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ON k-INDEPENDENT SETS IN SPECIAL PRODUCTS OF GRAPHS

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Abstract: A subset $S \subset V(G)$ is k-independent if for each two distinct vertices from S the distance between them is at least k. In this paper we determine the number of all k-independent sets in special products of graphs.

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1. Introduction

For general concepts we refer the reader to [1] and [2]. By a graph G we mean a finite, undirected, connected simple graph without loops and multiple edges. $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The length of the shortest path joining vertices x and y in G will be denoted by $d_G(x,y)$. By P_n and C_n, for $n \geq 2$, we mean graphs with the vertex set $V(P_n) = V(C_n) = \{t_1, ..., t_n\}$ and the edge sets $E(P_n) = \{\{t_i, t_{i+1}\};\ i = 1, ..., n-1\}$ and $E(C_n) = E(P_n) \cup \{t_n, t_1\}$, respectively. In addition $C_1 = P_1$, where P_1 is a graph consisting of one vertex. By K_n we will denote the complete graph on n vertices, $n \geq 1$. Let G be a graph on $V(G) = \{t_1, ..., t_n\},\ n \geq 2$, and H be a graph on $V(H) = \{y_1, ..., y_m\},\ m \geq 1$. By Cartesian product of two graphs G and H we mean a graph $G \times H$ such that $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{\{(t_i, y_p), (t_j, y_q)\};\ t_i = t_j$ and $\{y_p, y_q\} \in E(H)$ or $\{t_i, t_j\} \in E(G)$ and $y_p = y_q\}$.

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Let G be a graph on $V(G) = \{t_1, ..., t_n\}$, $n \geq 2$ and $h_n = (H_i)_{i\in\{1,...,n\}}$ be a sequence of vertex disjoint graphs on $V(H_i) = V = \{y_1, ..., y_x\}$, $x \geq 1$. By generalized lexicographic product of G and $h_n = (H_i)_{i\in\{1,...,n\}}$ we mean a graph $G[h_n]$ such that $V(G[h_n]) = V(G) \times V$ and $E(G[h_n]) = \{\{(t_i, y_p), (t_j, y_q)\};\ t_i = t_j$ and $\{y_p, y_q\} \in E(H_i)$ or $\{t_i, t_j\} \in E(G)\}$. If $H_i = H$, $i = 1, ..., n$ then $G[h_n] = G[H]$, where $G[H]$ is a lexicographic product of two graphs.

Let k be a fixed integer, $k \geq 2$. A subset $S \subseteq V(G)$ is said to be a k-independent set of G if for each two distinct vertices $x, y \in S, d_G(x, y) \geq k$. Moreover the empty set and a subset containing only one vertex also are k-independent sets of G. If $k = 2$, then the definition reduces to the definition of an independent set of the graph G. The number of k-independent sets in G is denoted by $N_kI(G)$. If $k = 2$ then $N_kI(G) = NI(G)$. In the chemical literature the graph parameter $NI(G)$ is referred to as the Merrifield-Simmons index, see Merrifield and Simmons [5]. This index is one of the most popular topological indices in chemistry. There is the correlation between this index and boiling points. Results concerning counting independent sets in graphs may have potential use in combinatorial chemistry. Prodinger et al [7] initiated the study of the number $NI(G)$ of independent sets in a graph. They also named the number $NI(G)$ as the Fibonacci number of a graph. The literature includes many paper dealing with the theory of counting of k-independent sets in graphs, see for instance [4, 6, 8, 9, 10].

If G is a graph on n vertices, then $f_G(k, n, p)$ denotes the number of all p-element k-independent sets in G. Consequently $N_kI = \sum_{p\geq 0} f_G(k, n, p)$. Kwaśnik at al [4] proved, that: $f_{P_n}(k, n, p) = \binom{n-p-(p-1)(k-2)+1}{p}$ and $f_{C_n}(k, n, 0) = 1$ and for $p \geq 1$, $f_{C_n}(k, n, p) = \frac{n}{p}\binom{n-p(k-1)-1}{p-1}$. Consequently:

$$N_kI(P_n) = \sum_{p\geq 0} \binom{n-p-(p-1)(k-2)+1}{p}$$

and

$$N_kI(C_n) = 1 + \sum_{p\geq 1} \frac{n}{p}\binom{n-p(k-1)-1}{p-1}.$$

These numbers generalize the Fibonacci numbers and the Lucas numbers, respectively.

Theorem 1. (see [3]) *Let $n \geq 1$, $p \geq 0$, $x \geq 1$ be integers. Then for an arbitrary graph G on n vertices $NI(G[K_x]) = \sum_{p\geq 0} f_G(n, p)x^p$.*

Theorem 2. (see [10]) *Let $k \geq 3,\ x \geq 1,\ p \geq 0$ be integers. Then for an arbitrary graph G on $n, n \geq 2$, vertices and for an arbitrary sequence $h_n = (H_i)_{i \in \{1,\dots,n\}}$ of vertex disjoint graphs such that $|V(H_i)| = x$ for $i = 1, ..., n$, $N_kI(G[h_n] = \sum_{p\geq 0} f_G(k,n,p)x^p$.*

2. Main Results

For convenience we use in this paper following notation:

$$f(k,n,p) = f_{P_n \times K_m}(k, mn, p),$$

$$f^*(k,n,p) = f_{C_n \times K_m}(k, mn, p),$$

B_n^p-the total number of p-element k-independent sets in $P_n \times K_m$ containing a fixed vertex (t_n, y_i), $1 \leq i \leq m$

In [9] it has been proved the following result.

Theorem 3. (see [9]) *Let $k \geq 2, n \geq 0, m \geq 1, p \geq 0$ be integers. Then the numbers $f(k,n,p)$ satisfy the following recurrence relations*

$f(k,n,0) = 1,\ f(k,n,1) = mn$ and for $p \geq 2$, $f(k,n,p) = 0$ if $n < (p-1)\tau + 1$.

For $n \geq (p-1)\tau + 1$ we have:

$f(k,n,p) = f(k,n-1,p) + mB_n^p$ and $B_n^p = f(k,n-k,p-1) + (m-1)B_{n-(k-1)}^{p-1}$, where $B_n^1 = 1$ and

$$\tau = \begin{cases} k-1\,, & \text{if } m > 1\,, \\ k\,, & \text{if } m = 1\,. \end{cases}$$

Now we consider the graph $C_n \times K_m$, $n \geq 1$, $m > 1$ and we present numbers $NI(C_n \times K_m)[K_x]$ and $N_kI(C_n \times K_m)[h_n]$, where $h_n = (H_i)_{i\in\{1,\dots,n\}}$ is an arbitrary sequence of vertex disjoint graphs. Then to prove the main result we need the following lemmas.

Lemma 1. *Let $n \geq 1,\ m > 1,\ k \geq 2,\ p \geq 1$ be integers. Let $S_{i,q}^p$ be a family of all p-element k-independent sets of $C_n \times K_m$ containing the fixed vertex (t_i, y_q), $1 \leq i \leq n$, $1 \leq q \leq m$. Then for each $1 \leq j \leq n$ and $1 \leq r \leq m$, $|S_{i,q}^p| = |S_{j,r}^p|$.*

Lemma 2. *Let $k \geq 2,\ m > 1,\ p \geq 2$ be integers. If S is a p-element k-independent set of $P_n \times K_m$ such that $(t_1, y_1), (t_n, y_1) \in S$, then $n \geq n_0$, where*

$$n_0 = \begin{cases} (p-1)(k-1) + 2\,, & \text{if } p \text{ is even and } m = 2\,, \\ (p-1)(k-1) + 1\,, & \text{otherwise.} \end{cases}$$

Proof. Let S be a p-elements k-independent set of $P_n \times K_m$ such that $(t_1, y_1), (t_n, y_1) \in S$. Then from the definition of the graph $P_n \times K_m$ we deduce that the set S can be constructed as follows:

$$S = \{(t_1, y_1), (t_k, y_j), ..., (t_{(p-2)(k-1)+1}, y_q), (t_n, y_1)\},$$

where if $(t_r, y_i), (t_{r+k-1}, y_j) \in S$, then $i \neq j$.

Consequently we obtain that S is a p-element k-independent set if $n \geq n_0$, where $n_0 = \begin{cases} (p-1)(k-1)+2\,, & \text{if } p \text{ is even and } m = 2\,, \\ (p-1)(k-1)+1\,, & \text{otherwise.} \end{cases}$ □

Denote by D_n^p the total number of p-element k-independent sets in $C_n \times K_m$ containing a fixed vertex (t_j, t_i), $1 \leq j \leq n$, $1 \leq i \leq m$.

Lemma 3. *Let $k \geq 2$, $p \geq 2$, $m > 1$, $n \geq n_0$ (n_0 as in Lemma 2) be integers. If $\tilde{f}(k, n, p)$ denotes the total number of p-element k-independent sets in $P_n \times K_m$ containing both of vertices (t_1, y_i), (t_n, y_i) for fixed $1 \leq i \leq m$, simultaneously, then $\tilde{f}(k, n, p) = D_{n-1}^{p-1}$.*

Proof. Let $S \subseteq V(P_n \times K_m)$ be an arbitrary p-element k-independent set of $P_n \times K_m$ containing the vertices $(t_1, y_i), (t_n, y_i)$, $1 \leq i \leq m$. Since $n \geq n_0$ by Lemma 2 such subset exists. For each $i = 1, ..., m$ we identify vertices (t_1, y_i) and (t_n, y_i), next we denote the identified vertices by (t_1, y_i). The obtained graph is isomorphic to $C_{n-1} \times K_m$. From the definition of the graph $P_n \times K_m$ and $C_{n-1} \times K_m$ the set $S \setminus \{(t_n, y_i)\}$ is a $(p-1)$-element k-independent set of $C_{n-1} \times K_m$ containing the vertex (t_1, y_i). Consequently we have D_{n-1}^{p-1} subset S, so $\tilde{f}(k, n, p) = D_{n-1}^{p-1}$. □

Theorem 4. *Let $k \geq 2$, $n \geq 1$, $p \geq 2$, $m > 1$ be integers. If $n < n_1$, then $f^*(k, n, p) = 0$, where*

$$n_1 = \begin{cases} p(k-1)\,, & \text{if } m = 2 \text{ and } p \text{ is even or } m > 2\,, \\ p(k-1)+1\,, & \text{if } m = 2 \text{ and } p \text{ is odd.} \end{cases}$$

Proof. From the definition of the graph $C_n \times K_m$ and by fact that K_m is a complete graph on m, $m > 1$ vertices we can construct a k-independent set S' of $C_n \times K_m$ as follows: $S' = \{(t_1, y_i), (t_k, y_j), ..., (t_{(p-1)(k-1)+1}, y_q)\}$, where $d_{C_n \times K_m}((t_1, y_i), (t_{(p-1)(k-1)+1}, y_q)) \geq k$ and if $(t_r, y_h), (t_s, y_l) \in S'$ and $s = r + k - 1$, then $h \neq l$. Consequently, if $n < n_1$, then it is not possible to construct the set S', so $f^*(k, n, p) = 0$.

Thus the theorem is proved. □

Theorem 5. *Let $k \geq 2$, $n \geq 1$, $m > 1$, $p \geq 0$ be integers and let $f(k,n,p)$ and B_p^n be as in Theorem 3. Then $f^*(k,n,p)$ satisfies the following recurrence relations:*

$f^*(k,n,0) = 1$,

$f^*(k,n,1) = mn$ and for $p \geq 2$,

$f^*(k,n,p) = 0$ if $n < n_1$ (n_1 as in Theorem 4),

and for $n \geq n_1$ we have $f^*(k,n,p) = f(k,n-(k-1),p) + m(k-1)D_n^p$ and

$$D_n^p = f(k,n-(2k-3),p) - \begin{cases} 1, & \text{for } n = 2k-2 \text{ and } p = 2,\\ 2, & \text{for } n > 2k-2 \text{ and } p = 2,\\ 2B_{n-(2k-3)}^{p-1}, & \text{for } p \geq 3 \text{ and } n < n_2,\\ 2B_{n-(2k-3)}^{p-1} - D_{n-(2k-2)}^{p-2}, & \text{for } p \geq 3 \text{ and } n \geq n_2,\end{cases}$$

where $n_2 = \begin{cases} p(k-1)+1, & \text{if } p \text{ is even},\\ p(k-1), & \text{if } p \text{ is odd}\end{cases}$ *and putting $D_n^1 = 1$.*

Proof. If $p = 0$, then the empty set is the unique k-independent set in the graph $C_n \times K_m$, so $f^*(k,n,0) = 1$. If $p = 1$, then every vertex of the graph $C_n \times K_m$ is a k-independent set in the graph $C_n \times K_m$, consequently $f^*(k,n,1) = mn$.

Let now $p \geq 2$. If $n < n_1$, where

$$n_1 = \begin{cases} p(k-1), & \text{if } m = 2 \text{ and } p \text{ is even or } m > 2,\\ p(k-1)+1, & \text{if } m = 2 \text{ and } p \text{ is odd},\end{cases}$$

then by Theorem 4 we have that $f^*(k,n,p) = 0$. Assume that $n \geq n_1$ and consider two families of sets. Let $\mathcal{S}_1$ be a family of all p-element k-independent sets $S \subseteq V(C_n \times K_m)$ such that for each $i = 1,...,k-1$, $j = 1,...,m$ the vertex $(t_i, y_j) \notin S$. Let $\mathcal{S}_2$ be a family of all p-element k-independent sets $S \subseteq V(C_n \times K_m)$ such that there exists $1 \leq i \leq k-1$ and $1 \leq j \leq m$ that $(t_i, y_j) \in S$. Evidently $f^*(k,n,p) = |\mathcal{S}_1| + |\mathcal{S}_2|$. By the definition of $C_n \times K_m$ it follows that $|\mathcal{S}_1| = f(k,n-(k-1),p)$. Now consider the family $\mathcal{S}_2$ and by Lemma 1, without loos of generality, assume that $(t_1, y_1) \in S$. So we have to determine D_n^p for $p \geq 2$. Evidently $(t_1, y_q) \notin S$, where $q = 2,...,m$ and $(t_i, y_j) \notin S$, where $i = 2,...,k-1,n,...,n-(k-2)$ and $j = 1,...,m$. Let S^* be an arbitrary $(p-1)$-element k-independent set of the graph $P_{n-(2k-3)} \times K_m$. Then $S^* \cup \{t_1, y_1\}$ can be a p-element k-independent set of the graph $C_n \times K_m$ and we have $f(k,n-(2k-3),p-1)$ such sets. Moreover $S^* \cup \{(t_1, y_1)\}$ is a p element k - independent set of $C_n \times K_m$ if and only if $(t_k, y_1) \notin S^*$ or

$(t_{n-(k-2)}, y_1) \notin S^*$. Hence from the number $f(k, n-(2k-3), p-1)$ we have to subtract the number r of all subsets S^* containing (t_k, y_1) or $(t_{n-(k-2)}, y_1)$. To calculate it consider the following cases:

If $p = 2$, then S^* has only one vertex, so if $n = 2k-2$, then $r = 1$ and if $n > 2k-2$, then $r = 2$. Let $p \geq 3$. Then we have $B^{p-1}_{n-(2k-3)}$ subsets containing (t_k, y_1) and also $B^{p-1}_{n-(2k-3)}$ subsets S^* containing $(t_{n-(k-2)}, y_1)$. But in some case the number $2B^{p-1}_{n-(2k-3)}$ double calculated subsets S^* containing (t_k, y_1) and $(t_{n-(k-2)}, y_1)$ simultaneously. Using Lemma 2 and by simple calculations we obtain that if $n < n_2$, where

$$n_2 = \begin{cases} p(k-1)+1, & \text{if } p \text{ is even,} \\ p(k-1), & \text{if } p \text{ is odd,} \end{cases}$$

then does not exist a subset S^* containing $(t_k, y_1), (t_{n-(k-2)}, y_1)$ simultaneously. Hence $r = 2B^{p-1}_{n-(2k-3)}$. If $n \geq n_2$, then from Lemma 3 we have $r = D^{p-2}_{n-(2k-2)}$.

Consequently $D^p_n = f(k, n-(2k-3), p-1) - (2B^{p-1}_{n-(2k-3)} - D^{p-2}_{n-(2k-2)})$. Since (t_1, y_1) is a fixed vertex from $m(k-1)$ vertices so we have that $|\mathcal{S}_2| = m(k-1)[f(k, n-(n-(2k-3), p-1) - 2B^{p-1}_{n-(2k-3)} + D^{p-2}_{n-(2k-2)}]$.

From the above cases we obtain that $f^*(k, n, p) = f(k, n-(k-1), p) + m(k-1)D^p_n$ and

$$D^p_n = f(k, n-(2k-3), p-1) - \begin{cases} 1, & \text{for } n = 2k-2 \text{ and } p = 2, \\ 2, & \text{for } n > 2k-2 \text{ and } p = 2, \\ 2B^{p-1}_{n-(2k-3)}, & \text{for } p \geq 3 \text{ and } n < n_2, \\ 2B^{p-1}_{n-(2k-3)} - D^{p-2}_{n-(2k-2)}, & \text{for } p \geq 3 \text{ and } n \geq n_2, \end{cases}$$

where $n_2 = \begin{cases} p(k-1)+1, & \text{if } p \text{ is even,} \\ p(k-1), & \text{if } p \text{ is odd.} \end{cases}$

Thus the theorem is proved. □

Corollary 1. *Let $k \geq 2$, $n \geq 1$, $m > 1$, $p \geq 0$ be integers. Then* $N_kI(C_n \times K_m) = \sum_{p \geq 0} f^*(k, n, p)$.

Corollary 2. *Let $n \geq 1$, $m > 1$, $x \geq 1$ be integers. Then* $F((C_n \times K_m)[K_x]) = \sum_{p \geq 0} f^*(2, n, p)x^p$.

Corollary 3. *Let* $k \geq 3$, $n \geq 2$, $m \geq 1$, $x \geq 1$ *be integers. Then for an arbitrary sequence* $h_n = (H_i)_{i \in \{1,...,\}}$ *of vertex disjoint graphs such that* $|V(H_i)| = x$, *for* $i = 1, ..., n$ *we have* $F_k((C_n \times K_m)[h_n]) = \sum_{p \geq 0} f^*(k, n, p)x^p$.

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