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THE EXACT MACROSCOPIC APPROACH TO EXTENDED THERMODYNAMICS WITH MANY MOMENTS

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Abstract: Extended thermodynamics is a very important theory: for example, it predicts hyperbolicity, finite speeds of propagation waves as well as continuous dependence on initial data. Therefore, it constitutes a significative improvement of ordinary thermodynamics. Here its methods are applied to the case of an arbitrary, but fixed, number of moments. The kinetic approach has already been developed in literature; then, the macroscopic approach is here considered and the constitutive functions appearing in the balance equations are determined up to whatever order with respect to thermodynamical equilibrium. The results of the kinetic approach are a particular case of the present ones.

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1. Introduction

Extended Thermodynamics is a well established and appreciated physical theory (see [8], [10] regarding the first pioneering paper on this subject and an

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exhaustive description of the results which has been subsequently found). More recent results regarding the kinetic approach are described in [1]-[4] and many interesting properties are there obtained and exposed. Originally, ET was studied with a macroscopic approach but the exact solution of the conditions which are present in the theory with many moments is still lacking. This gap is here filled and the constitutive functions appearing in the balance equations are determined up to whatever order with respect to thermodynamical equilibrium.

In the meanwhile, it has been studied with the kinetic approach which is more restrictive of the above one. But it leads to an ill-posed system of equations (see [5], for example). To avoid this difficulty, another approach to the problem has been developed and is denoted by COET (Consistent Order Extended Thermodynamics) (see [9], for example); speaking in simple words, it does not close the system by simply dropping the moments from a given order forwards. However, some problems remain open: In COET of order 2 we find extended thermodynamics with 13 moments, which has no sense in ordinary extended thermodynamics with the kinetic approach; moreover, the non relativistic limit of relativistic extended thermodynamics gives the 14 moments theory instead of the 13 moments one. This aspect has to be clarified.

In the meanwhile, we propose the present alternative approach. It also does not close the system by simply dropping the moments from a given order forwards, but retains also some traces of higher order moments, as suggested by the non-relativistic limit of the relativistic approach to this problem.

The balance equations of this extended thermodynamics with an arbitrary number of moments are

$$\begin{aligned}
&\partial_t F^{i_1 \dots i_n} + \partial_k F^{i_1 \dots i_n k} = S_{i_1 \dots i_n} \quad \text{for } n = 0, \cdots, N, \\
&\partial_t F^{i_1 \cdots i_r e_1 e_1 \cdots e_{\frac{N+M+1-2r}{2}} e_{\frac{N+M+1-2r}{2}}} + \partial_k F^{k i_1 \cdots i_r e_1 e_1 \cdots e_{\frac{N+M+1-2r}{2}} e_{\frac{N+M+1-2r}{2}}} \\
&\qquad = Q^{i_1 \cdots i_r} \quad \text{for } 0 \le r \le M \,,
\end{aligned} \tag{1}$$

where N and M are two given number such that $M < N$, $M + N$ odd, and we call F the tensor $F_{i_1 \dots i_n}$ when $n = 0$.

The entropy principle for this system, by using Liu's Theorem, [7], ensures the existence of the parameters $\lambda_{i_1 \dots i_n}$, $\mu_{i_1 \dots i_r}$ called Lagrange Multipliers, such that

$$dh = \lambda_{i_1 \dots i_n} dF^{i_1 \dots i_n} + \mu_{i_1 \dots i_r} dF^{i_1 \cdots i_r e_1 e_1 \cdots e_{\frac{N+M+1-2r}{2}} e_{\frac{N+M+1-2r}{2}}} \,, \tag{2}$$

$$dh^k = \lambda_{i_1 \dots i_n} dF^{k i_1 \dots i_n} + \mu_{i_1 \dots i_r} dF^{k i_1 \cdots i_r e_1 e_1 \cdots e_{\frac{N+M+1-2r}{2}} e_{\frac{N+M+1-2r}{2}}} \,,$$

where h is the entropy density and h^k its flux. From this result we see that, if we consider only equation $(1)_1$ but with $N + M + 1$ instead of N, the entropy principle becomes

$$dh = \sum_{n=0}^{N+M+1} \lambda_{i_1 \dots i_n} dF^{i_1 \dots i_n}\,, \quad dh^k = \sum_{n=0}^{N+M+1} \lambda_{i_1 \dots i_n} dF^{k i_1 \dots i_n}\,.$$

After that, in these equations we can substitute from

$$\lambda_{i_1 \dots i_n} = \mu_{(i_1 \dots i_{N+M+1-n}} \delta_{i_{N+M+2-n} i_{N+M+3-n}} \cdots \delta_{i_{n-1} i_n)} \quad \text{for} \quad n \geq N+1\,,$$

so obtaining equations (2). This result can be expressed by saying that the model with all the equations (1) can be obtained as a subsystem of that with only equation $(1)_1$, but with $N+M+1$ instead of N. Consequently, in the sequel, we will neglect equation $(1)_2$ for the sake of simplicity.

Now in equation $(1)_1$, the various tensors are symmetric and $F_{i_1 \dots i_N k}$ and $S_{i_1 \dots i_N}$ are supposed to be functions of the previous one, in order to obtain a closed system. In particular F, F_i, F_{ll}, F_{ill} denote the densities of mass, momentum, energy, and energy flux respectively. In this way equations (1) for $n = 0, 1$, and the trace of equations (1) for $n = 2$ are the conservation laws of mass, momentum and energy; obviously to this end it is necessary to assume that S=0, $S_i = 0$ and $S_{ll} = 0$.

Equation (1) can be rewritten in a more compact form using a 4-dimensional notation in a space that we suppose to be Euclidean (nothing will change if the space is pseudo-Euclidean with -+++ signature, so we have chosen the simpler case).

In particular, let us define the symmetric tensors $M^{\alpha_1 \cdots \alpha_{N+1}}$ and $S^{\alpha_1 \cdots \alpha_N}$ as follows:

1. the Greek indexes go from 0 to 3,
2. $M^{i_1 \dots i_n 0 \dots 0} = F_{i_1 \dots i_n}$ for $n = 0, \dots, N+1$,
3. $S^{i_1 \dots i_n 0 \dots 0} = S_{i_1 \dots i_n}$ for $n = 0, \dots, N$.

In that way the balance equations (1) can be simply written as

$$\partial_\alpha M^{\alpha \alpha_1 \dots \alpha_N} = S^{\alpha_1 \dots \alpha_N}, \tag{3}$$

where ∂_α for $\alpha = 0$ means the partial derivative with respect to time.

The entropy principle for this equations, by using Liu's Theorem, [5], ensures the existence of the parameters $L_{\alpha_1 \cdots \alpha_N}$, called Lagrange multipliers, such that

$$dH^\alpha = L_{\alpha_1 \cdots \alpha_N} dM^{\alpha_1 \cdots \alpha_N \alpha}, \quad L_{\alpha_1 \cdots \alpha_N} S^{\alpha_1 \cdots \alpha_N} \geq 0\,, \tag{4}$$

where H^0 is the entropy density and H^i its flux. A brilliant Ruggeri's idea is to define

$$H'^\alpha = -H^\alpha + L_{\alpha_1 \cdots \alpha_N} M^{\alpha_1 \cdots \alpha_N \alpha} \tag{5}$$

and to take the Lagrange multipliers as independent variables. In this way equation $(4)_1$ becomes $dH'^{\alpha} = M^{\alpha_1 \cdots \alpha_N \alpha} dL_{\alpha_1 \cdots \alpha_N}$, from which

$$M^{\alpha_1 \alpha_2 \ldots \alpha_{N+1}} = \frac{\partial H'^{\alpha_{N+1}}}{\partial L_{\alpha_1 \ldots \alpha_N}} \,. \tag{6}$$

In this way the tensors appearing in the balance equations (3) are found as functions of the parameters $L_{\alpha_1 \ldots \alpha_N}$, called also mean field, as soon as H'^{α} is known. Obviously $L_{\alpha_1 \ldots \alpha_N}$ is symmetric. By substituting (6) into equation (3) this takes the symmetric form

$$\frac{\partial^2 H'^{\alpha_{N+1}}}{\partial L_{\beta_1 \cdots \beta_N} \partial L_{\alpha_1 \cdots \alpha_N}} \partial_{\alpha_{N+1}} L_{\beta_1 \cdots \beta_N} = S^{\alpha_1 \cdots \alpha_N} ,$$

so that hyperbolicity is ensured provided that H'^{α} is a convex function of the mean field. By eliminating these parameters from equations (6) we obtain $F_{i_1 \ldots i_{N+1}}$ again, as function of F, F_i,..., $F_{i_1 \ldots i_N}$. If we want a model in which some among equations (1) is present only by means of one of its traces, it can be obtained from the present model with the method of the subsystems [10].

Note that equation (6) for $\alpha_1 \alpha_2 \ldots \alpha_{N+1} = i_1 \ldots i_n i_{n+1} 0 \ldots 0$ and for $\alpha_1 \alpha_2 \ldots \alpha_{N+1} = i_1 \ldots i_n 0 \ldots 0 i_{n+1}$ gives respectively

$$F_{i_1 \ldots i_n i_{n+1}} = \frac{\partial H'^{0}}{L_{i_1 \ldots i_n i_{n+1}}} \,, \qquad F_{i_1 \ldots i_n i_{n+1}} = \frac{\partial H'^{n+1}}{L_{i_1 \ldots i_n}} \tag{7}$$

as in the 3-dimensional notation.

So, to impose equation (6) we have to find the more general expression of H'^{α} such that $M^{\alpha_1 \alpha_2 \ldots \alpha_{N+1}}$ is symmetric. We will refer to this as "the symmetry condition". We will impose also the principle of galilean invariance; this has been exploited in [10], [9]-[14] for a generic system of balance laws. In Section 2 we will apply these results to our system, taking care of converting them in the present 4-dimensional notation, so obtaining further conditions.

In Section 3 these, together with the symmetry condition, will be investigated and their solution will be found up to whatever order with respect to thermodynamical equilibrium, except for two numerable families of constants arising from integration. In Section 4 it will be shown that the results of the kinetic approach are a particular case of the present one so that, as usual, the macroscopic approach is more general than the kinetic one. If we rewrite the present paper with $N-1$ instead of N, we find the model with less moments with the method used in this work, which we call the "direct method". But in [10] it has been shown that a solution (for the model with $N-1$ instead of N) can be obtained also with the "method of subsystems"; this consists in taking the costitutive functions of the model with N as maximum order of moments, and in calculating them in $\lambda_{i_1 \cdots i_N = 0}$, i.e., for zero value of the 3-dimensional Lagrange multiplier with the greatest order. In Section 5 we will see that the solution obtained with the method of subsystems is a particular one of that obtained with the direct method; more explicitly, it can be obtained from the latter by considering equal to zero one of the above mentioned families of constants. At last, conclusions will be drown.

2. The Galilean Relativity Principle

To impose this principle, it is firstly necessary to know how our variables transform under a change of galileanly equivalent frames Σ and Σ'. This problem has been studied by Ruggeri in [9] and we have only to write its results in our 4-dimensional form. This is easily achieved in the kinetic model because the kinetic counterpart of $M^{\alpha_1\cdots\alpha_{N+1}}$ is

$$M^{\alpha_1\cdots\alpha_{N+1}} = \int f c^{\alpha_1}\cdots c^{\alpha_{N+1}} d\underline{c} \tag{8}$$

with $c^0 = 1$, $d\underline{c} = dc^1 dc^2 dc^3$ and f is the distribution function. Consequently, in Σ' we have

$$m^{\alpha_1\cdots\alpha_{N+1}} = \int f c'^{\alpha_1}\cdots c'^{\alpha_{N+1}} dc'$$

and, if v^i is the constant velocity of each point of Σ' with respect to Σ, we have $c^\alpha = c'^\alpha + v^\alpha$, with $v^0 = 0$. It follows that

$$M^{\alpha_1\ldots\alpha_{N+1}} = \sum_{i=0}^{N+1} \binom{N+1}{i} v^{(\alpha_1}\ldots v^{\alpha_i} m^{\alpha_{i+1}\ldots\alpha_{N+1})},$$

or

$$M^{\alpha_1\ldots\alpha_{N+1}} = \sum_{i=0}^{N+1} \binom{N+1}{i} v^{(\alpha_1}\ldots v^{\alpha_i} m^{\alpha_{i+1}\ldots\alpha_{N+1})\beta_1\ldots\beta_i} t_{\beta_1}\ldots t_{\beta_i} \tag{9}$$

with $t_\mu \equiv (1,0,0,0)$ for our previous notation. We obtain the transformation of $M^{\alpha_1\ldots\alpha_N 0}$ (which was the initial independent variable) multiplying equation (9) by $t_{\alpha_{N+1}}$ so finding

$$M^{\alpha_1\ldots\alpha_N 0} = X^{\alpha_1\ldots\alpha_N}_{\beta_1\ldots\beta_N}(\underline{v}) m^{\beta_1\ldots\beta_N 0} \tag{10}$$

with

$$X^{\alpha_1\ldots\alpha_N}_{\beta_1\ldots\beta_N} = \sum_{i=0}^{N} \binom{N}{i} t_{(\beta_1}\ldots t_{\beta_i} v^{(\alpha_1}\ldots v^{\alpha_i} \delta^{\alpha_{i+1}}_{\beta_{i+1}}\ldots\delta^{\alpha_N)}_{\beta_N)}, \tag{11}$$

where we have taken into account of $v^0 = 0$, of the identity $\binom{N+1}{i}\frac{N+1-i}{N+1} = \binom{N}{i}$ and that the term with $i = N+1$ gives a null contribution. Comparison between (10) and (11) with (9) shows that $X^{\alpha_1\ldots\alpha_N}_{\beta_1\ldots\beta_N}$ could be obtained from $X^{\alpha_1\ldots\alpha_{N+1}}_{\beta_1\ldots\beta_{N+1}}$ simply replacing $N+1$ with N. From equation (11) it follows also

$$X^{\alpha_1\ldots\alpha_N\alpha}_{\beta_1\ldots\beta_N\beta} = X^{\alpha_1\ldots\alpha_N}_{(\beta_1\ldots\beta_N}t_{\beta)}v^\alpha + X^{\alpha_1\ldots\alpha_N}_{(\beta_1\ldots\beta_N}\delta^\alpha_{\beta)}. \tag{12}$$

Similarly, H^α transforms according to the rule

$$H^\alpha = h^0 v^\alpha + h^\alpha, \tag{13}$$

of which $H^0 = h^0$ is a component.

Equations (9) and (13) have been obtained with the kinetic model only for the sake of simplicity; it is obvious that they hold also in the macroscopic case. The transformation rule of the Lagrange multipliers can be obtained now from $(4)_1$ with $\alpha = 0$, i.e.

$$dh^0 = dH^0 = L_{\alpha_1\cdots\alpha_N} dM^{\alpha_1\cdots\alpha_N 0} = L_{\alpha_1\cdots\alpha_N} X^{\alpha_1\ldots\alpha_N}_{\beta_1\ldots\beta_N} dm^{\beta_1\cdots\beta_N 0}\,,$$

where (13) and (10) have been used. In other words we have

$$dh^0 = l_{\beta_1\cdots\beta_N} dm^{\beta_1\cdots\beta_N 0} \tag{14}$$

with

$$l_{\alpha_1\ldots\alpha_N} = X^{\beta_1\ldots\beta_N}_{\alpha_1\ldots\alpha_N} L_{\beta_1\ldots\beta_N}\,,$$

i.e.

$$l_{\alpha_1\ldots\alpha_N} = \sum_{i=0}^{N} \binom{N}{i} t_{(\alpha_1}\ldots t_{\alpha_i} v^{\beta_1}\ldots v^{\beta_i} L_{\alpha_{i+1}\ldots\alpha_N)\beta_1\ldots\beta_i}. \tag{15}$$

A consequence of this result can be obtained from (5) with $\alpha = 0$ and written in the frame Σ', i.e., $h'^0 = -h^0 + l_{\alpha_1\cdots\alpha_N} m^{\alpha_1\cdots\alpha_N 0}$; it follows $dh'^0 = m^{\alpha_1\cdots\alpha_N 0} dl_{\alpha_1\cdots\alpha_N}$ from which

$$m^{\alpha_1\cdots\alpha_N 0} = \frac{\partial h'^0}{\partial l_{\alpha_1\cdots\alpha_N}}\,, \tag{16}$$

as in Σ. Moreover, from (5), (13), (9), (12), (15) and again (5) and (13) it follows

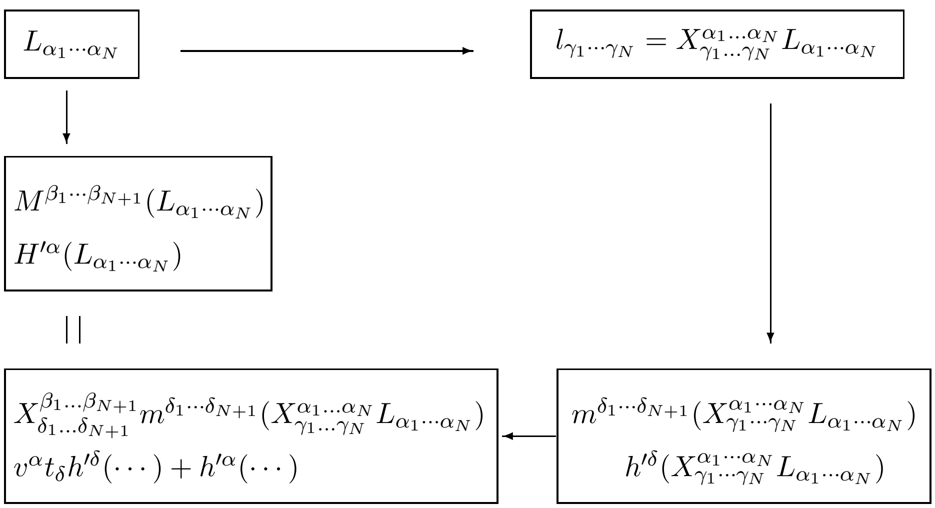
$$\begin{aligned} H'^\alpha &= -h^0 v^\alpha - h^\alpha + L_{\alpha_1\cdots\alpha_N} X^{\alpha_1\ldots\alpha_N\alpha}_{\beta_1\ldots\beta_N\beta} m^{\beta_1\cdots\beta_N\beta} \\ &= -h^0 v^\alpha - h^\alpha + l_{\beta_1\cdots\beta_N} m^{\beta_1\cdots\beta_N 0} v^\alpha + l_{\beta_1\cdots\beta_N} m^{\beta_1\cdots\beta_N\alpha}\,, \end{aligned}$$

i.e.

$$H'^\alpha = h'^0 v^\alpha + h'^\alpha \tag{17}$$

which is similar to (13).

We are now ready to consider the Galilean relativity principle. It imposes that the following diagram is commutative



In other words, we must have

$$H'^{\alpha}(L_{\alpha_1\cdots\alpha_N}) = v^{\alpha} t_{\delta} h'^{\delta}(X^{\alpha_1\ldots\alpha_N}_{\gamma_1\ldots\gamma_N}(\underline{v}) L_{\alpha_1\cdots\alpha_N}) + h'^{\alpha}(X^{\alpha_1\ldots\alpha_N}_{\gamma_1\ldots\gamma_N}(\underline{v}) L_{\alpha_1\cdots\alpha_N}),$$

$$M^{\beta_1\cdots\beta_{N+1}}(L_{\alpha_1\cdots\alpha_N}) = X^{\beta_1\ldots\beta_{N+1}}_{\delta_1\ldots\delta_{N+1}}(\underline{v}) m^{\delta_1\cdots\delta_{N+1}}(X^{\alpha_1\ldots\alpha_N}_{\gamma_1\ldots\gamma_N}(\underline{v}) L_{\alpha_1\cdots\alpha_N}) . \tag{18}$$

Equation $(18)_2$, by using equations (12) and (16) becomes

$$\begin{aligned} M^{\beta_1\cdots\beta_N\alpha} &= X^{\beta_1\ldots\beta_N}_{\delta_1\ldots\delta_N} m^{\delta_1\cdots\delta_N 0} v^{\alpha} + X^{\beta_1\ldots\beta_N}_{\delta_1\ldots\delta_N} m^{\delta_1\cdots\delta_N\alpha} \\ &= X^{\beta_1\ldots\beta_N}_{\delta_1\ldots\delta_N} \frac{\partial h'^0}{\partial l_{\delta_1\cdots\delta_N}} v^{\alpha} + X^{\beta_1\ldots\beta_N}_{\delta_1\ldots\delta_N} m^{\delta_1\cdots\delta_n\alpha} . \end{aligned}$$

Now the derivative of $(18)_1$ with respect to $L^{\beta_1\cdots\beta_N}$ is

$$\frac{\partial H'^{\alpha}}{\partial L^{\beta_1\cdots\beta_N}} = M^{\beta_1\cdots\beta_N\alpha} = v^{\alpha}\frac{\partial h'^0}{\partial l_{\gamma_1\cdots\gamma_N}} X^{\beta_1\cdots\beta_N}_{\gamma_1\cdots\gamma_N}(\underline{v}) + \frac{\partial h'^{\alpha}}{\partial l_{\gamma_1\cdots\gamma_N}} X^{\beta_1\cdots\beta_N}_{\gamma_1\cdots\gamma_N}(\underline{v}).$$

It follows that equation $(18)_2$, holds iff

$$\frac{\partial h'^{\alpha}}{\partial l_{\gamma_1\cdots\gamma_N}} X^{\beta_1\cdots\beta_N}_{\gamma_1\cdots\gamma_N} = m^{\delta_1\cdots\delta_N\alpha} X^{\beta_1\cdots\beta_N}_{\gamma_1\cdots\gamma_N} ,$$

i.e.

$$m^{\gamma_1\cdots\gamma_N\alpha} = \frac{\partial h'^{\alpha}}{\partial l_{\gamma_1\cdots\gamma_N}} \tag{19}$$

which is the counterpart of equation (6) in the frame Σ'.

It remains to impose equation $(18)_1$. Now it becomes an identity when calculated in $\underline{v} = 0$ (see equations (17) and (11) to this regard) so that it holds iff its derivative with respect to v_j is satisfied, i.e.,

$$0 = \frac{\partial h'^0}{\partial l_{\gamma_1\cdots\gamma_N}} \frac{\partial l_{\gamma_1\cdots\gamma_N}}{\partial v_j} \qquad \text{for } \alpha = 0,$$

$$0 = h'^0 \delta_j^\alpha + \frac{\partial h'^\alpha}{\partial l_{\gamma_1 \cdots \gamma_N}} \frac{\partial l_{\gamma_1 \cdots \gamma_N}}{\partial v_j} \quad \text{for } \alpha = 0, 1, 2, 3. \tag{20}$$

The second of this has been obtained by taking into account also equation $(20)_1$; on the other hand, this is included in $(20)_2$ with $\alpha = 0$. Equation $(20)_2$, by using equation $(15)_2$ now becomes

$$h'^0 \delta_j^\alpha + \frac{\partial h'^\alpha}{\partial l_{\alpha_1 \dots \alpha_N}} \sum_{i=1}^{N} \begin{pmatrix} N \\ i \end{pmatrix} i \cdot t_{(\alpha_1} \dots t_{\alpha_i} v^{\beta_1} \dots v^{\beta_{i-1}} L_{\alpha_{i+1} \dots \alpha_N) \beta_1 \dots \beta_{i-1} j} = 0.$$

We remove the symmetrization with respect to $\alpha_1 \cdots \alpha_N$ which is not necessary because of the contraction with $\frac{\partial h'^\alpha}{\partial l_{\alpha_1 \dots \alpha_N}}$ which is symmetric; for the same reason we can exchange α_i and α_N and then reintroduce the symmetrization with respect to $\alpha_1 \dots \alpha_{N-1}$, obtaining so

$$h'^0 \delta_j^\alpha + t_{\alpha_N} \frac{\partial h'^\alpha}{\partial l_{\alpha_1 \dots \alpha_N}} \sum_{i=1}^{N} \begin{pmatrix} N \\ i \end{pmatrix} i \cdot t_{(\alpha_1} \dots t_{\alpha_{i-1}} v^{\beta_1} \dots v^{\beta_{i-1}} L_{\alpha_i \dots \alpha_{N-1}) \beta_1 \dots \beta_{i-1} j} = 0.$$

We replace i with $i+1$ and we have

$$h'^0 \delta_j^\alpha + \frac{\partial h'^\alpha}{\partial l_{\alpha_1 \dots \alpha_N}} t_{\alpha_N} \cdot \sum_{i=0}^{N-1} \begin{pmatrix} N \\ i+1 \end{pmatrix} (i+1) \cdot t_{(\alpha_1} \dots t_{\alpha_i} v^{\beta_1} \dots v^{\beta_i} L_{\alpha_{i+1} \dots \alpha_{N-1}) \beta_1 \dots \beta_i j} = 0$$

or

$$h'^0 \delta_j^\alpha + \tfrac{\partial h'^\alpha}{\partial l_{\alpha_1 \dots \alpha_N}} t_{\alpha_N} \cdot \sum_{i=0}^{N-1} N \begin{pmatrix} N-1 \\ i \end{pmatrix} \cdot t_{(\alpha_1} \dots t_{\alpha_i} v^{\beta_1} \dots v^{\beta_i} L_{\alpha_{i+1} \dots \alpha_{N-1}) \beta_1 \dots \beta_i j} = 0. \tag{21}$$

But, by using equation (15) we have

$$\begin{aligned} l_{\alpha \cdots \alpha_{N-1} j} &= \sum_{i=1}^{N} \frac{i}{N} \binom{N}{i} t_j t_{(\alpha_1} \cdots t_{\alpha_{i-1}} v^{\beta_1} \cdots v^{\beta_i} L_{\alpha_i \cdots \alpha_{N-1}) \beta_1 \cdots \beta_i} \\ &+ \sum_{i=0}^{N-1} \frac{N-i}{N} \binom{N}{i} t_{(\alpha_1} \cdots t_{\alpha_i} v^{\beta_1} \cdots v^{\beta_i} L_{\alpha_{i+1} \cdots \alpha_{N-1}) j \beta_1 \cdots \beta_i} \\ &= \sum_{i=0}^{N-1} \binom{N-1}{i} t_{(\alpha_1} \cdots t_{\alpha_i} v^{\beta_1} \cdots v^{\beta_i} L_{\alpha_{i+1} \cdots \alpha_{N-1}) j \beta_1 \cdots \beta_i} \end{aligned} \tag{22}$$

because $t_j = 0$. This allows to rewrite equation (21) as

$$0 = h'^{\mu} t_{\mu} \delta^{\alpha}_{j} + N \frac{\partial h'^{\alpha}}{\partial l_{\alpha_1 \dots \alpha_N}} t_{\alpha_N} l_{\alpha_1 \dots \alpha_{N-1} j}. \tag{23}$$

Until now we have obtained that the entropy principle jointly with the Galilean relativity principle amounts to say that:

1. equations (6) are invariant under changes of Galileanly equivalent observers (see equation (19)),

2. the further condition (23) must hold.

For the sake of completeness, we note that equation $(18)_1$ might be satisfied also with H^{α} and h^{α}, i.e.

$$H^{\alpha}(L_{\alpha_1 \cdots \alpha_N}) = v^{\alpha} t_{\delta} h^{\delta}(X^{\alpha_1 \cdots \alpha_N}_{\gamma_1 \cdots \gamma_N} L_{\alpha_1 \cdots \alpha_N}) + h'^{\alpha}(X^{\alpha_1 \cdots \alpha_N}_{\gamma_1 \cdots \gamma_N} L_{\alpha_1 \cdots \alpha_N}).$$

But this is a consequence of (18) as it can be seen running over backwards the above passages which allowed to obtain equation (17) from equation (13). Moreover, in [7] and [14] it has be proved that the conditions here obtained are the same of the following approach:

1. consider equations (9), (13) and (17) but with $v_i = \frac{F_i}{F}$, instead of an arbitrary constant v_i; in this way $m^{\alpha_1 \cdots \alpha_{N+1}}$, h^{α} and h'^{α} become the non-convective parts of $M^{\alpha_1 \cdots \alpha_{N+1}}$, H^{α} and H'^{α}, respectively,

2. impose the conditions (19) and (23) but considering $l_{\gamma_1 \cdots \gamma_N}$ independent variables,

3. consider equations (19) with $\alpha = 0$ and $m^{i0 \cdots 0}$ as definition of $l_{\gamma_1 \cdots \gamma_N} = l_{\gamma_1 \cdots \gamma_N}(m^{\alpha_1 \cdots \alpha_N 0})$, and substitute this in the expressions of $m^{i_1 \cdots i_{N+1}}$, h^{α} and h'^{α} so obtaining the closure in terms of the non-convective quantities $m^{\alpha_1 \cdots \alpha_N 0}$.

In any case, we have to impose (19) and (23); in other words we have to find the quadri-vector $h'^{\alpha_{N+1}}$ such that the right hand side of equation (19) is symmetric and for which equation (23) holds; after that equation (19) gives $m^{\beta_1 \cdots \beta_N \beta_{N+1}}$. In this way we will find the required closure satisfying the entropy principle and that of Galilean relativity. This will be done in the next section.

3. Exploitation of the Conditions (19) and (23)

We want now to impose equations (19) and (23) up to whatever order with respect to thermodynamical equilibrium. This is defined as the state where

$$l_{\beta_1 \cdots \beta_N} = \lambda t_{\beta_1} \cdots t_{\beta_N} + \frac{1}{3} \lambda_{ll} h_{(\beta_1 \beta_2} t_{\beta_3} \cdots t_{\beta_N)} \tag{24}$$

holds, with $h_{\beta\gamma} = \delta_{\beta\gamma} - t_\beta t_\gamma = \mathrm{diag}\,(0,1,1,1)$,

$$\lambda = t^{\beta_1} \cdots t^{\beta_N} l_{\beta_1 \cdots \beta_N} \qquad \lambda_{ll} = \binom{N}{2} h^{\beta_1\beta_2} t^{\beta_3} \cdots t^{\beta_N} l_{\beta_1 \cdots \beta_N}. \tag{25}$$

We can consider the Taylor expansion for h'^α

$$h'^\alpha = \sum_{k=0}^{\infty} \frac{1}{k!} A^{\alpha B_1 \cdots B_k} \tilde{l}_{B_1} \cdots \tilde{l}_{B_k}, \tag{26}$$

with

$$\tilde{l}_{\beta_1 \cdots \beta_N} = l_{\beta_1 \cdots \beta_N} - \lambda t_{\beta_1} \cdots t_{\beta_N} - \frac{1}{3} \lambda_{ll} h_{(\beta_1\beta_2} t_{\beta_3} \cdots t_{\beta_N)}, \tag{27}$$

$$A^{\alpha B_1 \cdots B_k} = \left(\frac{\partial^k h'^\alpha}{\partial l_{B_1} \cdots \partial l_{B_k}} \right)_{eq}, \tag{28}$$

where the multi-index notation $B_i = \beta_i^1 \cdots \beta_i^N$ has been used. Thanks to equation (19) we can exchange α with each other index taken from those included in any B_i. So it is possible to exchange every index with all the others, i.e., $A^{\alpha B_1 \cdots B_k}$ is symmetric with respect to any couple of indexes. We note that there are 2 compatibility conditions between equations (26) and (28); they can be obtained as follows: let us consider the tensor $\frac{\partial^k h'^\alpha}{\partial l_{B_1} \cdots l_{B_k}}$ as function of $\tilde{l}_B, \lambda, \lambda_{ll}$, and take the derivatives with respect to $l_{\beta_1 \cdots \beta_N}$, calculating the result at equilibrium; we find

$$\begin{aligned} A^{\alpha B_1 \cdots B_k \beta_1 \cdots \beta_N} &= \left(\frac{\partial^{k+1} h'^\alpha}{\partial l_{B_1} \cdots \partial l_{B_k} \partial \tilde{l}_{\gamma_1 \cdots \gamma_N}} \right)_{eq} \frac{\partial \tilde{l}_{\gamma_1 \cdots \gamma_N}}{\partial l_{\beta_1 \cdots \beta_N}} \\ &+ \left(\frac{\partial^{k+1} h'^\alpha}{\partial l_{B_1} \cdots \partial l_{B_k} \partial \lambda} \right)_{eq} \frac{\partial \lambda}{\partial l_{\beta_1 \cdots \beta_N}} + \left(\frac{\partial^{k+1} h'^\alpha}{\partial l_{B_1} \cdots \partial l_{B_k} \partial \lambda_{ll}} \right)_{eq} \frac{\partial \lambda_{ll}}{\partial l_{\beta_1 \cdots \beta_N}}. \end{aligned}$$

If we multiply this by $t_{\beta_1} \cdots t_{\beta_N}$ and by $h_{\beta_1\beta_2} t_{\beta_3} \cdots t_{\beta_N}$ we find, respectively

$$\begin{cases} A^{\alpha B_1 \cdots B_k \beta_1 \cdots \beta_N} t_{\beta_1} \cdots t_{\beta_N} = \frac{\partial}{\partial \lambda} A^{\alpha B_1 \cdots B_k}, \\ A^{\alpha B_1 \cdots B_k \beta_1 \cdots \beta_N} h_{\beta_1\beta_2} t_{\beta_3} \cdots t_{\beta_N} = 3 \frac{\partial}{\partial \lambda_{ll}} A^{\alpha B_1 \cdots B_k}, \end{cases} \tag{29}$$

where we have taken into account that from equations (25) and (27) it follows

$$\begin{aligned} \frac{\partial \lambda}{\partial l_{\beta_1 \cdots \beta_N}} &= t^{\beta_1} \cdots t^{\beta_N}, \quad \frac{\partial \lambda_{ll}}{\partial l_{\beta_1 \cdots \beta_N}} = \binom{N}{2} h^{(\beta_1\beta_2} t^{\beta_3} \cdots t^{\beta_N)}, \\ \frac{\partial \tilde{l}_{\gamma_1 \cdots \gamma_N}}{\partial l_{\beta_1 \cdots \beta_N}} &= g_{\gamma_1}^{(\beta_1} \cdots g_{\gamma_N}^{\beta_N)} - t^{\beta_1} \cdots t^{\beta_N} t_{\gamma_1} \cdots t_{\gamma_N} \\ &- \frac{1}{3} \binom{N}{2} h^{(\beta_1\beta_2} t^{\beta_3} \cdots t^{\beta_N)} h_{(\gamma_1\gamma_2} t_{\gamma_3} \cdots t_{\gamma_N)}, \end{aligned}$$

from which

$$\frac{\partial\lambda}{\partial l_{\beta_1\cdots\beta_N}} t_{\beta_1}\cdots t_{\beta_N} = 1\,, \quad \frac{\partial\lambda}{\partial l_{\beta_1\cdots\beta_N}} h_{\beta_1\beta_2} t_{\beta_3}\cdots t_{\beta_N} = 0\,,$$
$$\frac{\partial\lambda_{ll}}{\partial l_{\beta_1\cdots\beta_N}} t_{\beta_1}\cdots t_{\beta_N} = 0\,, \quad \frac{\partial\lambda_{ll}}{\partial l_{\beta_1\cdots\beta_N}} h_{\beta_1\beta_2} t_{\beta_3}\cdots t_{\beta_N} = 3\,,$$
$$\frac{\partial\tilde{l}_{\gamma_1\cdots\gamma_N}}{\partial l_{\beta_1\cdots\beta_N}} t_{\beta_1}\cdots t_{\beta_N} = 0\,, \quad \frac{\partial\tilde{l}_{\gamma_1\cdots\gamma_N}}{\partial l_{\beta_1\cdots\beta_N}} h_{\beta_1\beta_2} t_{\beta_3}\cdots t_{\beta_N} = 0.$$

It will be useful in the sequel to note a consequence of the condition (29). By using also equation (26) we have

$$\begin{aligned}
\frac{\partial h'^{\alpha}}{\partial l_{\beta_1\cdots\beta_N}} &= \sum_{k=1}^{\infty}\frac{1}{(k-1)!}A^{\alpha B_1\cdots B_{k-1}\gamma_1\cdots\gamma_N}\tilde{l}_{B_1}\cdots\tilde{l}_{B_{k-1}} \\
&\quad \Big(g^{(\beta_1}_{\gamma_1}\cdots g^{\beta_N)}_{\gamma_N} - t^{\beta_1}\cdots t^{\beta_N} t_{\gamma_1}\cdots t_{\gamma_N} \\
&- \frac{1}{3}\binom{N}{2} h^{(\beta_1\beta_2}t^{\beta_3}\cdots t^{\beta_N)} h_{(\gamma_1\gamma_2}t_{\gamma_3}\cdots t_{\gamma_N)}\Big) \\
&+ \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\partial}{\partial\lambda}A^{\alpha B_1\cdots B_k}\right)\tilde{l}_{B_1}\cdots\tilde{l}_{B_k} t^{\beta_1}\cdots t^{\beta_N} \\
&+ \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\partial}{\partial\lambda_{ll}}A^{\alpha B_1\cdots B_k}\right)\tilde{l}_{B_1}\cdots\tilde{l}_{B_k} h^{(\beta_1\beta_2}t^{\beta_3}\cdots t^{\beta_N)}\binom{N}{2} \\
&= \sum_{k=1}^{\infty}\frac{1}{(k-1)!}A^{\alpha B_1\cdots B_{k-1}\beta_1\cdots\beta_N}\tilde{l}_{B_1}\cdots\tilde{l}_{B_{k-1}},
\end{aligned}$$

where conditions (29) have been used in the last passage. So we have proved that derivation of equation (26) with respect to $l_{\beta_1\cdots\beta_N}$ is equivalent to its derivation with respect to $\tilde{l}_{\beta_1\cdots\beta_N}$, but considering independent the components of this tensor, except for the symmetry. Proceeding with the subsequent derivatives and calculating the result at equilibrium, we find equation (28). In other words we can forget equation (28) but we have to retain equations (29). We have then to transform equations (19), (23) and (29) in conditions for the tensor $A^{\alpha B_1\cdots B_k}$; the above mentioned symmetry of this tensor ensures that equation (19) is satisfied. Before imposing equations (23) and (29), we note that the most general expression for a symmetric tensor depending on the scalars λ, λ_{ll} and on t^{α} is

$A^{\alpha_1\alpha_2\cdots\alpha_{Nk+1}}$

$$= \sum_{s=0}^{\left[\frac{Nk+1}{2}\right]} \binom{Nk+1}{2s} g_{k,2s}(\lambda, \lambda_{ll}) h^{(\alpha_1\alpha_2} \cdots h^{\alpha_{2s-1}\alpha_{2s}} t^{\alpha_{2s+1}} \cdots t^{\alpha_{Nk+1})} , \quad (30)$$

where the binomial factor has been introduced for later convenience. Thanks to this, equations (29) become

$$\begin{cases} g_{k+1,2s} = \frac{\partial}{\partial\lambda} g_{k,2s} , \\ \\ g_{k+1,2s+2} = \frac{2s+1}{2s+3} 3 \frac{\partial}{\partial\lambda_{ll}} g_{k,2s} , \end{cases} \quad \text{for } s = 0, \cdots , \left[\tfrac{Nk+1}{2}\right] . \quad (31)$$

It remains to consider equation (23); thanks to equation (26), (24) and (30), its value at equilibrium is

$$0 = g_{0,0} + \frac{2}{3} \lambda_{ll} g_{1,2}$$

which, thanks to equation $(31)_2$, becomes

$$0 = g_{0,0} + \frac{2}{3} \lambda_{ll} \frac{\partial}{\partial\lambda_{ll}} g_{0,0}.$$

Its solution is

$$g_{0,0} = \lambda_{ll}^{-\frac{3}{2}} G_{0,0}(\lambda), \quad (32)$$

with $G_{0,0}(\lambda)$ an arbitrary single variable function.

But equation (23) is equivalent to its value at equilibrium, and to its r-th derivatives with respect to l_{B_i} calculated at equilibrium, for all values of r. The r-th derivatives of equation (23) with respect to l_{B_i} is

$$\begin{aligned} 0 \;=\; & \delta_j^\alpha \frac{\partial^r h'^\mu t_\mu}{\partial l_{B_1} \cdots \partial l_{B_r}} + N \frac{\partial^{r+1} h'^\alpha}{\partial l_{B_1} \cdots \partial l_{B_r} \partial l_{\alpha_1 \cdots \alpha_N}} t_{\alpha_N} l_{\alpha_1 \cdots \alpha_{N-1} j} \\ + \; & N r t_{\alpha_N} \frac{\partial^r h'^\alpha}{\partial l_{\alpha_1 \cdots \alpha_N} \partial l_{(B_1} \cdots \partial l_{B_{r-1}}} \frac{\partial l_{\alpha_1 \cdots \alpha_{N-1} j}}{\partial l_{B_r)}} , \end{aligned} \quad (33)$$

where the indicated symmetrization is treated as the multi-index B_i was a single index. The equation (33) can be easily proved with the iterative procedure. Now we have to calculate this expression at equilibrium. Let us evaluate each single term of this relation.

— Thanks to equations (26) and (24), we have for the first term

$$\delta_j^\alpha \left(\frac{\partial^r h'^\mu t_\mu}{\partial l_{B_1} \cdots \partial l_{B_r}} \right)_{eq} = \delta_j^\alpha A^{\mu B_1 \cdots B_r} t_\mu .$$

— The second term at equilibrium, thanks to equation (24), is

$$\left(N \frac{\partial^{r+1} h'^\alpha}{\partial l_{B_1} \cdots \partial l_{B_r} \partial l_{\alpha_1 \cdots \alpha_N}} t_{\alpha_N} l_{\alpha_1 \cdots \alpha_{N-1} j} \right)_{eq}$$

$$= NA^{\alpha B_1\cdots B_r\alpha_1\cdots\alpha_N} t_{\alpha_N}\frac{1}{3}\lambda_{ll}\frac{2}{N}h_{j(\alpha_1}t_{\alpha_2}\cdots t_{\alpha_{N-1})}\,.$$

The symmetrization in the right hand side can be omitted because the term is contracted with a symmetric tensor. Now we use equation (30). We see that the terms containing the factor t^{α_1} gives zero contribute, so that the above expression can be written as

$$\sum_{s=1}^{\left[\frac{N(r+1)+1}{2}\right]} g_{r+1,2s}\binom{N(r+1)+1}{2s}\frac{2s}{N(r+1)+1}$$

$$h^{\alpha_1(\alpha_2}\cdots h^{\alpha_{2s-1}\alpha_{2s}}t^{\alpha_{2s+1}}\cdots t^{\alpha_{N(r+1)}}t^{\alpha)}t_{\alpha_N}\cdot\frac{1}{3}\lambda_{ll}2h_{j\alpha_1}t_{\alpha_2}\cdots t_{\alpha_{N-1}}\,,$$

where the indexes in $B_1\cdots B_r$ and α_N are included into the α_i; after the contraction with $t_{\alpha_2}\cdots t_{\alpha_N}$ this expression becomes

$$\sum_{s=1}^{\left[\frac{Nr+2}{2}\right]}\binom{Nr+1}{2s-1}\frac{2}{3}\lambda_{ll}g_{r+1,2s}h_j^{(\gamma_2}\cdots h^{\gamma_{2s-1}\gamma_{2s}}t^{\gamma_{2s+1}}\cdots t^{\gamma_{Nr+1}}t^{\alpha)}\,,$$

where the indexes $\gamma^{\cdot}$ represent $B_1\cdots B_r$.

— Let us evaluate now the contribute of the last term in equation (33), i.e.

$$Nrt_{\alpha_N}\left(\frac{\partial^r h'^\alpha}{\partial l_{\alpha_1\cdots\alpha_N}\partial l_{B_1}\cdots\partial l_{B_{r-1}}}\right)_{eq}\frac{\partial l_{\alpha_1\cdots\alpha_{N-1}j}}{\partial l_{B_r}}$$

$$= Nrt_{\alpha_N}A^{\alpha B_1\cdots B_{r-1}\alpha_1\cdots\alpha_N}g_{\alpha_1}^{(\beta_1^r}\cdots g_{\alpha_{N-1}}^{\beta_{N-1}^r}h_j^{\beta_N^r)}$$

$$= Nrt_{\alpha_N}A^{\alpha\alpha_N B_1\cdots B_{r-1}(\beta_1^r\cdots\beta_{N-1}^r}h_j^{\beta_N^r)}\,,$$

where we have exploited $B_r=\beta_1^r\cdots\beta_N^r$. We can now prove that

$$Nrt_{\alpha_N}\left(\frac{\partial^r h'^\alpha}{\partial l_{\alpha_1\cdots\alpha_N}\partial l_{(B_1}\cdots\partial l_{B_{r-1}}}\right)_{eq}\frac{\partial l_{\alpha_1\cdots\alpha_{N-1}j}}{\partial l_{B_r)}}$$

is symmetric with respect to two generic indexes β_i^s and β_q^t, with $s\le t=1,\cdots,r$. In fact it can be written as

$$\sum_{k=1}^{r}Nt_{\alpha_N}\frac{\partial^r h'^\alpha}{\partial l_{\alpha_1\cdots\alpha_N}\partial l_{B_1}\cdots\partial l_{B_{k-1}}\partial l_{B_{k+1}}\cdots\partial l_{B_r}}\frac{\partial l_{\alpha_1\cdots\alpha_N j}}{\partial B_k}$$

$$=\sum_{k=1,\cdots,r}^{k\neq s,k\neq t}Nt_{\alpha_N}A^{\alpha\alpha_N B_1\cdots B_{k-1}B_{k+1}\cdots B_r(\beta_1^k\cdots\beta_{N-1}^k}h_j^{\beta_N^k)}$$

$$+Nt_{\alpha_N}A^{\alpha\alpha_N B_1\cdots B_{s-1}B_{s+1}\cdots B_{t-1}\beta_1^t\cdots\beta_N^t B_{t+1}\cdots B_r(\beta_1^s\cdots\beta_{N-1}^s}h_j^{\beta_N^s)}$$

$$+Nt_{\alpha_N}A^{\alpha\alpha_N B_1\cdots B_{s-1}\beta_1^s\cdots\beta_N^s B_{s+1}\cdots B_{t-1}B_{t+1}\cdots B_r(\beta_1^t\cdots\beta_{N-1}^t}h_j^{\beta_N^t)}\,.$$

The first of these terms is clearly symmetric with respect to β_i^s and β_q^t, while the sum of the last two is

$$t_{\alpha_N} A^{\alpha\alpha_N B_1\cdots B_{s-1}B_{s+1}\cdots B_{t-1}\beta_1^t\cdots\beta_k^t\cdots\beta_N^t B_{t+1}\cdots B_r\beta_1^s\cdots\beta_{i-1}^s\beta_{i+1}^s\cdots\beta_N^s} h_j^{\beta_i^s}$$
$$+t_{\alpha_N} A^{\alpha\alpha_N B_1\cdots B_{s-1}\beta_1^s\cdots\beta_i^s\cdots\beta_N^s B_{s+1}\cdots B_{t-1}B_{t+1}\cdots B_r\beta_1^t\cdots\beta_{k-1}^t\beta_{k+1}^t\beta_N^t} h_j^{\beta_k^t}$$
$$+\text{terms like } t_{\alpha_N} A^{\alpha\alpha_N\beta_i^s\cdots\beta_k^t\cdots} h_j^{\cdot}$$

that is obviously symmetric with respect to β_i^s and β_k^t.

Consequently our tensor is symmetric with respect to every couple of indexes taken between $B_1\cdots B_r$, so that it can be expressed as

$$Nr t_{\alpha_N} A^{\alpha\alpha_N(\beta_1^1\cdots\beta_N^1\cdots\beta_1^r\cdots\beta_{N-1}^r} h_j^{\beta_N^r)}$$
$$= \sum_{s=0}^{\left[\frac{Nr}{2}\right]} 2s \binom{Nr}{2s} g_{r,2s} h^{\alpha(\gamma_2}\cdots h^{\gamma_{2s-1}\gamma_{2s}} t^{\gamma_{2s+1}}\cdots t^{\gamma_{Nr}} h_j^{\gamma_{Nr+1})}$$
$$+ \sum_{s=0}^{\left[\frac{Nr}{2}\right]} (Nr-2s)\binom{Nr}{2s} g_{r,2s} t^{\alpha} h^{(\gamma_2\gamma_3}\cdots h^{\gamma_{2s}\gamma_{2s+1}} t^{\gamma_{2s+2}}\cdots t^{\gamma}_{Nr} h_j^{\gamma_{Nr+1})}. \quad (34)$$

Here we have calculated firstly $t_{\alpha_N} A^{\alpha\alpha_N\beta_1^1\cdots\beta_N^1\cdots\beta_1^r\cdots\beta_{N-1}^r}$ by using equation (30) and then distinguishing the terms in which α is index of an $h^{\cdot\cdot}$ from those in which it is an index of a $t^{\cdot}$; finally we have multiplied the result times $h_j^{\gamma_{Nr+1}}$ and symmetrized with respect to $\gamma_2\cdots\gamma_{Nr+1}$.

Until now we have finished to evaluate the three terms of equation (33) calculated at equilibrium; so it becomes

$$0 = \sum_{s=1}^{\left[\frac{Nr+2}{2}\right]} g_{r+1,2s}\binom{Nr+1}{2s-1}\frac{2}{3}\lambda_{ll} h_j^{(\gamma_2}\cdots h^{\gamma_{2s-1}\gamma_{2s}} t^{\gamma_{2s+1}}\cdots t^{\gamma_{Nr+1}} t^{\alpha)}$$
$$+ \sum_{s=1}^{\left[\frac{Nr+2}{2}\right]} g_{r+1,2s}\binom{Nr+1}{2s-1}\frac{2}{3}\lambda_{ll} h_j^{(\gamma_2}\cdots h^{\gamma_{2s-1}\gamma_{2s}} t^{\gamma_{2s+1}}\cdots t^{\gamma_{Nr+1}} t^{\alpha)}$$
$$+ \sum_{s=0}^{\left[\frac{Nr}{2}\right]} 2s\binom{Nr}{2s} g_{r,2s} h^{\alpha(\gamma_2}\cdots h^{\gamma_{2s-1}\gamma_{2s}} t^{\gamma_{2s+1}}\cdots t^{\gamma_{Nr}} h_j^{\gamma_{Nr+1})}$$
$$+ \sum_{s=0}^{\left[\frac{Nr}{2}\right]} (Nr-2s)\binom{Nr}{2s} g_{r,2s} t^{\alpha} h^{(\gamma_2\gamma_3}\cdots h^{\gamma_{2s}\gamma_{2s+1}} t^{\gamma_{2s+2}}\cdots t^{\gamma_{Nr}} h_j^{\gamma_{Nr+1})}$$
$$= \sum_{s=0}^{\left[\frac{Nr}{2}\right]} (Nr+1)\binom{Nr}{2s} g_{r,2s} h^{(\alpha\gamma_2}\cdots h^{\gamma_{2s-1}\gamma_{2s}} t^{\gamma_{2s+1}}\cdots t^{\gamma_{Nr}} h_j^{\gamma_{Nr+1})}$$

$$+\sum_{s=0}^{\left[\frac{Nr}{2}\right]} \binom{Nr+1}{2s+1} \frac{2}{3} \lambda_{ll} g_{r+1,2s+2} h_j^{(\gamma_2} \cdots h^{\gamma_{2s+1}\gamma_{2s+2}} t^{\gamma_{2s+3}} \cdots t^{\gamma_{Nr+1}} t^{\alpha)} \,, \quad (35)$$

where in the second term we have changed the summation index s according to $s = S + 1$.

Note that this equation is automatically symmetric. In [13] was proved that $\frac{\partial \phi_{[k}}{\partial v_{i]}} = 0$ is an identity for the case of 13 moments; here we find that this property is valid also for an arbitrary number of moments.

So we have proved that equation (33) amounts to

$$0 = (Nr+1) \binom{Nr}{2s} g_{r,2s} + \binom{Nr+1}{2s+1} \frac{2}{3} \lambda_{ll} g_{r+1,2s+2} \quad , \text{ i.e.,}$$

$$g_{r,2s} + \frac{2}{3} \lambda_{ll} \frac{1}{2s+1} g_{r+1,2s+2} = 0 \quad \text{for } s = 0, \cdots, \left[\frac{Nr}{2}\right]. \quad (36)$$

Consequently, all our conditions are equivalent to the scalar equations (31), (32) and (36) which are constraints on the scalars $g_{r,2s}$ of the expansion (30). It remains to exploit them. For $s = 0, \cdots, \left[\frac{Nr}{2}\right]$ we can substitute $g_{r+1,2s+2}$ from equation (31) into equation $(36)_2$ which now becomes

$$\frac{2}{2s+3} \lambda_{ll} \frac{\partial}{\partial \lambda_{ll}} g_{r,2s} + g_{r,2s} = 0 \quad (37)$$

whose solution is

$$g_{r,2s} = \lambda_{ll}^{-\frac{2s+3}{2}} G_{r,2s}(\lambda) \quad \text{for } s = 0, \cdots, \left[\frac{Nr}{2}\right]. \quad (38)$$

In this way equation $(31)_2$ is exhausted, except for $s = \frac{Nr+1}{2}$ but only for the case with Nr odd.

If Nr is even equation (38) holds for all $g_{r,2s}$, while if Nr is odd the validity of equation (38) is not still proved for $g_{r,Nr+1}$. But for Nr odd we can use equations (31) with $k = r$, $s = \frac{Nr+1}{2}$, i.e.,

$$\begin{cases} \dfrac{\partial}{\partial \lambda} g_{r,Nr+1} = g_{r+1,Nr+1} \,, \\ \dfrac{\partial}{\partial \lambda_{ll}} g_{r,Nr+1} = \dfrac{Nr+4}{Nr+2} \dfrac{1}{3} g_{r+1,Nr+3}. \end{cases} \quad (39)$$

In the right hand sides we can use equation (38) because $\frac{Nr+1}{2} \leq \left[\frac{N(r+1)}{2}\right]$ and $\frac{Nr+3}{2} \leq \left[\frac{N(r+1)}{2}\right]$ holds, except for the trivial cases $N = 1, 2$. In this way the system (39) becomes

$$\begin{cases} \frac{\partial}{\partial \lambda} g_{r,Nr+1} = \lambda_{ll}^{-\frac{Nr+4}{2}} G_{r+1,Nr+1}(\lambda) \,, \\ \frac{\partial}{\partial \lambda_{ll}} g_{r,Nr+1} = \frac{Nr+4}{Nr+2} \frac{1}{3} \lambda_{ll}^{-\frac{Nr+6}{2}} G_{r+1,Nr+3}(\lambda). \end{cases} \quad (40)$$

The integrability conditions for this system gives

$$G'_{r+1,Nr+3} = \frac{-3}{2}(Nr+2)G_{r+1,Nr+1}. \tag{41}$$

After that the system (40) can be integrated and gives

$$g_{r,Nr+1} = \lambda_{ll}^{-\frac{Nr+4}{2}} G_{r,Nr+1}(\lambda) + c_{r,Nr+1}, \tag{42}$$

with

$$G_{r,Nr+1} = -\frac{2}{3}\frac{1}{Nr+2}G_{r+1,Nr+3}, \tag{43}$$

while $c_{r,Nr+1}$ is an arbitrary constant arising from integration. So equation (38) is a valid solution also in the case Nr odd and $s = \frac{Nr+1}{2}$, except to add the arbitrary constant $c_{r,Nr+1}$.

Now we can see that this constant does not occur in equation $(31)_1$ (because the right hand side is differentiated, while in the left hand side and in the case $N(k+1)$ odd, we have $2s \leq 2\left[\frac{Nk+1}{2}\right]$ from which $2s < N(k+1)+1$). Nor it occurs in equations (32), (38), (41), (43) and (36) (the proof for this last equation amounts to verify that $\left[\frac{Nr}{2}\right] < \left[\frac{Nr+1}{2}\right]$ for Nr odd and $\left[\frac{Nr}{2}\right]+1 < \left[\frac{N(r+1)+1}{2}\right]$ for $N(r+1)$ odd; obviously, in both of them we have N odd. If r is odd too, we have to verify only the first one, i.e. $\frac{Nr-1}{2} < \frac{Nr+1}{2}$, which is an identity; if r is even, we have to verify only the second one, i.e. $\frac{Nr}{2}+1 < \frac{N(r+1)+1}{2}$ which is true, at least for $N > 1$).

On the other hand, the contribute of this constant to the tensor $A^{\alpha_1\alpha_2\cdots\alpha_{Nk+1}}$ is $h^{(\alpha_1\alpha_2}\cdots h^{\alpha_{N_k}\alpha_{N_k+1})}\cdot c_{k,Nk+1}$, as it can be seen from equation (30).

The contribute of all these constants to h'^{α} follows from equation (26) and reads

$$\sum_{r=0}^{\infty}\frac{1}{(2r+1)!}c_{2r+1,N(2r+1)+1}$$
$$h^{\alpha(\beta_1^1}\cdots h^{\beta_{N-1}^1\beta_N^1}\cdots h^{\beta_N^{N-1}\beta_1^N}\cdots h^{\beta_{N-1}^N\beta_N^N)}\cdot l_{\beta_1^1\cdots\beta_N^1}\cdots l_{\beta_1^N\cdots\beta_N^N}, \tag{44}$$

where we have put $k = 2r+1$.

It is easy to verify that this additional term satisfies identically the symmetry conditions for equation (19) and (23) (in fact t_{α_N} is contracted with an h^{α_N}, for this additional term). In other words, we can assume equation (38) for all $g_{r,2s}$ (also for $s = \left[\frac{Nr+1}{2}\right]$), except that, in the case with N odd, we have to add to h'^{α} the additional term (44).

Let us then substitute from equation (38) into equation $(31)_1$ and (36); so they become

$$G_{k+1,2s} = G'_{k,2s} \quad \text{for } s = 0, \cdots, \left[\frac{Nk+1}{2}\right], \tag{45}$$

$$G_{r+1,2s+2} = -3\frac{2s+1}{2}G_{r,2s} \quad \text{for } s = 0, \cdots \left[\frac{Nr}{2}\right]. \tag{46}$$

But this last equation holds also for $s = 0, \cdots, \left[\frac{Nr+1}{2}\right]$; this is obvious when Nr is even, while it is just equation (43) when Nr is odd (remember that we have equation (43) only for the case with Nr odd).

After that, we see that equation (32) is contained in (38) for $r = s = 0$, while equation (41), by using equation (43), becomes $G'_{N,Nr+1} = G_{r+1,Nr+1}$ which is just equation (45) with $k = r$ and $s = \left[\frac{Nr+1}{2}\right]$ (remember that equation (41) holds only for Nr odd).

There remain equations (45) and (46). To this end, let us define $H_{r,s}$ from

$$G_{r,2s} = \left(\frac{-3}{2}\right)^r \frac{(2s)!}{2^s s!} H_{r,s}. \tag{47}$$

In this way equations (45) and (46) become

$$H_{r+1,s+1} = H_{r,s}, \qquad H'_{r,s} = \frac{-3}{2}H_{r+1,s} \quad \text{for } s = 0, \cdots, \left[\frac{Nr+1}{2}\right]. \tag{48}$$

Equation $(48)_1$ suggests to define $H_{r,s}$ also for $s > \left[\frac{Nr+1}{2}\right]$. In fact, let h be a number such that $s + h \leq \left[\frac{N(r+h)+1}{2}\right]$ (for example, $h = \left[\frac{2s-Nr+1}{N-2}\right]$); we can define $H_{r,s} = H_{r+h,s+h}$. In this way equation $(48)_1$ holds for all r and s. Regarding equation $(48)_2$ we have

$$H'_{r,s} = H'_{r+h,s+h} = \frac{-3}{2}H_{r+h+1,s+h} = \frac{-3}{2}H_{r+1,s};$$

in other words, also $(48)_2$ holds for all r and s.

After that:

— if $r \geq s$ we have

$$H_{r,s} = H_{r-s,0} = \left(\frac{-2}{3}\right)^{r-s} \frac{d^{r-s}H_{0,0}}{d\lambda^{r-s}}, \tag{49}$$

— if $r < s$ we have

$$H_{r,s} = H_{0,s-r}. \tag{50}$$

In this way $H_{r,s}$ is known except for $H_{0,p}$.

On the other hand, it is easy to see that (49) and (50) satisfy equation $(48)_1$. Regarding $(48)_2$, we see that:

— if $r \geq s \;\Rightarrow\; r+1 \geq s$, we have to use equation (49) for both sides of equation $(48)_2$ and it becomes an identity,

— if $r = s-1$, we have to use equation (50) for the left hand side of equation $(48)_2$ and equation (49) for the right hand side. The result is $H'_{0,1} = \frac{-3}{2} H_{0,0}$,

— if $r < s-1$, we have to use equation (50) for both sides of equation $(48)_2$ which becomes $H'_{0,s-r} = \frac{-3}{2} H_{0,s-r-1}$.

In conclusion, $H_{0,0}$ is arbitrary and $H_{0,p}$ is defined by

$$H'_{0,p} = \frac{-3}{2} H_{0,p-1}, \tag{51}$$

except for a constant arising from integration. After that, equation (49) and (50) give all the other functions $H_{r,s}$.

4. The Kinetic Approach

Let us now search a solution, for conditions (19) and (23), of the form

$$h'^{\alpha} = \int F\left(l_{\beta_1 \cdots \beta_N} c'^{\beta_1} \cdots c'^{\beta_N}\right) c'^{\alpha} d\underline{c}', \tag{52}$$

where F is an arbitrary single variable function; it is related to the distribution function, but this relation does not affect the following considerations, so that we choose to omit it.

The symmetry for the left hand side of equation (19) is certainly ensured; remembering that $c'^0 = 1$, equation (23) becomes

$$0 = \int \frac{\partial}{\partial c'_j} \left[F\left(l_{\beta_1 \cdots \beta_N} c'^{\beta_1} \cdots c'^{\beta_N}\right) c'^{\alpha}\right] d\underline{c}' \tag{53}$$

which is certainly true. The expansion of equation (52) with respect to equilibrium is

$$h'^{\alpha} = \sum_{k=0}^{\infty} \frac{1}{k!} \int F^{(k)}\left(\lambda + \frac{1}{3}\lambda_{ll} c'^2\right) c'^{\alpha} c'^{B_1} \cdots c'^{B_k} d\underline{c}' \tilde{l}_{B_1} \cdots \tilde{l}_{B_k},$$

where equations (25), (27) and the multi-index notation have been used. Then we have obtained equation (26) with

$$A^{\alpha B_1 \cdots B_k} = \frac{\partial^k B^{\alpha B_1 \cdots B_k}}{\partial \lambda_k}, \tag{54}$$
$$B^{\alpha B_1 \cdots B_k} = \int F\left[\lambda + \frac{1}{3}\lambda_{ll} c'^2\right] c'^{\alpha} c'^{B_1} \cdots c'^{B_k} d\underline{c}'.$$

It is easy to verify that equations (29) are satisfied with this expression. The integral in equation $(54)_2$ can be calculated with a well known procedure. To reach faster the result, let us consider the tensor

$$B^{\beta_1\cdots\beta_s\beta_{s+1}\cdots\beta_r} h^{\gamma_1}_{\beta_1}\cdots h^{\gamma_s}_{\beta_s} t_{\beta_{s+1}}\cdots t_{\beta_r} = \int F\left[\lambda + \frac{1}{3}\lambda_{ll} c'^2\right] c'^{\beta_1}\cdots c'^{\beta_s} h^{\gamma_1}_{\beta_1}\cdots h^{\gamma_s}_{\beta_s} d\underline{c'}.$$

The above tensor depends only on scalar quantities and is symmetric, so it is equal to

$$\begin{cases} 0\,, & \text{if s is odd,} \\ g_s(\lambda,\lambda_{ll}) h^{(\gamma_1\gamma_2}\cdots h^{\gamma_{s-1}\gamma_s)}\,, & \text{if s is even.} \end{cases}$$

To know $g_s(\lambda,\lambda_{ll})$ it suffices to multiply both members by $h_{\gamma_1\gamma_2}\cdots h_{\gamma_{s-1}\gamma_s}$ obtaining

$$\int_0^\infty F\left[\lambda + \frac{1}{3}\lambda_{ll} c'^2\right] c'^{s+2}\left(\int_0^\pi \sin\theta d\theta\right)\left(\int_0^{2\pi} d\phi\right) dc' = g_s(\lambda,\lambda_{ll})(s+1)\,,$$

where we have changed the integration variables according to the rule

$$c'^1 = c'\sin\theta\cos\phi,\quad c'^2 = c'\sin\theta\sin\phi,\quad c'^3 = c'\cos\theta\,,$$
$$c' \in [0,+\infty[,\qquad \theta\in[0,\pi],\qquad \phi\in[0,2\pi[.$$

We obtain

$$g_s(\lambda,\lambda_{ll}) = \frac{4\pi}{s+1}\int_0^\infty F\left[\lambda + \frac{1}{3}\lambda_{ll} c'^2\right] c'^{s+2} dc' = (\lambda_{ll})^{-\frac{s+3}{2}} G_s(\lambda)$$

with

$$G_s(\lambda) = \frac{4\pi}{s+1}\int_0^\infty F\left[\lambda + \frac{1}{3}\eta^2\right]\eta^{s+2} d\eta,\qquad \eta = \sqrt{\lambda_{ll}}c'. \tag{55}$$

For the sequel it will be useful to note that

$$G'_s(\lambda) = \frac{4\pi}{s+1}\int_0^\infty \left\{\frac{d}{d\eta}F\left[\lambda + \frac{1}{3}\eta^2\right]\right\}\eta^{s+1} d\eta \frac{3}{2} = 4\pi\frac{-3}{2}\int_0^\infty F\left[\lambda + \frac{1}{3}\eta^2\right]\eta^s d\eta = \frac{-3}{2}(s-1)G_{s-2} \tag{56}$$

provided that $F\eta^{s+1}$ is infinitesimal for η going to infinity. After that, we have

$$B^{\gamma_1\cdots\gamma_r} = B^{\beta_1\beta_2\cdots\beta_r}\left(h^{\gamma_1}_{\beta_1} + t_{\beta_1}t^{\gamma_1}\right)\left(h^{\gamma_2}_{\beta_2} + t_{\beta_2}t^{\gamma_2}\right)\cdots\left(h^{\gamma_r}_{\beta_r} + t_{\beta_r}t^{\gamma_r}\right)$$
$$= \sum_{s=0}^{r}\binom{r}{s} B^{\beta_1\cdots\beta_r} h^{(\gamma_1}_{\beta_1}\cdots h^{\gamma_s}_{\beta_s} t_{\beta_{s+1}} t^{\gamma_{s+1}}\cdots t_{\beta_r} t^{\gamma_r)}$$
$$= \sum_{q=0}^{\left[\frac{r}{2}\right]}\binom{r}{2q} h^{(\gamma_1\gamma_2}\cdots h^{\gamma_{2q-1}\gamma_{2q}} t^{\gamma_{2q+1}}\cdots t^{\gamma_r)}\lambda_{ll}^{-\frac{2q+3}{2}} G_{2q}(\lambda).$$

This allows to rewrite equation (54) as

$$A^{\alpha B_1\cdots B_k} = \frac{\partial^k B^{\alpha B_1\cdots B_k}}{\partial \lambda^k}$$
$$= \sum_{q=0}^{\left[\frac{kN+1}{2}\right]} \binom{kN+1}{2q} h^{(\gamma_1\gamma_2}\cdots h^{\gamma_{2q-1}\gamma_{2q}} t^{\gamma_{2q+1}}\cdots t^{\gamma_{kN}} t^{\alpha)} \lambda_{ll}^{-\frac{2q+3}{2}} G_{2q}^{(k)}(\lambda).$$

This result confirms equation (30) also in the kinetic case, but with $g_{k,2s}(\lambda, \lambda_{ll}) = \lambda_{ll}^{-\frac{2s+3}{2}} G_{2s}^{(k)}(\lambda)$, and it is easy to see that these functions $g_{k,2s}$ satisfy equation (31), as consequence of equation (56). Also equations (38) and (42) are confirmed, with $G_{r,2s}(\lambda) = G_{2s}^{(r)}(\lambda)$, $c_{r,Nr+1} = 0$. In this way we see that the additional term (44) is not present in the kinetic approach. Moreover, the matrix $H_{r,s}$ defined in equation (47) becomes, in this approach

$$H_{r,s} = \left[\frac{-2}{3}\right]^r \frac{2^s s!}{(2s)!} G_{2s}^{(r)}(\lambda),$$

and equation (51) becomes a consequence of equation (56). But the constants arising from integration of equation (51) are not present in the kinetic approach, because all the functions $G_s(\lambda)$ are defined by (55) in terms of the single variable function F.

5. On Subsystems

We aim to obtain now the model with $N-1$ instead of N through the method of subsystems. To this end we firstly need the relation between the 4-dimensional Lagrange multipliers and the 3-dimensional ones. The first of these are defined by equation (4), from which we obtain

$$dH^\alpha = l^N_{\alpha_1\cdots\alpha_N} dM_N^{\alpha\alpha_1\cdots\alpha_N} = l_N^{\alpha_1\cdots\alpha_N} dM_N^{\alpha\beta_1\cdots\beta_N} g_{\alpha_1\beta_1}\cdots g_{\alpha_N\beta_N}$$
$$= l_N^{\alpha_1\cdots\alpha_N} dM_N^{\alpha\beta_1\cdots\beta_N} (h_{\alpha_1\beta_1} + t_{\alpha_1}t_{\beta_1})\cdots(h_{\alpha_N\beta_N} + t_{\alpha_N}t_{\beta_N})$$
$$= \sum_{r=0}^{N} \binom{N}{r} l_N^{\alpha_1\cdots\alpha_r\alpha_{r+1}\cdots\alpha_N} dM_N^{\alpha\beta_1\cdots\beta_r\beta_{r+1}\cdots\beta_N}$$
$$h_{\alpha_1\beta_1}\cdots h_{\alpha_r\beta_r} t_{\alpha_{r+1}}\cdots t_{\alpha_N} t_{\beta_{r+1}}\cdots t_{\beta_N} = \sum_{r=0}^{N} \lambda^N_{j_1\cdots j_r} dF_N^{\alpha j_1\cdots j_r}$$

with $\lambda^N_{j_1\cdots j_r} = \binom{N}{r} l_N^{\alpha_1\cdots\alpha_r\alpha_{r+1}\cdots\alpha_N} h_{\alpha_1 j_1}\cdots h_{\alpha_r j_r} t_{\alpha_{r+1}}\cdots t_{\alpha_N}$

and $$F_N^{\alpha j_1\cdots j_r} = M_N^{\alpha\beta_1\cdots\beta_r\beta_{r+1}\cdots\beta_N} h^{j_1}_{\beta_1}\cdots h^{j_r}_{\beta_r} t_{\beta_{r+1}}\cdots t_{\beta_N}. \qquad (57)$$

Equation $(57)_1$ gives the 3-dimensional Lagrange multipliers in terms of the 4-dimensional ones. We have introduced the index N to remember that we are considering the model with N as maximum order of moments. In this way it will be distinguished from that with $N-1$ instead of N.

The inverse of equation $(57)_1$ is

$$\begin{aligned} l_N^{\alpha_1\cdots\alpha_N} &= l_N^{\beta_1\cdots\beta_N} g_{\beta_1}^{\alpha_1}\cdots g_{\beta_N}^{\alpha_N} = l_N^{\beta_1\cdots\beta_N}(h_{\beta_1}^{\alpha_1}+t^{\alpha_1}t_{\beta_1})\cdots(h_{\beta_N}^{\alpha_N}+t^{\alpha_N}t_{\beta_N}) \\ &= \sum_{s=0}^{N}\binom{N}{s} l_N^{\beta_1\cdots\beta_s\cdots\beta_N} h_{\beta_1}^{(\alpha_1}\cdots h_{\beta_s}^{\alpha_s} t^{\alpha_{s+1}}\cdots t^{\alpha_N)} t_{\beta_{s+1}}\cdots t_{\beta_N} \\ &= \sum_{s=0}^{N}\lambda_N^{(\alpha_1\cdots\alpha_s} t^{\alpha_{s+1}}\cdots t^{\alpha_N)}. \end{aligned} \tag{58}$$

The model with $N-1$ instead of N can be obtained as subsystem of the above one by taking

$$\begin{aligned} \lambda_N^{\alpha_1\cdots\alpha_N} &= 0, \\ \lambda_N^{\alpha_1\cdots\alpha_s} &= \lambda_{N-1}^{\alpha_1\cdots\alpha_s} \quad \text{for } s=0,\cdots,N-1. \end{aligned} \tag{59}$$

We have now to express these relations in terms of the 4-dimensional Lagrange multipliers; to this end we see that

$$l_N^{\alpha_1\cdots\alpha_N} = \sum_{s=0}^{N-1}\lambda_{N-1}^{(\alpha_1\cdots\alpha_s} t^{\alpha_{s+1}}\cdots t^{\alpha_N)}, \tag{60}$$

while equation $(57)_1$, with $N-1$ instead of N, is

$$\lambda_{j_1\cdots j_r}^{N-1} = \binom{N-1}{r} l_{N-1}^{\alpha_1\cdots\alpha_r\alpha_{r+1}\cdots\alpha_{N-1}} h_{\alpha_1 j_1}\cdots h_{\alpha_r j_r} t_{\alpha_{r+1}}\cdots t_{\alpha_{N-1}}. \tag{61}$$

Then, by substituting equation (61) in equation (60) we find

$$l_N^{\alpha_1\cdots\alpha_N} = \sum_{s=0}^{N-1} t^{(\alpha_{s+1}}\cdots t^{\alpha_N} h_{\gamma_1}^{\alpha_1}\cdots h_{\gamma_s}^{\alpha_s)} l_{N-1}^{\gamma_1\cdots\gamma_s\gamma_{s+1}\cdots\gamma_{N-1}} t_{\gamma_{s+1}}\cdots t_{\gamma_{N-1}}\binom{N-1}{s}$$

from which

$$l_N^{\alpha_1\cdots\alpha_N} = l_{N-1}^{(\alpha_1\cdots\alpha_{N-1}} t^{\alpha_N)}, \tag{62}$$

because

$$\begin{aligned} l_{N-1}^{\alpha_1\cdots\alpha_{N-1}} t^{\alpha_N} &= l_{N-1}^{\gamma_1\cdots\gamma_{N-1}} t^{\alpha_N}(h_{\gamma_1}^{\alpha_1}+t^{\alpha_1}t_{\gamma_1})\cdots(h_{\gamma_{N-1}}^{\alpha_{N-1}}+t^{\alpha_{N-1}}t_{\gamma_{N-1}}) \\ &= \sum_{s=0}^{N-1}\binom{N-1}{s} l_{N-1}^{\gamma_1\cdots\gamma_s\cdots\gamma_{N-1}} h_{\gamma_1}^{(\alpha_1}\cdots h_{\gamma_s}^{\alpha_s} t_{\gamma_{s+1}} t^{\alpha_{s+1}}\cdots t_{\gamma_{N-1}} t^{\alpha_{N-1})} t^{\alpha_N}. \end{aligned}$$

Now, from equation (24), we have

$$l_{N-1\ \text{equation}}^{\alpha_1\cdots\alpha_{N-1}} = \lambda t^{\alpha_1}\cdots t^{\alpha_{N-1}} + \frac{1}{3}\lambda_{ll} h^{(\alpha_1\alpha_2} t^{\alpha_3}\cdots t^{\alpha_{N-1})}.$$

This and equation (24) yield

$$l_{N\ \text{equation}}^{\alpha_1\cdots\alpha_N} = l_{N-1\ \text{equation}}^{(\alpha_1\cdots\alpha_{N-1}}t^{\alpha_N)},$$

that is, equation (62) holds also when we calculate it at equilibrium. The deviation of equation (62) from its value at equilibrium is

$$\tilde{l}_N^{\alpha_1\cdots\alpha_N} = \tilde{l}_{N-1}^{(\alpha_1\cdots\alpha_{N-1}}t^{\alpha_N)}; \tag{63}$$

in other words, equation (62) holds when we substitute the Lagrange multipliers with their deviation with respect to equilibrium. We can now substitute equation (63) into equation (26); in this way we find the counterpart of (26) with $N-1$ instead of N. To this end we have to contract an index of each $B_1, \cdots, B_k$ with a $t_.$; in other words, we have to contract the expression (30) with $t_{\alpha_{(N-1)k+2}} \cdots t_{\alpha_{Nk+1}}$. It is easy to verify that in this way equation (30) remains unchanged except that now $N-1$ replaces N; obviously this is true also for $g_{k,2s}$, where s now goes from 0 to $\left[\frac{(N-1)k+1}{2}\right]$. This property is transferred to $G_{k,2s}$ for equation (38) and to $H_{r,s}$ for equation (47). But $H_{r,s}$ is defined by equations (49) and (50) in terms of $H_{0,p}$ which are determined by equation (51). Therefore, the family of constants arising by integrating equation (51), is inherited also by the subsystem.

We have only to notice that from equation (50) it follows that $H_{0,p}$ is useful for $H_{r,r+p}$ which, for equation (47) is useful for $G_{r,2(r+p)}$. It follows that $H_{0,p}$ is present in the subsystem when $r+p \leq \left[\frac{(N-1)r+1}{2}\right]$, that is $p \leq \left[\frac{(N-3)r+1}{2}\right]$, while for the initial system was useful when $p \leq \left[\frac{(N-2)r+1}{2}\right]$. Now, for a fixed value of p, it is always possible to find r such that both of the previous inequalities are satisfied. The only difference is that in the subsystem, $H_{0,p}$ occurs only in terms of higher order with respect to equilibrium, than in the initial system. This is true, provided that $N > 3$, that is if neither the system, nor the subsystem are the 10 moments model.

But what happens to the other family of constants, that is for the supplementary term (44)?

If N is even, the model has not this term and, consequently, it cannot be inherited by the subsystem.

If N is odd, this term is present; but when we substitute equation (62) in (44) we obtain zero because each $t_.$ is contracted with a projector $h^{..}$. We expected this result because, with N odd, we have $N-1$ even in which case the term (44) is not present. We may conclude that the other family of constants, or the supplementary term (44), disappears in the subsystem. Only the other family of constants is inherited.

This can be seen also from the following viewpoint: the family of constants arising from integration of equation (51), in the case $N = 3$, will perpetuate also for the subsequent values of N; equivalently, we can say that the closure in the model with a generic $N > 3$ is exactly determined in terms of that with $N = 3$, except for the supplementary term (44).

Conclusions

In this paper we have applied the method of Extended thermodynamics to the case of an arbitrary but fixed number of moments. This case has already been developed in the kinetic approach. Here we have considered the macroscopic approach and the costitutive functions have been determined up to whatever order with respect to thermodynamical equilibrium. In this way we have been able to find the exact general solution of the conditions present in extended thermodynamics for the case of many moments. The results founded in the kinetic case, as we expected to found, are a particular case of the present ones. Moreover, we have introduced an innovative 4-dimensional notation that simplifies very much the form of the equations. Finally, the present results in the case $N = 3$ are equivalent to those found in [12].

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