\begin{cases} \mathbf{C}u_{tt} = \mathcal{A}u & \text{in} \quad Q\,, \\ \mathcal{B}u = \kappa & \text{on} \quad \Sigma_0\,, \\ u = 0 & \text{on} \quad \Sigma_1\,, \\ (u(0), u_t(0)) = (u^0, u^1) & \text{in} \quad \Omega\,. \end{cases} \tag{3}$$

Here $\partial_n = n_1\frac{\partial}{\partial x} + n_2\frac{\partial}{\partial y}$ and we have written ψ_x instead of $\frac{\partial\psi}{\partial x}$, φ_{xy} for $\frac{\partial^2\varphi}{\partial x\partial y}$ etc., for brevity. Also note that we only apply κ to the part Γ_0 of the boundary.

We will usually refer to (3) as the control system since it is the system we want to control by acting on the boundary, i.e. the task is to find a control κ such that we can steer any initial data to any final state in some suitable function space. The next task is therefore to derive appropriate well-posedness results for (3) and then eventually apply modern control theory to the system. This program is persued in [14], applying the modern formulation of HUM (see [15], [13]).

We will now state a Green's formula associated to system (3), this is the fundamental formula which we will use repeatedly throughout the rest of this paper.

We introduce the bilinear forms a, a_0 and a_1 by

$$\begin{aligned} a(u,v) &= a_0(u,v) + Ka_1(u,v) \\ &= -\int_\Omega \tilde{\psi}\left[D\left(\frac{\partial^2\psi}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2\psi}{\partial y^2} + \frac{1+\mu}{2}\frac{\partial^2\varphi}{\partial x\partial y}\right) - K\left(\psi + \frac{\partial w}{\partial x}\right)\right] \end{aligned}$$

$$+\tilde{\varphi}\left[D\left(\frac{\partial^2\varphi}{\partial y^2}+\frac{1-\mu}{2}\frac{\partial^2\varphi}{\partial x^2}+\frac{1+\mu}{2}\frac{\partial^2\psi}{\partial x\partial y}\right)-K\left(\varphi+\frac{\partial w}{\partial y}\right)\right]$$
$$+\tilde{w}K\left(\frac{\partial}{\partial x}\left(\psi+\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\varphi+\frac{\partial w}{\partial y}\right)\right)dxdy$$
$$+\int_\Gamma \tilde{\psi}D\left[\left(\frac{\partial\psi}{\partial x}+\mu\frac{\partial\varphi}{\partial y}\right)n_1+\frac{1-\mu}{2}\left(\frac{\partial\psi}{\partial y}+\frac{\partial\varphi}{\partial x}\right)n_2\right]$$
$$+\tilde{\varphi}D\left[\left(\frac{\partial\varphi}{\partial y}+\mu\frac{\partial\psi}{\partial x}\right)n_2+\frac{1-\mu}{2}\left(\frac{\partial\psi}{\partial y}+\frac{\partial\varphi}{\partial x}\right)n_1\right]$$
$$+\tilde{w}K\left[\frac{\partial w}{\partial n}+n_1\psi+n_2\varphi\right]d\Gamma.$$

Here we recognize $\mathcal{A}u$ and $\mathcal{B}u$ and we get the (halfways) Green's formula:

$$-\int_\Omega \mathcal{A}uvdxdy = a(u,v)-\int_{\Gamma_0}\mathcal{B}uvd\Gamma\,, \tag{4}$$

where we have assumed $v=0$ on Γ_1. The complementary boundary operator is then

$$\mathcal{C}v = v|_{\Gamma_0}.$$

1.1. The Variational Form

In order to formulate problem (3) in a variational form we introduce spaces V and H, such that we have a Gelfand triple

$$V \hookrightarrow H \hookrightarrow V'\,, \tag{5}$$

where as usual $\hookrightarrow$ denotes a continuous, dense injection. We define:

$$H = [L^2(\Omega)]^3, \tag{6}$$

$$H^1_{\Gamma_1}(\Omega) = \left\{ f\in H^1(\Omega) \middle| f=0 \text{ on } \Gamma_1 \right\} \tag{7}$$

and equip $H^1_{\Gamma_1}$ with the H^1 norm, here $f=0$ of course has to be taken in sense of traces. Notice that if $\Gamma_1=\emptyset$ then $H^1_{\Gamma_1}(\Omega)=H^1(\Omega)$ and $\Gamma_1=\Gamma$ implies $H^1_{\Gamma_1}=H^1_0(\Omega)$. Now we can define V as

$$V = [H^1_{\Gamma_1}(\Omega)]^3, \tag{8}$$

corresponding to the homogeneous Dirichlet boundary condition in (3). Equipping V and H with their respective product topologies we have a continuous, dense injection $V \hookrightarrow H$ and thus the Gelfand triple $V \hookrightarrow H \hookrightarrow V'$. Here we have as usual identified the dual of H with itself. Now it is convenient to introduce the spaces

$$\mathcal{H} = V \times H, \qquad \mathcal{H}' = H \times V'.$$

Now, following the general approach from Pedersen [12], we can formulate (3) in a variational form. From the Green's formula we have

$$\int_\Omega \mathcal{A}uv dxdy + a(u,v) - \int_{\Gamma_0} \mathcal{B}uv d\Gamma = 0$$

using (3) and the expression for c, we get

$$\int_\Omega \mathbf{C}u_{tt}v dxdy + a(u,v) - \int_{\Gamma_0} \mathcal{B}uv d\Gamma = c(u_{tt},v) + a(u,v) - \int_{\Gamma_0} \mathcal{B}uv d\Gamma = 0.$$

From this expression and the spaces defined above we can now write the variational form of the system (3) as

$$\begin{cases} (u,u_t) \in C([0,T];\mathcal{H}), \\ c(u_{tt},v) + a(u,v) - \int_{\Gamma_0} \mathcal{B}uv d\Gamma = 0, \qquad \forall v \in V,\ 0 < t < T, \\ (u(0),u_t(0)) = (u^0,u^1) \in \mathcal{H}. \end{cases} \tag{9}$$

2. The Adjoint System

Following the HUM-method we define the adjoint system to be

$$\begin{cases} \mathbf{C}v_{tt} = \mathcal{A}v & \text{in} \quad Q, \\ \mathcal{B}v = 0 & \text{on} \quad \Sigma_0, \\ v = 0 & \text{on} \quad \Sigma_1, \\ (v(0),v_t(0)) = (v^0,v^1) & \text{in} \quad \Omega, \end{cases} \tag{10}$$

which is simply the homogeneous control system.

The variational formulation of the adjoint system is then

$$\begin{cases} (v,v_t) \in C([0,T];\mathcal{H}), \\ c(v_{tt},v) + a(u,v) = 0, \quad \forall v \in V,\ 0 < t < T, \\ (v(0),v_t(0)) = (v^0,v^1) \in \mathcal{H}. \end{cases} \tag{11}$$

2.1. Well-Posedness of the Adjoint System

In order to establish the well-posedness of the adjoint system we will study the coercivity of the forms $c(\cdot,\cdot)$ and $a(\cdot,\cdot)$. We recall that

$$\begin{aligned} a(u,v) &= a_0(\psi,\varphi,\tilde{\psi},\tilde{\varphi}) + K a_1(u,v)\,, \\ c(u,v) &= \int_\Omega \rho h \left[\frac{h^2}{12}\psi\tilde{\psi} + \frac{h^2}{12}\varphi\tilde{\varphi} + w\tilde{w}\right] dxdy\,, \end{aligned}$$

where

$$\begin{aligned} a_0(\psi,\varphi,\tilde{\psi},\tilde{\varphi}) = \int_\Omega D\Big[&\psi_x\tilde{\psi}_x + \varphi_y\tilde{\varphi}_y + \mu\tilde{\psi}_x\varphi_y + \mu\psi_x\tilde{\varphi}_y \\ &+\frac{1-\mu}{2}\left(\psi_y+\varphi_x\right)\left(\tilde{\psi}_y+\tilde{\varphi}_x\right)\Big]dxdy \end{aligned} \tag{12}$$

and

$$a_1(u,v) = \int_\Omega \left(w_x+\psi\right)\left(\tilde{w}_x+\tilde{\psi}\right) + \left(w_y+\varphi\right)\left(\tilde{w}_y+\tilde{\varphi}\right)dxdy. \tag{13}$$

Lemma 1. *The symmetric bilinear form $c(\cdot,\cdot)$ is continuous on H and defines an inner product and a norm $\|\cdot\|_c$ on H, equivalent to the usual norm.*

Proof. First we prove continuity. By a direct calculation we have for all $u,v\in H$

$$\begin{aligned} |c(u,v)| = \left|\rho h\left[\frac{h^2}{12}\left(\psi \mid \tilde{\psi}\right)_{L^2} + \frac{h^2}{12}\left(\varphi\mid\tilde{\varphi}\right)_{L^2} + \left(w\mid\tilde{w}\right)_{L^2}\right]\right| \\ \le \max\{\rho h, \frac{\rho h^3}{12}\}|(u\mid v)_H| \le C\|u\|_H\|v\|_H \end{aligned}$$

for some constant $C>0$, from the Cauchy-Schwarz' inequality.

Now, since

$$c(u,u) \ge \min\{\rho h, \frac{\rho h^3}{12}\}(u\mid u)_H = \lambda\|u\|_H^2 \qquad \text{for all } u\in H,$$

where $\lambda>0$, $c(\cdot,\cdot)$ is also H-elliptic and hence defines a norm on H. □

Lemma 2. *The symmetric bilinear form $a(\cdot,\cdot)$ is continuous on V.*

Proof. We consider a_0 and a_1 separately, so a_0 and a_1 must be continuous on $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ and V, respectively.

For a_0 we have

$$\begin{aligned} |a_0(\psi,\varphi,\tilde{\psi},\tilde{\varphi})| &= \Big| D\int_\Omega \psi_x\tilde{\psi}_x + \varphi_y\tilde{\varphi}_y + \mu\psi_x\tilde{\varphi}_y \\ &\quad + \mu\varphi_y\tilde{\psi}_x + \frac{1-\mu}{2}(\psi_y+\varphi_x)(\tilde{\psi}_y+\tilde{\varphi}_x)dxdy\Big| \\ &\le \max\{D,\mu,\frac{1-\mu}{2}\}|((\psi,\varphi)\mid(\tilde{\psi},\tilde{\varphi}))|_{[H^1_{\Gamma_1}(\Omega)]^2} \\ &\le C_0\|(\psi,\varphi)\|_{H^1_{\Gamma_1}(\Omega)}\|(\tilde{\psi},\tilde{\varphi})\|_{H^1_{\Gamma_1}(\Omega)} \end{aligned}$$

for a constant $C_0 > 0$. Similarly for a_1 we have

$$\begin{aligned} |a_1(u,v)| &= \left|\int_\Omega (w_x+\psi)(\tilde{w}_x+\tilde{\psi}) + (w_y+\varphi)(\tilde{w}_y+\tilde{\varphi})dxdy\right| \\ &\le C_1|(u\mid v)_V| \le C_1\|u\|_V\|v\|_V, \end{aligned}$$

for a constant $C_1 > 0$. □

We also see that $a(\cdot,\cdot)$ is V-elliptic:

Lemma 3. *Assume $\Gamma_1 \neq \emptyset$, then:*
(i) there exists a constant $\alpha_0 > 0$ such that

$$a_0(\psi,\varphi,\psi,\varphi) \ge \alpha_0\left(\|\psi\|^2_{H^1(\Omega)} + \|\varphi\|^2_{H^1(\Omega)}\right)$$

for all $(\psi,\varphi) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$.
(ii) for all $K > 0$ there is an $\alpha > 0$ (depending on K) such that

$$a_0(\psi,\varphi,\psi,\varphi) + Ka_1(u,u) \ge \alpha\|u\|^2_V$$

for all $u \in V$, i.e. $a(\cdot,\cdot)$ is V-elliptic.

Lemma 4. *Assume $\Gamma_1 = \emptyset$, then:*
(i) there exists , for every $\lambda > 0$ a constant $\alpha_0 > 0$ such that

$$a_0(\psi,\varphi,\psi,\varphi) \ge \alpha_0\left(\|\psi\|^2_{H^1(\Omega)} + \|\varphi\|^2_{H^1(\Omega)}\right) - \lambda(\|\psi\|^2_{L^2(\Omega)} + \|\varphi\|^2_{L^2(\Omega)})$$

for all $(\psi,\varphi) \in H^1(\Omega) \times H^1(\Omega)$.
(ii) for all $K > 0, \lambda > 0$, there is an $\alpha > 0$ (depending on K and λ) such that

$$a_0(\psi,\varphi,\psi,\varphi) + Ka_1(u,u) \ge \alpha\|u\|^2_V - \lambda\|u\|^2_H$$

for all $u \in V$, i.e. $a(\cdot,\cdot)$ is V-coercive.

Proof. The inequalities in (i) and (ii) are versions of Korn's inequality, due to Gobert [4]. □

As an immediate corollary of Lemmas 2 and 3 we have

Corollary 5. *If $\Gamma_1 \neq \emptyset$ then $a(\cdot,\cdot)$ defines a norm on V equivalent to the usual norm*

The well-posedness of (11) now follows from standard variational theory.

2.2. Semigroup Formulation

The desired well-posedness results of (11) come quite natural if we instead take a semigroup approach. This is studied from a stabilization point of view in Lagnese [6], but for a general study of semigroup formulations and well-posedness results for structural models we refer to Banks [1]. We will here briefly give an outline of the setting and refer to e.g. Pazy, [11] for further information on the theory. Moreover, we will only consider the case when $\Gamma_1 \neq \emptyset$.

Lemma 6. *Let H and V be equipped with the inner-products $c(\cdot,\cdot)$ and $a(\cdot,\cdot)$, respectively. Then there exist unique isomorphisms $C_c : H \to H$ and $A_a : V \to V'$ such that*

$$\begin{aligned} (C_c v \mid u)_H = c(v,u) \qquad & \text{for all } u,v \in H\,, \\ \langle A_a v, u\rangle_{V',V} = a(v,u) \qquad & \text{for all } u,v \in V\,. \end{aligned}$$

Proof. The existence and uniqueness of $C_c : H \to H$ follows from the Lax-Milgram Theorem.

Since a is V-elliptic it follows again from the Lax-Milgram Theorem that there exists a unique isomorphism $L : V \to V$ with

$$a(v,u) = (Lv \mid u)_V \text{ for all } u,v \in V. \tag{14}$$

On the other hand, from the Riesz' Representation Theorem there exist an isometric isomorphism $R : V \to V'$ such that

$$(v \mid u)_V = \langle Rv, u\rangle_{V',V}. \tag{15}$$

Combining (14) and (15) gives us

$$a(v,u) = \langle RLv, u\rangle_{V',V} \text{ for all } u,v \in V$$

so we see that $A = RL$. □

Using the above lemma we can rewrite (11) as

$$(C_c v_{tt}, u)_H + \langle A_a v, u\rangle_{V',V} = 0 \text{ for all } u \in V$$

which is equivalent to the "first order" formulation

$$\frac{d}{dt} C_c v_t + A_a v = 0, \quad v \in V. \tag{16}$$

We will formally formulate (16) as a first order system in t in the following way:

$$\bar{C}\begin{pmatrix} v \\ v_t \end{pmatrix}_t + \bar{A}\begin{pmatrix} v \\ v_t \end{pmatrix} = 0\,, \tag{17}$$

where

$$\bar{C} = \begin{pmatrix} A_a & 0 \\ 0 & C_c \end{pmatrix} \qquad \bar{A} = \begin{pmatrix} 0 & -A_a \\ A_a & 0 \end{pmatrix}.$$

It is clear from Lemma 6 that $\bar{C} : V \times H \to V' \times H$ is an isomorphism and therefore has an inverse $\bar{C}^{-1} : V' \times H \to V \times H$. If we define the domain of $\bar{A}$ as

$$D(\bar{A}) = \{v \in V | A_a v \in H\} \times V$$

and equip it with the norm

$$\|(v^0, v^1)\|_{D(\bar{A})} = \|A_a v^0\|_H + \|v^1\|_V \tag{18}$$

and use Lemma 6 again we see that $\bar{A} : D(\bar{A}) \to V' \times H$. Thus we are able to rewrite (17) as

$$\begin{pmatrix} v \\ v_t \end{pmatrix}_t + \bar{C}^{-1}\bar{A}\begin{pmatrix} v \\ v_t \end{pmatrix} = 0. \tag{19}$$

Now in order to obtain again the well-posedness results of (11) we will show that $-\bar{C}^{-1}\bar{A}$ is the infinitesimal generator of a C_0-semigroup of contractions in $V \times H$, by the Lumer-Phillips Theorem (see Chen and Zhou [2, p. 236]) this is the case when $-\bar{C}^{-1}\bar{A}$ is densely defined, dissipative and there exist a $\lambda > 0$ such that $R(\lambda I + \bar{C}^{-1}\bar{A}) = V \times H$.

Lemma 7. *Let $\Gamma_1 \neq \emptyset$ then $-\bar{C}^{-1}\bar{A}$ generates a C_0-semigroup of contractions in $V \times H$, or equivalently*

(i) $D(\bar{A})$ is dense in $V \times H$.

(ii) $-\bar{C}^{-1}\bar{A}$ is dissipative, i.e.

$$(-\bar{C}^{-1}\bar{A}(v,u) \mid (v,u))_{V\times H} \leq 0 \qquad \text{for all } v, u \in D(\bar{A})\,.$$

(iii) $R(\lambda I + \bar{C}^{-1}\bar{A}) = \mathcal{H}$ for all $\lambda > 0$.

Proof. (i) Let $(v,u) \in D(\bar{A})$. From Lemma 6 and Green's formula (4) we have

$$\langle A_a v, u\rangle_{V',V} = a(v,u) = -\int_\Omega \mathcal{A} v u dx dy$$

but the integral is only well-defined if we restrict v to be in $V \cap [H^2(\Omega)]^3$, thus we have

$$\{(v,u) \in (V \cap [H^2(\Omega)]^3) \times V\} \subseteq D(\bar{A}). \tag{20}$$

Since the inclusion $(V \cap [H^2(\Omega)]^3) \times V \subset V \times H$ is dense, the inclusion in (20) is also dense.

(ii) We have

$$\begin{aligned}(-\bar{C}^{-1}\bar{A}(v,u) \mid (v,u))_{V\times H} &= (-\bar{C}^{-1}(-A_a u, A_a v) \mid (v,u))_{V\times H} \\ = ((u, -C_c^{-1}A_a v) \mid (v,u))_{V\times H} &= a(v,u) - c(C_c^{-1}A_a v, u) \\ &= \langle A_a v, u\rangle_{V',V} - \langle A_a v, u\rangle_{V',V} = 0\end{aligned}$$

for all $v, u \in D(\bar{A})$, where we have used that V and H are equipped with the inner-products $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$, respectively.

(iii) $R(\lambda I + \bar{C}^{-1}\bar{A}) = \mathcal{H}$ for all $\lambda > 0$ is equivalent to $R(\lambda^2 C_c + A_a) = V'$ but we see that this clearly holds from the previous discussion since $\bar{A}$ is skew-adjoint. □

If we by $S(t) : V \times H \to V \times H$ denote the semigroup of contractions generated by $-\bar{C}^{-1}\bar{A}$, we have that solutions to (19) with initial data $(v^0, v^1) \in V \times H$ is given by

$$(v(t), v_t(t)) = S(t)(v^0, v^1). \tag{21}$$

This is called a mild solution. It can be proved that the mild solution (21) is in fact also a solution to the variational problem (11), and vice versa, see Banks [1]. We have the the following theorem.

Theorem 8. *Let $\Gamma_1 \neq \emptyset$. Then the problem*

$$\begin{cases}(v, v_t) \in C([0,T]; \mathcal{H}), \\ c(u, v_{tt}) + a(u,v) = 0, \quad \forall u \in V,\ 0 < t < T, \\ (v(0), v_t(0)) = (v^0, v^1) \in \mathcal{H},\end{cases} \tag{22}$$

is well-posed and the map $(v^0, v^1) \to (v, v_t) : \mathcal{H} \to C([0,T]; \mathcal{H})$ is continuous and an isomorphism.

Proof. It follows from a combination of Theorem 4.1 and Theorem 4.13 and the remark on p. 103 in [1]. □

If we consider initial data in $D(\bar{A})$ we have a solution $(v, v_t) \in C([0,T];\mathcal{H})$, but we are able to improve this result:

Theorem 9. *Let $\Gamma_1 \neq \emptyset$. Then*

$$\begin{cases} (v, v_t) \in C^1([0,T];\mathcal{H}) \cap C([0,T]; D(\bar{A})) \,, \\ c(u, v_{tt}) + a(u, v) = 0, \quad \forall u \in V,\ 0 < t < T \,, \\ (v(0), v_t(0)) = (v^0, v^1) \in D(\bar{A}) \,, \end{cases} \tag{23}$$

is well-posed and the map $(v^0, v^1) \to (v, v_t) : D(\bar{A}) \to C([0,T]; D(\bar{A}))$ is continuous and an isomorphism, when $D(\bar{A})$ is equipped with the norm

$$\|(v^0, v^1)\|_{D(\bar{A})} = \|\bar{C}^{-1}\bar{A}(v^0, v^1)\|_{\mathcal{H}}.$$

Proof. We will use two properties of C_0-semigroups. First, by the definition of strong continuity we have for any $s \geq 0$ that

$$\lim_{t \downarrow s} S(t)(v^0, v^1) = S(s)(v^0, v^1) \quad \text{for all } (v^0, v^1) \in \mathcal{H}. \tag{24}$$

Now, for all $(v^0, v^1) \in D(\bar{A})$ we have

$$\bar{C}^{-1}\bar{A}S(t)(v^0, v^1) = S(t)\bar{C}^{-1}\bar{A}(v^0, v^1). \tag{25}$$

We notice that $(v, v_t) \in C([0,T]; D(\bar{A}))$ is equivalent to

$$\lim_{t \downarrow s} \|(v(t), v_t(t)) - (v(s), v_t(s))\|_{D(\bar{A})} = 0 \,.$$

Using the norm on $D(\bar{A})$ we get

$$\begin{aligned} &\lim_{t \downarrow s} \|(v(t), v_t(t)) - (v(s), v_t(s))\|_{D(\bar{A})} \\ &= \lim_{t \downarrow s} \|\bar{C}^{-1}\bar{A}(v(s), v_t(s)) - \bar{C}^{-1}\bar{A}(v(t), v_t(t))\|_{\mathcal{H}}. \end{aligned}$$

Inserting $(v(t), v_t(t)) = S(t)(v^0, v^1)$ gives us

$$\lim_{t \downarrow s} \|\bar{C}^{-1}\bar{A}S(s)(v^0, v^1) - \bar{C}^{-1}\bar{A}S(t)(v^0, v^1)\|_{\mathcal{H}}$$

and using (25) we find that this equals

$$\lim_{t \downarrow s} \|(S(s) - S(t))\bar{C}^{-1}\bar{A}(v^0, v^1)\|_{\mathcal{H}}$$

that vanishes as $t \downarrow s$ due to the strong continuity property (24).

That we also have $(v, v_t) \in C^1([0,T];\mathcal{H})$ follows in the same manner from (24) and (25). □

2.3. Domains with Corners

First we state a classical result. We will assume that $\Gamma_1 \neq \emptyset$. If we assume $\Gamma \in C^{1,1}$ (that is, Γ is locally the graph of a C^1-function with Lipschitz continuous first derivatives, with Ω locally on one side of Γ) and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$, classical PDE theory applies. We have

Theorem 10. *Assume $\Gamma \in C^{1,1}$ and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and let v be a solution to* (23) *with initial data $(v^0, v^1) \in D(\bar{A})$. Then $v \in [H^2(\Omega)]^3$, i.e. v is a classical solution.*

We will also consider boundaries with corners, which is natural from an applications point of view. But now the theory becomes rather tricky and we will only consider the situation where Γ has a finite number of corners $P_1, \dots, P_N$, each formed by a pair of line segments in Γ, furthermore we assume that Γ is $C^{1,1}$ except at the corners, and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$, then no corners have a Dirichlet condition on one of the line segments forming a corner and a Neumann condition on the other line segment.

Boundary value problems with nonsmooth boundary are studied in Grisvard [5], the results we refer to here are also quoted in Lagnese [6, pp. 33-35].

Theorem 11. *Assume $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and $\Gamma \in C^{1,1}$ except at a finite number of corners and let v be a solution to* (23) *with the initial data $(v^0, v^1) \in D(\bar{A})$. Then there exists $\frac{3}{2} < s \leq 2$ such that $v \in [H^s(\Omega)]^3$. If we assume furthermore that Ω is convex then $v \in [H^2(\Omega)]^3$.*

We now consider the case where $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 \neq \emptyset$.

Theorem 12. *Assume $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 \neq \emptyset$ and $\Gamma \in C^{1,1}$ except at a finite number of corners. Assume also that the line segments from Γ_0 and Γ_1 always meet at a strictly convex corner. Let v be a solution to* (23) *with initial data $(v^0, v^1) \in D(\bar{A})$. Then there exists $\frac{3}{2} < s \leq 2$ such that $v \in [H^s(\Omega)]^3$. If we assume furthermore that Ω is convex then $v \in [H^2(\Omega)]^3$.*

Under the conditions specified in Theorems 11 or 12 the solution $v = (\psi, \varphi, w)$ to the adjoint problem will satisfy:

$$\begin{cases} v \in C([0,\infty[; [H^s(\Omega)]^3 \cap V)\,, \\ v' \in C([0,\infty[; V)\,, \\ v'' \in C([0,\infty[; H). \end{cases} \tag{26}$$

These are the well-posedness results for the adjoint system that we need in order to apply the modern HUM-procedure.

3. Wellposedness of the Control System

To obtain well-posedness results for the control system we will use the method of transposition as follows. Let v be a (smooth) solution to the adjoint system with smooth initial data $v^0, v^1 \in C_0^\infty(\Omega)$ and consider the control system also with smooth initial data. We now multiply (3) with $v(s)$ and integrate over $\Omega \times]0,t[$.

$$\begin{aligned} 0 &= \int_0^t \int_\Omega (\mathcal{C}u_{tt}(s) - \mathcal{A}u(s))v(s)dxdyds \\ &= \int_0^t \int_\Omega \mathcal{C}u_{tt}(s)v(s)dxdy + a(u(s),v(s)) - \int_{\Gamma_0} \mathcal{B}u(s)v(s)d\Gamma dt\,, \end{aligned}$$

where we have used Green's formula (4). Recognizing $c(\cdot,\cdot)$ and $\mathcal{B}u = \kappa$ we get

$$0 = \int_0^t c(u_{tt}(s),v(s)) + a(u(s),v(s)) - \int_{\Gamma_0} \kappa(s) \cdot v(s)d\Gamma dt$$

which after integration by parts in t yields

$$\begin{aligned} 0 &= c(u_t(s),v(s))|_0^t - c(u(s),v_t(s))|_0^t \\ &+ \int_0^t c(u(s),v_{tt}(s)) + a(u(s),v(s)) - \int_{\Gamma_0} \kappa(s) \cdot v(s)d\Gamma dt. \end{aligned}$$

Using the variational form of the adjoint system, $c(u(s),v_{tt}(s)) + a(u(s),v(s)) = 0$, this is reduced to

$$\begin{aligned} 0 = c(u_t(t),v(t)) - c(u^1,v^0) - c(u(t),v_t(t)) + c(u^0,v^1) \\ - \int_0^t \int_{\Gamma_0} \kappa(s) \cdot v(s)d\Gamma dt. \end{aligned}$$

By the density of $C_0^\infty(\Omega)$ in H and from Lemma 1, $c(\cdot,\cdot)$ is a norm on H, thus

$$\begin{aligned} 0 &= (u_t(t) \mid v(t))_H - (u^1 \mid v^0)_H \\ &- (u(t) \mid v_t(t))_H + (u^0 \mid v^1)_H - \int_0^t \int_{\Gamma_0} \kappa(s) \cdot v(s)d\Gamma dt\,. \end{aligned} \tag{27}$$

By the Gelfand triple (5) we extend the inner-product on H to a duality product on $V' \times V$ and introduce the duality product

$$\langle (u^0,u^1),(v^0,v^1)\rangle_{\mathcal{H}',\mathcal{H}} = \langle u^1,v^0\rangle_{V',V} - (u^0 \mid v^1)_H\,.$$

(27) can now be expressed as

$$\langle (u(t), u_t(t)), (v(t), v_t(t))\rangle_{\mathcal{H}',\mathcal{H}} - \langle (u^0, u^1), (v^0, v^1)\rangle_{\mathcal{H}',\mathcal{H}} = \int_0^t \int_{\Gamma_0} \kappa(s)\cdot v(s) d\Gamma dt \,. \quad (28)$$

Due to the isomorphism $(v^0, v^1) \mapsto (v(t), v_t(t)) : \mathcal{H} \to \mathcal{H}$, we also have the isomorhism $(u^0, u^1) \mapsto (u(t), u_t(t)) : \mathcal{H}' \to \mathcal{H}'$ when the integral on the RHS is well-defined. By the trace theorem the restriction of $v(s) \in V$ to the boundary is in $[H^{\frac{1}{2}}(\Gamma_0)]^3$ thus $\kappa(s) \in ([H^{\frac{1}{2}}(\Gamma_0)]^3)'$. This means that $\kappa \in (L^2(0,T; [H^{\frac{1}{2}}(\Gamma_0)]^3)'$. We have the following theorem.

Theorem 13. *For any* $(u^0, u^1) \in \mathcal{H}'$ *and* $\kappa \in (L^2(0,T; [H^{\frac{1}{2}}(\Gamma_0)]^3))'$ *the control system* (3) *has a unique (weak) solution*

$$(u, u_t) \in C([0,T]; \mathcal{H}').$$

The mapping $(u^0, u^1, \kappa) \to (u, u_t)$ *is linear and continuous*

$$\mathcal{H}' \times (L^2(0,T; [H^{\frac{1}{2}}(\Gamma_0)]^3))' \to C([0,T]; \mathcal{H}')$$

and there exists $C(T) > 0$ *such that*

$$\|(u, u_t)\|_{L^\infty(0,T;\mathcal{H}')} \le C(T)(\|(u^0, u^1)\|_{\mathcal{H}'} + \|\kappa\|_{(L^2(0,T;[H^{\frac{1}{2}}(\Gamma_0)]^3))'}). \quad (29)$$

Remark 14. The continuity (29) is not an estimate made using (28), instead it is a property of the method used here, we refer to [10, Volume II, Chapter 5].

In the discussion leading to Theorem 13 we could just as well have made the arguments using $D(\bar{A})$ and $D(\bar{A})'$, instead of $\mathcal{H}$ and $\mathcal{H}'$, respectively. Recall that $D(\bar{A})'$ is the space

$$D(\bar{A})' = V' \times \{v \in V | A_a v \in H\}'$$

and (28) must be replaced by

$$\langle (u(t), u_t(t)), (v(t), v_t(t))\rangle_{D(\bar{A})',D(\bar{A})} - \langle (u^0, u^1), (v^0, v^1)\rangle_{D(\bar{A})',D(\bar{A})} \quad (30)$$

$$= \int_0^t \int_{\Gamma_0} \kappa(s)\cdot v(s) d\Gamma dt \,. \quad (31)$$

From this relation and Theorem 9 we get

Theorem 15. *For any* $(u^0, u^1) \in D(\bar{A})'$ *and* $\kappa \in (H^1(0,T;[H^{\frac{1}{2}}(\Gamma_0)]^3))'$, *the control system* (3) *has a unique (weak) solution*

$$(u, u_t) \in C([0,T]; D(\bar{A})').$$

The mapping $(u^0, u^1, \kappa) \to (u, u_t)$ *is linear and continuous*

$$D(\bar{A})' \times (H^1(0,T;[H^{\frac{1}{2}}(\Gamma_0)]^3))' \to C([0,T]; D(\bar{A})'),$$

and there exists $C(T) > 0$ *such that*

$$\|(u,u_t)\|_{L^\infty(0,T;D(\bar{A})')} \le C(T)(\|(u^0,u^1)\|_{D(\bar{A})'} + \|\kappa\|_{(H^1(0,T;[H^{\frac{1}{2}}(\Gamma_0)]^3))'}).$$

3.1. The Controllability Spaces $\mathcal{F}_0$ and $\mathcal{F}_1$

We will introduce a subspace of $\mathcal{H}'$

$$\mathcal{F}_0 = \{(v^0, v^1) \in \mathcal{H}' \big| v|_{\Sigma_0} \in [L^2(\Sigma_0)]^3\} \tag{32}$$

equipped with semi-norm

$$\|(v^0,v^1)\|^2_{\mathcal{F}_0} = \int_0^T \int_{\Gamma_0} v^2 d\Gamma dt, \tag{33}$$

where v is a solution to the adjoint system (10) with initial data (v^0, v^1). It is obvious that we have the algebraic inclusions $\mathcal{F}_0 \subset \mathcal{H}'$ and $\mathcal{H} \subset \mathcal{F}_0$ for any $T > 0$; that the last inclusion is also continuous follows immediately from trace theory, hence $\mathcal{H} \hookrightarrow \mathcal{F}_0$. Hence we have

$$\mathcal{H} \hookrightarrow \mathcal{F}_0 \subset \mathcal{H}',$$
$$\mathcal{H} \subset \mathcal{F}_0{}' \hookrightarrow \mathcal{H}'.$$

If (33) is a norm we can prove that all these inclusions are continuous.

If we assume that (33) actually defines a norm on $\mathcal{F}_0$, and take initial data $(u^0, u^1) \in \mathcal{F}_0{}'$ and a control $\kappa \in [L^2(\Sigma_0)]^3$ we have the following estimate from (28)

$$\begin{aligned} &|\langle (u(t), u_t(t)), (v(t), v_t(t))\rangle_{\mathcal{H}',\mathcal{H}}| \\ &\qquad = |\langle (u^0,u^1), (v^0,v^1)\rangle_{\mathcal{H}',\mathcal{H}} + \int_0^t \int_{\Gamma_0} \kappa(s)\cdot v(s) d\Gamma dt| \\ &\qquad \le \|(u^0,u^1)\|_{\mathcal{H}'} \|(v^0,v^1)\|_{\mathcal{H}} + \|\kappa\|_{[L^2(\Sigma_0)]^3} \|(v^0,v^1)\|_{\mathcal{F}_0} \end{aligned}$$

$$\leq C(\|(u^0,u^1)\|_{\mathcal{F}_0{}'} + \|\kappa\|_{[L^2(\Sigma_0)]^3})\|(v^0,v^1)\|_{\mathcal{H}}\,, \quad (34)$$

where $C > 0$.

We now have the following theorem.

Theorem 16. *Assume that $(u^0,u^1) \in \mathcal{F}_0{}'$. Then for any $\kappa \in [L^2(\Sigma_0)]^3$ the control system (3) has a unique (weak) solution*

$$(u,u_t) \in C([0,T];\mathcal{H}').$$

Furthermore, there exists a constant $C > 0$ such that

$$\|(u,u_t)\|_{L^\infty(0,T;\mathcal{H}')} \leq C(\|(u^0,u^1)\|_{\mathcal{F}_0{}'} + \|\kappa\|_{[L^2(\Sigma_0)]^3}).$$

Following the same lines we introduce a subspace of $\mathcal{H}$

$$\mathcal{F}_1 = \{(v^0,v^1) \in \mathcal{H} \mid v_t|_{\Sigma_0} \in [L^2(\Sigma_0)]^3\}, \quad (35)$$

with semi-norm

$$\|(v^0,v^1)\|^2_{\mathcal{F}_1} = \int_0^T \int_{\Gamma_0} v_t^2 d\Gamma dt, \quad (36)$$

where v is the solution to (10) with initial data (v^0,v^1). We now have for any $T > 0$

$$D(\bar{A}) \hookrightarrow \mathcal{F}_1 \subset \mathcal{H}\,,$$
$$\mathcal{H}' \subset \mathcal{F}_1{}' \hookrightarrow D(\bar{A})\,,$$

again we have to prove that all these inclusions are continuous in order to verify that $\|\cdot\|_{\mathcal{F}_1}$ is in fact a norm.

We can now choose initial data $(u^0,u^1) \in \mathcal{F}_1{}'$ and control $\kappa \in (H^1(0,T;[L^2(\Gamma_0)]^3))'$ and we have the following estimate from (30)

$$\begin{aligned}
&|\langle (u(t),u_t(t)),(v(t),v_t(t))\rangle_{D(\bar{A})',D(\bar{A})}| \\
&\qquad = |\langle (u^0,u^1),(v^0,v^1)\rangle_{D(\bar{A})',D(\bar{A})} + \int_0^t \int_{\Gamma_0} \kappa(s)\cdot v(s) d\Gamma dt| \\
&\leq \|(u^0,u^1)\|_{D(\bar{A})'}\|(v^0,v^1)\|_{D(\bar{A})} + \|\kappa\|_{(H^1(0,T;[L^2(\Gamma_0)]^3))'}\|(v^0,v^1)\|_{H^1(0,T;[L^2(\Gamma_0)]^3)} \\
&\qquad \leq C_1(\|(u^0,u^1)\|_{\mathcal{F}_1{}'} + \|\kappa\|_{(H^1(0,T;[L^2(\Gamma_0)]^3))'})\|(v^0,v^1)\|_{D(\bar{A})},
\end{aligned}$$

with a constant $C_1 > 0$. We formulate this as a uniqueness theorem.

Theorem 17. *Assume that* $(u^0, u^1) \in \mathcal{F}_1{}'$. *Then for any* $\kappa \in (H^1(0,T;[L^2(\Gamma_0)]^3))'$ *the control system* (3) *has a unique (weak) solution*

$$(u, u_t) \in C([0,T]; D(\bar{A})').$$

Furthermore, there exists a constant $C > 0$ *such that*

$$\|(u, u_t)\|_{L^\infty(0,T;D(\bar{A})')} \leq C(\|(u^0, u^1)\|_{\mathcal{F}_1{}'} + \|\kappa\|_{(H^1(0,T;[L^2(\Gamma_0)]^3))'}).$$

The spaces $\mathcal{F}_0$ and $\mathcal{F}_1$ were first introduced by Lions in [7], [8] and [9]. These spaces seem hard to characterize in other ways than through their definition and other spaces could be considered. These are, however, natural choices in the context of plate theory and in the variational setup used in this work. We should also mention that as in the general presentation of the classical HUM method, the abovementioned semi-norms can only be norms for T large enough, due to the finite speed of propagation. Thus the method is heavily influenced by both the size and the shape of the spatial domains considered. There are many open problems still to be solved along these lines, many seem to be closely connected to the unique continuation properties of the solutions to the equations.

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