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ON THE HILBERT FUNCTIONS OF DISJOINT LINES IN $\mathbf{P}^n$ AND OF THEIR SPANNING SUBSETS

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Abstract: Here we discuss the following questions. Fix integers $n \geq 3$ and $x > 0$. Let $\{P_i, Q_i\}$, $1 \leq i \leq x$, x non-ordered set of unordered pairs of distinct points of $\mathbf{P}^n$. Set $S := \cup_{i=1}^{x} \{P_i, Q_i\}$, $D_i := \langle \{P_i, Q_i\} \rangle$ and $X := \cup_{i=1}^{x} D_i \subset \mathbf{P}^n$. Assume that X is the union of x distinct lines, i.e. assume $\sharp(S) = 2x$, $\{P_j, Q_j\} \cap D_i = \emptyset$ for all $i \neq j$ and that no plane contains $\{P_i, Q_i, P_j, Q_j\}$ for some $i \neq j$. What can be said about the Hilbert function h_X of X in terms of the Hilbert function H_S of X (and viceversa)? Can the base locus of $|\mathcal{I}_S(t)|$ contain a line?

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Here we discuss the following two questions.

Question 1. Fix integers $n \geq 3$ and $x > 0$. Let $\{P_i, Q_i\}$, $1 \leq i \leq x$, x non-ordered set of unordered pairs of distinct points of $\mathbf{P}^n$. Set $S := \cup_{i=1}^{x} \{P_i, Q_i\}$, $D_i := \langle \{P_i, Q_i\} \rangle$ and $X := \cup_{i=1}^{x} D_i \subset \mathbf{P}^n$. Assume that X is the union of x distinct lines, i.e. assume $\sharp(S) = 2x$, $\{P_j, Q_j\} \cap D_i = \emptyset$ for all $i \neq j$ and that no plane contains $\{P_i, Q_i, P_j, Q_j\}$ for some $i \neq j$. What can be said about the Hilbert function h_X of X in terms of the Hilbert function H_S of X?

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Question 2. Let $X = D_1 \cup \cdots \cup D_x \subset \mathbf{P}^n$, $n \geq 3$, a disjoint union of x lines $D_1, \cdots, D_x$. Fix $\{P_i, Q_i\} \subset D_i$, $1 \leq i \leq x$, such that $P_i \neq Q_i$ for all i. Set $S := \cup_{i=1}^{x} \{P_i, Q_i\}$. What can be said about the Hilbert function H_S of S in terms of the Hilbert function H_X of X? Can the base locus of $|\mathcal{I}_S(t)|$ contain a line?

For some results concerning the Hilbert function of disjoint lines in $\mathbf{P}^n$, see [3] and [1].

We will only use the following result easily proved as in [3].

Proposition 1. *Fix integers n, d, x such that $n \geq 3$ and $0 < x \leq n - d$. Set z be the minimal integer $t > 0$ sucht that $\binom{n+t}{n} \geq d(t+1)$. Let $D \subset \mathbf{P}^n$ be a line and a general $A \subset D$ such that $\sharp(A) = x$. Let $X \subset \mathbf{P}^n$ be a general union of d lines with the only restriction that $X \cap D = A$. Then X has the best possible postulation with the constraint $X \cap D = A$, i.e. if $x \leq z+1$, then $h^1(\mathbf{P}^n, \mathcal{I}_X(t)) = 0$ for all $t \geq z$ and $h^0(\mathbf{P}^n, \mathcal{I}_X(t)) = 0$ for all $t < z$, while if $x \geq t+2$, then $h^1(\mathbf{P}^n, \mathcal{I}_X(t)) = 0$ for all $t \geq x-1$ and $h^1(\mathbf{P}^n, \mathcal{I}_X(t)) = x-t-1$ for all $z \leq t \leq x-2$.*

Remark 1. Fix an integer $d > 0$, a closed subscheme $Z \subset \mathbf{P}^n$ and a line D such that $h^0(\mathbf{P}^n, \mathcal{I}_{Z\cup D}) \leq h^0(\mathbf{P}^n, \mathcal{I}_Z) - 2$. Then $h^0(\mathbf{P}^n, \mathcal{I}_{Z\cup\{P,Q\}}) = h^0(\mathbf{P}^n, \mathcal{I}_Z) - 2$ for a general $(P, Q) \in D \times D$.

Proposition 2. *Fix, n, d, x, D, A, X as in Proposition 1 and write $D = D_1 \cup \cdots \cup D_x$ with each D_i a line. Let $S \subset X$ be a general subset such that $\sharp(S \cap D_i) = 2$ for all i. Then S has maximal rank, i.e. $h^1(\mathbf{P}^n, \mathcal{I}_X(t)) = 0$ if $\binom{n+t}{n} \geq 2d$ and $h^0(\mathbf{P}^n, \mathcal{I}_X(t)) = 0$ if $\binom{n+t}{n} \leq 2d$.*

Proof. Two general points of a general line of $\mathbf{P}^n$ are two general points of $\mathbf{P}^n$. Hence it is sufficient to do the case $x = 2$. We will do induction on the integer d, the case $d = 1$ being obvious. Assume $d \geq 2$ and take a general union $Y \subset \mathbf{P}^n$ of $d-1$ lines such that $\sharp(Y \cap D) = d-1$. Let $E \subset Y$ be a general subset such that $\sharp(E \cap T) = 2$ for every line $T \subseteq Y$. Fix a general line $R \subset \mathbf{P}^n$ such that $R \cap D \neq \emptyset$ $F \subset R$ such that $\sharp(F) = 2$. It is sufficient to prove $h^0(\mathbf{P}^n, \mathcal{I}_{E\cup F}(t)) = \max\{0, h^0(\mathbf{P}^n, \mathcal{I}_E(t)) - 2\}$ for all $t > 0$. Fix an integer $t > 0$. Since R contains a general point of $\mathbf{P}^n$, it is sufficient to do the case $h^0(\mathbf{P}^n, \mathcal{I}_E(t)) \geq 2$. By Remark 1 it is sufficient to prove that $h^0(\mathbf{P}^n, \mathcal{I}_{E\cup R}(t)) \leq h^0(\mathbf{P}^n, \mathcal{I}_E(t)) - 2$. Assume that this is not true. Since R contains a general point of $\mathbf{P}^n$, our assumption implies $h^0(\mathbf{P}^n, \mathcal{I}_{E\cup R}(t)) = h^0(\mathbf{P}^n, \mathcal{I}_E(t)) - 1$. Fix a general $P \in \mathbf{P}^n$ and let M be any degree t hypersurface containing $E \cup \{P\}$. Since we may take as R a general line through P and contained in the plane $\langle \{P\} \cup D \rangle$, we get $\langle \{P\} \cup D \rangle \subseteq M$. Varying P in M we get that M is a cone with D contained in its vertex. In particular we get $Y \cup R \subset M$. Hence (with the notation of Proposition 1) we have $z \geq t$ and $h^0(\mathbf{P}^n, \mathcal{I}_X(t)) = \binom{n+t}{n} - 2d + 1$, contradicting Proposition 1 or a very easy form of its proof. $\square$

Remark 2. Fix integers $n \geq 3$, $t \geq 2$, and an integral hypersurface $M \subset \mathbf{P}^n$ contained infinitely many pairwise disjoint lines. Such hypersurfaces exist for all pairs (n, t). Fix an integer d such that $2d \geq \binom{n+t}{n}$ and let $X \subset M$ a union of d disjoint lines. For all $S \subset X$ such that $\sharp(S) = 2d$ we have $h^0(\mathbf{P}^n, \mathcal{I}_S(t)) \neq 0$ and $h^1(\mathbf{P}^n, \mathcal{I}_S(t)) \neq 0$. Hence the maximal rank property stated in Proposition 2 is not true for an arbitrary X.

Proposition 3. *Fix integers $d > 0$ and $t \geq 2$ and an integral degree t surface $M \subset \mathbf{P}^3$ containing infinitely many lines. Then there exists a union $X \subset M$ of d distinct lines and $S \subset X_{reg}$ such that $\sharp(S \cap R) = 2$ for every irreducible conponent R of X and $h^0(\mathbf{P}^3, \mathcal{I}_S(t-1)) = \max\{0, \binom{t+2}{3} - 2d\}$.*

Proof. Since the Hilbert schemes of all lines in M has finitely many component, there is an integral positive dimensional family T of lines contained in M. Take $X = R_1 \cup \cdots \cup R_d$ with $(R_1, \ldots, R_d)$ general in T^d. If M is a cone, then Proposition 3 is easy. Hence we assume that M is not a cone. The generality of $(R_1, \ldots, R_d)$ and the assumption that M is not a cone imply $R_i \cap R_j = \emptyset$ for all $i \neq j$. We will use induction on d, the case $d = 1$ being obvious. Assume $d \geq 2$ and set $Y := R_1 \cup \cdots \cup R_{d-1}$. Let E be a general subset of Y such that $\sharp(E \cap R_i) = 2$ for all $i = 1, \ldots, d-1$. The inductive assumption implies $h^0(\mathbf{P}^3, \mathcal{I}_E(t-1)) = \max\{0, \binom{t+2}{3} - 2d + 2\}$. Since a general $R \in T$ pass through a general point of M, M is integral and $\deg(M) = t$, it is sufficient to do the case $\binom{t+2}{3} - 2d + 2 \geq 2$. By Remark 1 it is sufficient to prove $h^0(\mathbf{P}^3, \mathcal{I}_{E \cup R}(t-1)) \leq \binom{t+2}{3} - 2d$ for a general $R \in T$. Assume $h^0(\mathbf{P}^3, \mathcal{I}_{E \cup R}(t-1)) = \binom{t+2}{3} - 2d + 1$ for a general $R \in T$. Fix a degree $t-1$ surface W such that $E \subset W$ and a general $P \in W \cap M$. The generality of P implies $h^0(\mathbf{P}^3, \mathcal{I}_{E \cup R}(t-1)) = \binom{t+2}{3} - 2d + 1$. Thus W contains the line $R_P \in T$ containing P. Since this is true for a general $P \in W \cap M$ and $W \cap M$ is infinite, we get $M \subseteq W$, contradiction. $\square$

Proposition 4. *Fix integers $d > 0$, $t \geq 4$ and $\binom{t+3}{3} > 2d$. Let $X \subset \mathbf{P}^3$ a union of d lines. Let B be the set of all $S \subset X_{reg}$ such that $\sharp(S \cap R) = 2$ for every irreducible component R of X. Assume $h^0(\mathbf{P}^3, \mathcal{I}_S(t)) = \binom{t+3}{3} - 2d$ for a general $S \in B$. Then a general degree x surface W containing a general $S \in B$ contains no line.*

Proof. Fix a general $A \subset \mathbf{P}^3$ such that $sharp(A) = \binom{t+3}{3} - 2d - 1$. By assumption for a general $S \in B$ there is a unique degree x surface W_S containing $A \cup S$. A very particular case of a theorem of Max Noether ([2]) says that a general degree x surface $W \subset \mathbf{P}^3$ contains no line. Since $\dim(B) = 2d$, a dimensional count conclude the proof. □

Every surface $W \subset \mathbf{P}^3$ such that $\deg(W) \leq 3$ contains a line. This observation explains the restriction "$t \geq 4$" in the statement of Proposition 4.

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