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THE POISSON SUBALGEBRAS OF COMPACT SEMISIMPLE LIE BIALGEBRAS

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Abstract: We begin by reviewing the definition of a Lie bialgebra and recall some standard results. The Lie bialgebra structures on a compact semisimple Lie bialgebra are then determined using Soibelman's classification of such structures for a compact simple Lie bialgebra. We then determine the Poisson subalgebras for these structures. We conclude with some straightforward remarks on the notion of Poisson homogeneous spaces.

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1. Introduction

A Lie bialgebra is a Lie algebra structure on a vectorspace together with a Lie algebra structure on the dual space. The two structures cannot be arbitrary but must satisfy a relation which can be formulated in various ways. This relation was first introduced by Drinfel'd [4] and studied by Semenov-Tian-Shansky [12], and Lu-Weinstein [11] to name a few. The classification of such structures when one of them is complex simple was provided in [1] and the classification for compact simple was given by Soibelman in [14]. Lie bialgebras form a category and an arrow injective at the vectorspace level gives rise to the notion of a sub Lie bialgebra.

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At the group level, the corresponding notion is that of a Poisson Lie group. This is a Lie group equipped with a Poisson structure such that the multiplication map is a Poisson map. Here too we have sub-objects given by injective arrows. It turns out [12] that the quotient of a Poisson Lie group G by a closed Poisson Lie subgroup H is naturally a Poisson manifold and the action map

$$G \times G/H \to G/H$$

is a Poisson map. As shown in [5], [10], and [7] however, not all homogeneous Poisson Lie spaces arrive as such quotients. In this letter we will extend Soibelman's classification to cover all compact semisimple Lie bialgebras nondegenerate in a certain sense, and determine all their sub-bialgebras. We will conclude with a simple type of duality between homogeneous Poisson submanifolds of homogeneous Poisson Lie manifolds and compute the homogeneous Poisson structures for some simple examples.

2. Definitions and Basic Theory

We begin with notation. Let $\mathfrak{u}$ be a finite dimensional real vectorspace with dual space $\mathfrak{u}^\star$. We will use Latin letters for elements of $\mathfrak{u}$, Greek letters for elements of $\mathfrak{u}^\star$, and will denote the pairing by $< \alpha, x >$ for $\alpha \in \mathfrak{u}^\star$ and $x \in \mathfrak{u}$. We define

$$\wedge^{jk} = \wedge^j \mathfrak{u}^\star \otimes \wedge^k \mathfrak{u}$$

and identify $\wedge^{11}$ with $gl(\mathfrak{u})$. $\wedge^{jk}$ is dual to $\wedge^{kj}$ and we normalize $<>$ by extension via

$$< \alpha \wedge \beta, x \wedge y >=< \alpha, x >< \beta, y > - < \alpha, y >< \beta, y > ,$$

where $\alpha, \beta \in \mathfrak{u}^\star$ and $x, y \in \mathfrak{u}$. We can consider any $r \in \wedge^{02}$ as a map $r : \mathfrak{u}^\star \to \mathfrak{u}^\star$ by $< \beta, r(\alpha) >=< \alpha \wedge \beta, r >$.

Now suppose that $p \in \wedge^{21}$ is a Lie algebra structure for $\mathfrak{u}$ and $\pi \in \wedge^{12}$ is a Lie algebra structure for $\mathfrak{u}^\star$. For easy of notation we will forgo the tradition Lie bracket notation in favor of the dot convention, which is that all natural Lie actions are denoted by a dot, with the specific action determined by the elements involved and all actions are left actions. Thus we see that $x \cdot y = p(x \wedge y)$ is the adjoint action of x on y, $x \cdot \alpha$ is the coadjoint action of x on α, and $\alpha \cdot x$ is the coadjoint action of α on x. If we identify

$$\wedge^{nk} = \mathrm{Hom}(\wedge^n \mathfrak{u}, \wedge^k \mathfrak{u}) ,$$

as the space of n cochains on the Lie algebra $(\mathfrak{u}, p)$ with values in the $\mathfrak{u}$-space $\wedge^k\mathfrak{u}$, we can define a coboundary operator

$$d : \wedge^{nk} \to \wedge^{n+1,k} .$$

Similarly if we identify

$$\wedge^{nk} = \mathrm{Hom}(\wedge^k\mathfrak{u}^\star, \wedge^n\mathfrak{u}^\star) ,$$

as the space of k cochains on the Lie algebra $(\mathfrak{u}^\star, \pi)$ with values in the $\mathfrak{u}^\star$-space $\wedge^n\mathfrak{u}^\star$, we can define a coboundary operator

$$\delta : \wedge^{nk} \to \wedge^{n,k+1} .$$

Finally, let $\mathfrak{d} = \mathfrak{u} \oplus \mathfrak{u}^\star$ and define a bracket on $\mathfrak{d}$ by

$$[x + \alpha, y + \beta] = x \cdot y + \alpha \cdot \beta + \alpha \cdot y + x \cdot \beta - y \cdot \alpha - \beta \cdot x ,$$

$\forall x, y \in \mathfrak{u}$ and $\forall \alpha, \beta \in \mathfrak{u}^\star$.

Theorem 1. (see [8] [9]) *The following are equivalent:*
— $x \cdot (\alpha \cdot \beta) = (x \cdot \alpha) \cdot \beta - (\alpha \cdot x) \cdot \beta - (x \cdot \beta) \cdot \alpha + (\beta \cdot x) \cdot \alpha,$
$\forall x \in \mathfrak{u}$ *and* $\forall \alpha, \beta \in \mathfrak{u}^\star$.
— *The bracket on* $\mathfrak{d}$ *defined above satisfies the Jacobi identity.*
($\mathfrak{d}$ *is a Lie algebra).*
— $d\pi = 0$.
— $\delta p = 0$.
— $d\delta + \delta d = 0$ *i.e.* d *and* δ *form a bicomplex.*

The proof is a straightforward computation with the last item resulting from the observation that

$$\delta + d : \wedge^{**} \to \wedge^{**}$$

can be identified with the Lie algebra cohomology operator of $\mathfrak{d}$ with values in the trivial representation. The first condition is known as the Drinfel'd identity.

An ordered pair $(\mathfrak{u}, \mathfrak{u}^\star)$ satisfying the above is known as a Lie bialgebra. The collection of finite dimensional Lie bialgebras forms a category with an arrow $\phi : (\mathfrak{u}_1, \mathfrak{u}_1^\star) \to (\mathfrak{u}_2, \mathfrak{u}_2^\star)$ given by a Lie map $\phi : \mathfrak{u}_1 \to \mathfrak{u}_2$ such that the dual map, also denoted by ϕ, is also a Lie map. For $x \in \mathfrak{u}_1$ such a ϕ will give $\phi(\pi_1(x)) = \pi_2(\phi(x))$ and for $\alpha \in \mathfrak{u}_2^\star$, $\phi(\phi(x) \cdot \alpha) = x \cdot \phi(\alpha)$. If $\pi = dr$ for some $r \in \wedge^{02}$ then the Lie bialgebra is called exact and r is known as the (classical) r-matrix. In this case the Poisson structure on the group is given by

$$\Pi_g = R_g^* r - L_g^* r ,$$

with R_g and L_g the right and left translations with respect to $g \in G$. The inner automorphism Lie algebra is defined as the subalgebra of $gl(\mathfrak{u})$ generated by the adjoint action of $\ker(\pi) \subset \mathfrak{u}$ and the coadjoint action of $\ker(p) \subset \mathfrak{u}^\star$.

The defining theorem for sub-objects is

Theorem 2. (see [13]) *For $\mathfrak{h}$ a subalgebra of $\mathfrak{u}$ the following are equivalent:*
— $\mathfrak{h}^\perp$ *is an ideal of* $\mathfrak{u}^\star$.
— $\mathfrak{u}^\star \cdot \mathfrak{h} \subset \mathfrak{h}$.
— $\mathfrak{h}^\perp \cdot \mathfrak{h} = 0$.
— $\pi(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}$.
When these hold, the pair $(\mathfrak{h}, \mathfrak{u}^\star/\mathfrak{h}^\perp)$ *forms a Lie bialgebra.*

Clearly the injection $\mathfrak{h} \to \mathfrak{u}$ is a bialgebra map and we call $\mathfrak{h}$ a Poisson subalgebra or simply a sub-bialgebra of $\mathfrak{u}$. Interestingly, it can happen that $(\mathfrak{u}, \mathfrak{u}^\star)$ is exact while $(\mathfrak{h}, \mathfrak{u}^\star/\mathfrak{h}^\perp)$ is not and we define a Lie bialgebra to be totally exact if every sub-bialgebra is exact. It is straightforward to see that if every sub-bialgebra $\mathfrak{h}$ of $\mathfrak{u}$ normalizes a complement V

$$\mathfrak{u} = \mathfrak{h} \oplus V \qquad \mathfrak{h} \cdot V \subset V ,$$

then $\mathfrak{u}$ is totally exact.

The defining theorem for direct sums is

Theorem 3. *Let $\phi : (\mathfrak{u}_1^\star, \mathfrak{u}_1) \to (\mathfrak{u}_2, \mathfrak{u}_2^\star)$ be a Lie bialgebra arrow. Then for $\alpha_1 \in \mathfrak{u}_1^\star$ and $\alpha_2 \in \mathfrak{u}_2^\star$, the action*

$$\alpha_1 \cdot \alpha_2 = \phi(\alpha_1) \cdot \alpha_2 - \phi(\alpha_2) \cdot \alpha_1 ,$$

together with the given Lie algebra structures on $\mathfrak{u}_1^\star$ and $\mathfrak{u}_2^\star$ give $\mathfrak{u}_1^\star \bowtie \mathfrak{u}_2^\star$, a twisted Lie algebra structure on the sum $\mathfrak{u}_1^\star \oplus \mathfrak{u}_2^\star$ such that $(\mathfrak{u}_1 \oplus \mathfrak{u}_2, \mathfrak{u}_1^\star \bowtie \mathfrak{u}_2^\star)$ is a Lie bialgebra. On the other hand, let $\mathfrak{u}_1$ and $\mathfrak{u}_2$ be two Lie algebras with trivial centers and let $(\mathfrak{u}_1 \oplus \mathfrak{u}_2, \mathfrak{u}_1^\star \oplus \mathfrak{u}_2^\star)$ be an exact Lie bialgebra with r-matrix

$$r = r_1 + r_2 + r_{12} \in \mathfrak{u}_1 \wedge \mathfrak{u}_1 \oplus \mathfrak{u}_2 \wedge \mathfrak{u}_2 \oplus \mathfrak{u}_1 \wedge \mathfrak{u}_2 ,$$

then each r_i is an r-matrix and

$$r_{12} : (\mathfrak{u}_1^\star, \mathfrak{u}_1) \to (\mathfrak{u}_2, \mathfrak{u}_2^\star)$$

is a Lie bialgebra map.

Proof. Let $\phi : (\mathfrak{u}_1^\star, \mathfrak{u}_1) \to (\mathfrak{u}_2, \mathfrak{u}_2^\star)$ be an arrow and let $\alpha \in \mathfrak{u}_1^\star$, $\beta_1, \beta_2 \in \mathfrak{u}_2^\star$. Then

$$\alpha \cdot (\beta_1 \cdot \beta_2) = \phi(\alpha) \cdot (\beta_1 \cdot \beta_2) - \phi(\beta_1 \cdot \beta_2) \cdot \alpha$$

$$= (\phi(\alpha)\cdot\beta_1)\cdot\beta_2 - (\beta_1\cdot\phi(\alpha))\cdot\beta_2 - (\phi(\alpha)\cdot\beta_2)\cdot\beta_1 + (\beta_2\cdot\phi(\alpha))\cdot\beta_1$$

$$-\phi(\beta_1)\cdot(\phi(\beta_2)\cdot\alpha) + \phi(\beta_2)\cdot(\phi(\beta_2)\cdot\alpha)$$

$$= (\phi(\alpha)\cdot\beta_1)\cdot\beta_2 - \phi(\phi(\beta_1)\cdot\alpha)\cdot\beta_2 + \phi(\beta_2)\cdot(\phi(\beta_1)\cdot\alpha)$$

$$-(\phi(\alpha)\cdot\beta_2)\cdot\beta_1 + \phi(\phi(\beta_2)\cdot\alpha)\cdot\beta_1 - \phi(\beta_1)\cdot(\phi(\beta_2)\cdot\alpha)$$

$$= (\phi(\alpha)\cdot\beta_1)\cdot\beta_2 - (\phi(\beta_1)\cdot\alpha)\cdot\beta_2 - (\phi(\alpha)\cdot\beta_2)\cdot\beta_1 + (\phi(\beta_2)\cdot\alpha)\cdot\beta_1$$

$$= (\alpha\cdot\beta_1)\cdot\beta_2 - (\alpha\cdot\beta_2)\cdot\beta_1\,.$$

A similar equality holds for $\alpha \in \mathfrak{u}_2^\star$ and $\beta_1, \beta_2 \in \mathfrak{u}_1^\star$. This shows that $\mathfrak{u}_1^\star \bowtie \mathfrak{u}_2^\star$ is indeed a Lie algebra. To verify that we do have a Lie bialgebra, It suffices to check that the Drinfel'd identity holds. This is a straight forward computation once we note that for $x \in \mathfrak{u}_1$ and $\alpha \in \mathfrak{u}_2^\star$, $\alpha \cdot x = \phi(\alpha)\cdot x$. For the converse, it is clear that $(\mathfrak{u}_1, \mathfrak{u}_1^\star)$ and $(\mathfrak{u}_2, \mathfrak{u}_2^\star)$ are Lie bialgebras, and a computation similar to the above shows that the map

$$Ad \circ r_{12} : \mathfrak{u}_1^\star \to \mathfrak{u}_2 \to gl(\mathfrak{u}_2)$$

is a Lie algebra map. Thus if $\mathfrak{u}_2$ is centerless $r_{12} : \mathfrak{u}_1^\star \to \mathfrak{u}_2$ is a Lie algebra map. Similarly for $r_{12} : \mathfrak{u}_2^\star \to \mathfrak{u}_1$. This gives the theorem. □

3. Compact Semisimple Lie Bialgebras

Now let $\mathfrak{u}$ be a compact simple Lie algebra with complexification $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, Δ^+ i times a fixed set of positive roots, and $e_\alpha \in \mathfrak{g}_\alpha$ a choice of standard elements, see [6]. For v an element of an arbitrary $\mathfrak{u}$ module we define the isotropy of v to be the set $\{x \in \mathfrak{u} | x \cdot v = 0\}$. We denote by $x \mapsto \tilde{x}$ the Killing map for $\mathfrak{u}$ i.e.

$$< \tilde{x}, y >= -\mathrm{trace}(ad_x ad_y)\,.$$

Let $\mathfrak{z} = \mathfrak{u} \cap \mathfrak{h}$, $x_\alpha = e_\alpha - e_{-\alpha}$, $y_\alpha = \mathrm{i}(e_\alpha + e_{-\alpha})$, h_α the vector with $2h_\alpha = x_\alpha \cdot y_\alpha$, and $\mathfrak{u}_\alpha = \mathrm{span}\{x_\alpha, y_\alpha\}$. We have chosen this nonstandard definition for Δ^+ so that $\Delta^+ \subset \mathfrak{z}^\star$. We will assume that $\mathfrak{u}$ is given as

$$\mathfrak{u} = \mathfrak{z} \oplus \sum_{\alpha\in\Delta^+} \mathfrak{u}_\alpha\,.$$

Since this decomposition is orthogonal with respect to the Killing form, we can write $\mathfrak{u}^\star$ as

$$\mathfrak{u}^\star = \mathfrak{z}^\star \oplus \sum_{\alpha\in\Delta^+} \mathfrak{u}^\star_\alpha \,,$$

where $\mathfrak{z}^\star = \tilde{\mathfrak{z}}$ and $\mathfrak{u}^\star_\alpha = \tilde{\mathfrak{u}}_\alpha$. As shown in [14], [3], and [2] any r-matrix with isotropy $\mathfrak{z}$ has the form $r = r_1 + r_0$, where r_1 is an element of $\mathfrak{z} \wedge \mathfrak{z}$ and r_0 has the form

$$r_0 = c \sum_{\alpha\in\Delta^+} x_\alpha \wedge y_\alpha$$

and we will call r nondegenerate if $c \neq 0$ in which case we will assume that $c = 1$. The adjoint and coadjoint actions of $\mathfrak{u}^\star$ are given as

$$\xi \cdot \eta = r(\xi) \cdot \eta - r(\eta) \cdot \xi \,,$$

$$\xi \cdot x = -(x \cdot r)(\xi) = r(\xi) \cdot x + r(x \cdot \xi) \,.$$

The fact that $\mathfrak{z}$ and $\mathfrak{z}^\star$ act as derivations on $\mathfrak{u}^\star$ allows us to observe that $\mathfrak{u}^\star_\alpha \cdot \mathfrak{u}^\star_\beta \subset \mathfrak{u}^\star_{\alpha+\beta}$ and $\mathfrak{u}^\star_\alpha \cdot \mathfrak{u}_\beta \subset \mathfrak{u}_{\beta-\alpha}$.

Let $\Phi = \{\alpha_1, ..., \alpha_n\}$ be the simple roots for Δ^+. Then any positive root α can be written uniquely as a sum

$$\alpha = \sum_i n_i \alpha_i$$

with the n_i nonnegative integers. Let $\Psi \subset \Phi$, and define $\Delta^+_0 \subset \Delta^+$ to be the set of all roots which have $n_i = 0$ for every i with $\alpha_i \in \Psi$. Let $\mathfrak{t}$ be a subspace of $\mathfrak{z}$ containing h_α for every $\alpha \in \Delta^+_0$ and with $r_1(\mathfrak{t}^\perp) \perp \mathrm{span}(\Delta^+_0)$. We define $\mathfrak{k}$ a subalgebra of $\mathfrak{u}$ by

$$\mathfrak{k} = \mathfrak{t} \oplus \sum_{\alpha\in\Delta^+_0} \mathfrak{u}_\alpha$$

and note that the perpendicular space

$$\mathfrak{k}^\perp = (\mathfrak{z}^\star \cap \mathfrak{t}^\perp) \oplus \sum_{\alpha\notin\Delta^+_0} \mathfrak{u}^\star_\alpha$$

is an ideal of $\mathfrak{u}^\star$. Thus $\mathfrak{k}$ is a Poisson subalgebra of $\mathfrak{u}$ and we claim that every Poisson subalgebra is of this form. To see this, we will need the following lemma.

Lemma 4. *Let $V = \bigoplus_i V_i$ be a direct sum of vector spaces with $\mathcal{A} \subset \mathrm{End}(V)$ a subalgebra such that for all $a \in \mathcal{A}$, $aV_i \subset V_i$ and $a|_{V_i}$ is either 0 or invertible. If $\mathcal{A}$ satisfies the property that for any $i \neq j$ there exists an $a \in \mathcal{A}$ such that $aV_i = 0$ and $aV_j = V_j$ then $\mathcal{A}$ contains the projections $\pi_i : V \to V_i$.*

Proof. Fix an i and for each $j \neq i$ pick $a_j \in \mathcal{A}$ so that $a_j V_j = 0$ and $a_j V_i = V_i$. Let $a = \Pi_{j \neq i} a_j$. Then $\ker(a) = \bigoplus_{j \neq i} V_j$ and $\hat{a} \in \mathrm{End}(V_i)$, the restricted of a to V_i is invertible. We can find a polynomial $q(t)$ such that $q(\hat{a})$ is the inverse of $\hat{a}$ which will give $\pi_i = aq(a) \in \mathcal{A}$ as desired. □

Let $\mathcal{A}$ be the subalgebra of $\mathrm{End}(\mathfrak{u}^\star)$ generated by the identity and ad_ξ for $\xi \in \mathfrak{z}^\star$. Then $\mathcal{A}$ preserves every ideal of $\mathfrak{u}^\star$ and we claim that $\mathcal{A}$ contains the projections onto the factors of the decomposition

$$\mathfrak{u}^\star = \mathfrak{z}^\star \oplus \sum_{\alpha \in \Delta^+} \mathfrak{u}^\star_\alpha .$$

To see this we verify the hypothesis of our lemma. For $\xi \in \mathfrak{z}^\star$, $\xi \cdot \mathfrak{z}^\star = 0$, $\xi \cdot \mathfrak{u}^\star_\alpha \subset \mathfrak{u}^\star_\alpha$ and in the basis $\{\tilde{x}_\alpha, \tilde{y}_\alpha\}$, ad_ξ has the form

$$ad_\xi|_{\mathfrak{u}^\star_\alpha} = \begin{pmatrix} <\xi, h_\alpha> & <\xi, r(\alpha)> \\ -<\xi, r(\alpha)> & <\xi, h_\alpha> \end{pmatrix},$$

with determinant $<\xi, h_\alpha>^2 + <\xi, r(\alpha)>^2$. We see that $\mathcal{A}$ restricted to $\mathfrak{u}^\star_\alpha$ is isomorphic to $\mathbb{R}$ if $r(\alpha) = 0$ or $\mathbb{C}$ if $r(\alpha) \neq 0$. Now fix $\alpha \neq \beta$. We wish to find an $a \in \mathcal{A}$ with $a(\mathfrak{u}^\star_\alpha) = \mathfrak{u}^\star_\alpha$ and $a(\mathfrak{u}^\star_\beta) = 0$. If α and β have the same length then $<\beta, h_\beta>><\alpha, h_\beta>$. If not, we can assume that β is the longer root and we still get $<\beta, h_\beta>><\alpha, h_\beta>$. We can take a to be $a =<\beta, h_\beta> 1 - ad_\beta$. Applying the lemma we see that $\mathcal{A}$ contains the desired projections. Let I be an ideal of $\mathfrak{u}^\star$. If we let $I_\alpha = I \cap \mathfrak{u}^\star_\alpha$ and $I_0 = I \cap \mathfrak{z}^\star$ then

$$I = I_0 \oplus \sum I_\alpha$$

and if $I_\alpha \neq 0$ then $I_{\alpha+\beta} = \mathfrak{u}^\star_{\alpha+\beta}$ for all $\beta \in \Delta^+$ with $\alpha + \beta \in \Delta^+$. Moreover, if $\xi \in I_0$ and $<\xi, h_\alpha> \neq 0$ then $I_\alpha = \mathfrak{u}^\star_\alpha$. It is possible to have some $I_\alpha \neq 0$ and $I_\alpha \neq \mathfrak{u}^\star_\alpha$. However, if $I^\perp$ is a subalgebra of $\mathfrak{u}$ this cannot happen. For if $I_\alpha \neq \mathfrak{u}^\star_\alpha$ then $h_\alpha \in \mathfrak{z} \cap I_0^\perp \subset I^\perp$ and $I^\perp$ contains the subalgebra generated by h_α and $\mathfrak{u}_\alpha \cap I_\alpha^\perp$. But this subalgebra contains $\mathfrak{u}_\alpha$ so that $I_\alpha = 0$. Thus, for any ideal I of $\mathfrak{u}^\star$ such that $I^\perp$ is a subalgebra of $\mathfrak{u}$, there is an $\Omega \subset \Delta^+$ such that

$$I = I_0 + \sum_{\alpha \in \Omega} \mathfrak{u}^\star_\alpha ,$$

and $\alpha+\beta \in \Omega$ whenever $\alpha \in \Omega$ and $\alpha+\beta \in \Delta^+$. Now suppose $\alpha = \sum n_i\alpha_i \in \Omega$ is minimal, i.e. $\alpha \neq \alpha' + \beta$ for any $\alpha' \in \Omega$ and suppose j is such that $\alpha' = \alpha - \alpha_j \in \Delta^+$. Then $\mathfrak{u}_{\alpha'}, \mathfrak{u}_{\alpha_j} \subset I^\perp$ and

$$x_{\alpha'} \cdot x_{\alpha_j} = N_1 x_\alpha + N_2 x_{\alpha'-\alpha_j}\,,$$

where N_1 is a nonzero constant and N_2 is nonzero if $\alpha' - \alpha_j \in \Delta^+$, see [6]. This implies that $x_{\alpha'} \cdot x_{\alpha_j} \notin I^\perp$ contradicting the fact that $I^\perp$ is a subalgebra. Thus we conclude such a j does not exist and hence, that α must be simple. To determine I_0 we note that $I \cdot I^\perp = 0$. For $\xi \in I_0$ and $\alpha \in \Delta_0^+$ we have

$$\xi \cdot x_\alpha = r(\xi) \cdot x_\alpha + r(x_\alpha \cdot \xi) = 0$$

and in fact both terms are 0 since they are perpendicular in the Killing norm. This gives $r(\xi) \perp \mathrm{span}(\Delta_0^+)$. Since we have $\xi \in I_0$ we already have $< \xi, h_\alpha >= 0$ for all $\alpha \in \Delta_0^+$. We restate our result.

Theorem 5. *Every Poisson subalgebra of $\mathfrak{u}$ nondegenerate has the form*

$$\mathfrak{k} = \mathfrak{t} \oplus \sum_{\alpha \in \Delta_0^+} \mathfrak{u}_\alpha\,,$$

where $\mathfrak{u}$, $\mathfrak{t}$ and Δ_0^+ are given as above.

We now assume that $\mathfrak{u}$ is compact semisimple. Then $\mathfrak{u}$ can be written as a direct sum of compact simple

$$\mathfrak{u} = \mathfrak{u}_1 \oplus \ldots \oplus \mathfrak{u}_l$$

and the dual space is given as

$$\mathfrak{u}^\star = \mathfrak{u}_1^\star \oplus \ldots \oplus \mathfrak{u}_l^\star\,.$$

Since $(\mathfrak{u}, \mathfrak{u}^\star)$ is exact, we know from Theorem 3 that each $(\mathfrak{u}_i, \mathfrak{u}_i^\star)$ is an exact Lie bialgebra and we call $\mathfrak{u}$ nondegenerate if each $\mathfrak{u}_i$ is nondegenerate. The mixing matrices are given by

Proposition 6. *Let $(\mathfrak{u}_1, \mathfrak{u}_1^\star), (\mathfrak{u}_2, \mathfrak{u}_2^\star)$ be two Lie bialgebras with $\mathfrak{u}_i$ compact simple and $c_i \neq 0$. Then the set of Lie bialgebra maps*

$$\phi : (\mathfrak{u}_1^\star, \mathfrak{u}_1) \to (\mathfrak{u}_2, \mathfrak{u}_2^\star)$$

is the space $\mathfrak{z}_1 \otimes \mathfrak{z}_2$.

To prove this, we note that $\mathfrak{u}_1^\star$ is solvable which gives $\phi(\mathfrak{u}_1^\star)$ solvable and hence abelian since $\mathfrak{u}_2$ is compact. Thus $\mathfrak{z}_1^\perp$, the derived algebra of $\mathfrak{u}_1^\star$, is contained in the kernel of ϕ. This gives $\phi(\mathfrak{u}_2^\star) \subset \mathfrak{z}_1$. Similarly $\phi(\mathfrak{u}_1^\star) \subset \mathfrak{z}_2$. It is also now clear that any element of $\mathfrak{z}_1 \otimes \mathfrak{z}_2$ gives a bialgebra map ϕ.

We can now find the sub-bialgebras for $\mathfrak{u}$ nondegenerate by noting that the r-matrix is given as

$$r = \sum_i r_i + \sum_{i \neq j} r_{ij}$$

and the isotropy of r as

$$\mathfrak{z} = \sum_i \mathfrak{z}_i \,,$$

with $r_{ij} \in \mathfrak{z}_i \wedge \mathfrak{z}_j$. As before we let $\mathcal{A}$ be the subalgebra generated by ad_ξ for $\xi \in \mathfrak{z}^\star$ and it is clear that $\mathcal{A}$ will again contain the projections onto each $\mathfrak{u}_{i,\alpha}^\star$. This implies that any ideal I of $\mathfrak{u}^\star$ will be a direct sum $I = I_1 + \ldots + I_l$ where each I_i is an ideal of $\mathfrak{u}_i$.

Theorem 7. *A Poisson subalgebra $\mathfrak{k}$ of $\mathfrak{u} = \bigoplus_i \mathfrak{u}_i$ nondegenerate is a direct sum of Poisson subalgebras $\mathfrak{k}_i$ of $\mathfrak{u}_i$ such that $r(\mathfrak{k}^\perp)$ is perpendicular to the roots of $\mathfrak{k}$.*

4. Homogeneous Spaces

Now let H and K be closed connected Poisson Lie subgroups of U, and $g \in U$. Suppose that KgH is a Poisson submanifold of U. Letting Π_m denote the Poisson tensor at a point, we see that for all $k \in K$ and $h \in H$

$$\Pi_{kgh} \in \wedge^2 T_{kgh} KgH \,.$$

Left translating to the identity gives

$$(kgh)^{-1} \cdot r - r \in \wedge^2((kgh)^{-1} \cdot \mathfrak{k} + \mathfrak{h}) \,, \quad \forall k \in K \,, \quad \forall h \in H \,.$$

Since K and H are Poisson Lie subgroups we have

$$k \cdot r - r \in \mathfrak{k} \wedge \mathfrak{k} \,, \quad h \cdot r - r \in \mathfrak{h} \wedge \mathfrak{h} \,.$$

So we see that KgH is a Poisson submanifold of U if and only if

$$r - g \cdot r \in \wedge^2(\mathfrak{k} + g \cdot \mathfrak{h}) \,.$$

Thus we only need to check the Poisson tensor at the one point g. Since $M \subset U$ is a Poisson submanifold if and only if M^{-1} is a Poisson submanifold we also see that

Proposition 8. *$K(gH)$ is a Poisson submanifold of U/H if and only if $H(g^{-1}K)$ is a Poisson submanifold of U/K.*

Now let Π^0 denote the Poisson bracket on U, and suppose that Π is a homogeneous Poisson structure on U/H, where H is a Poisson Lie subgroup. Letting $\sigma : U \to U/H$ be the projection, we have a nonhomogeneous equation for Π

$$\Pi_{gH} = \sigma^*(\Pi^0_g) + L^*_g(\Pi_{eH}) .$$

We can find an $\mathfrak{h}$ invariant complement V for $\mathfrak{h}$ in $\mathfrak{u}$ and note that $s = \Pi_{eH} \in V \wedge V$ is $\mathfrak{h}$ invariant. We define a bivectorfield Π on U by $\Pi_g = \Pi^0_g + L^*_g(s)$. Then we have $\sigma^*([\Pi, \Pi]) = 0$, where $[\Pi, \Pi]$ is the Schouten bracket of Π. Recalling that left invariant vector fields commute with right invariant vector fields, we can pull the resulting equations back to the identity to obtain

Proposition 9. *The Poisson homogeneous structures on U/H are given by H invariant elements $s \in V \wedge V$ such that*

$$2r \cdot s + s \cdot s \in \mathfrak{h} \wedge \mathfrak{u} \wedge \mathfrak{u} ,$$

where the action is the Schouten bracket on $\wedge^*\mathfrak{u}$.

If H is maximal then U/H will be a hermitian symmetric space. The H fixed subspace of $V \wedge V$ will be 1 dimensional and $V \cdot V \subset \mathfrak{h}$ so we see that the space of homogeneous Poisson structures on U/H is 1-dimensional.

At the other extreme, for $H = 0$, we see that the Poisson homogeneous structures on U itself are given by $s \in \mathfrak{u} \wedge \mathfrak{u}$ so that

$$2r \cdot s - s \cdot s = 0 .$$

Noting that $s + r$ will define another r-matrix on U allows us to identify the homogeneous Poisson structures on U with the space $U/Z \times \wedge^2 \mathfrak{z}$.

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