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REPRESENTATION THEOREMS FOR SOLUTIONS OF AN ITERATIVE FUNCTIONAL INEQUALITY

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Abstract: In this paper there are proved two representation theorems for continuous solutions of a some functional inequality of n-th order (1) with functional coefficients.

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1. Introduction

In the present paper we are concerned with the following functional inequality of n-th order

$$\psi \circ f^n \leq (-g \circ f^{n-1})(\psi \circ f^{n-1}) + (G_{n-1} \circ f)(\psi \circ f) + G_n \cdot \psi \,, \tag{1}$$

where $n \geq 2$ is a fixed integer, ψ is unknown function, f and g are given functions, f^i denotes the i-th iterate (cf. [4]) of the function f and the functions G_n are defined by the recurence formula

$$G_0(x) = 1, \qquad G_{k+1}(x) = g[f^k(x)]G_k(x), \quad k \in \mathbf{N}\,. \tag{2}$$

We shall be interested in real, continuous solutions of inequality (1).

We make the following assumptions about the given functions f and g:

(H_1) The function f is continuous and strictly increasing in $I = [0, a)$, $a > 0$. Moreover $0 < f(x) < x$ for $x \in I^\star = I \setminus \{0\}$.

(H_2) The function g is continuous in I and $g(x) < 0$ for $x \in I^\star$.

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First let us note that the form of inequality (1) implies that one can study instead, the equivalent system

$$\psi \circ f^{n-1} = G_{n-1}\psi + G_{n-1}\overline{\psi}\,, \tag{3}$$

$$\overline{\psi} \circ f + g\overline{\psi} \leq 0\,. \tag{4}$$

Inequality (4) is related to the functional equation

$$\varphi[f(x)] = -g(x)\varphi(x) \tag{5}$$

Under hypotheses (H_1), (H_2) inequality (4) was studied by D. Brydak in [1], cf. also D. Brydak, B. Choczewski [2] and M. Kuczma, B. Choczewski, R. Ger [5]. Some results about functional inequalities of the second order were obtained by M. Stopa in [6] and D. Brydak, B. Choczewski in [3]. In the present paper we make use of results from [1].

2. Sufficient Condition

We remind the following definition and lemma (cf. [6]):

Definition 1. Function $\eta : I \longrightarrow \mathbf{R}$ is called $\{f\}$-decreasing in I iff $\eta[f(x)] \leq \eta(x)$ for $x \in I$.

Lemma 2. *If $\varphi : I \longrightarrow \mathbf{R}_+$ is solution of the equation (5), η is a $\{f\}$-decreasing function in I, then the function*

$$\overline{\psi}(x) = \eta(x)\varphi(x) \tag{6}$$

satisfies inequality (4).

We prove the following generalization of this lemma for the case of the inequality (1).

Theorem 3. *Assume that (H_1) and (H_2) hold. Let $\varphi : I \longrightarrow \mathbf{R}^+$ be a continuous solution of the equation (5). Then each function*

$$\psi(x) = -\varphi(x)\sum_{i=0}^{\infty}(-1)^{i(n-1)}\eta[f^{i(n-1)}(x)]\,, \tag{7}$$

where η is a continuous, $\{f\}$-decreasing function such that

$$\eta(x) \neq 0 \qquad x \in I^{\star}, \tag{8}$$

$$\lim_{x\to 0^+} \frac{\eta[f^{n-1}(x)]}{\eta(x)} < 1\,, \tag{9}$$

and

$$\eta(0) = 0 \tag{10}$$

is a continuous solution of the inequality (1).

Proof. It is easy to note that condition (9) implies that the series in formula (7) is almost uniformly convergent in I ([4] Chapter II, p. 53, Theorem 2.7, [5] Chapter 3.1C). We prove that the function ψ given by (7) satisfies inequality (1).

It is sufficient to show that the function $\overline{\psi}$ given by (3), where ψ is given by (7), satisfies (4).

Let us introduce notation $\eta_i^f := \eta \circ f^{i(n-1)}$ for $i \in \mathbf{N} \cup \{0\}$ and put $\alpha_i := (-1)^{i(n-1)}$.

We have

$$\begin{aligned}
\overline{\psi}(x) &= \frac{\psi[f^{n-1}(x)]}{G_{n-1}(x)} - \psi(x) = -\frac{\varphi[f^{n-1}(x)]}{G_{n-1}(x)} \sum_{i=0}^{\infty} \alpha_i \eta_{i+1}^f(x) + \varphi(x) \sum_{i=0}^{\infty} \alpha_i \eta_i^f(x) \\
&= \varphi(x) \left(-\frac{|G_{n-1}(x)|}{G_{n-1}(x)} \sum_{i=0}^{\infty} \alpha_i \eta_{i+1}^f(x) + \sum_{i=0}^{\infty} \alpha_i \eta_i^f(x) \right) \\
&= \varphi(x) \left(-(-1)^{n-1} \sum_{i=0}^{\infty} \alpha_i \eta_{i+1}^f(x) + \sum_{i=0}^{\infty} \alpha_i \eta_i^f(x) \right) \\
&= \varphi(x) \left(-\sum_{i=0}^{\infty} \alpha_{i+1} \eta_{i+1}^f(x) + \sum_{i=0}^{\infty} \alpha_i \eta_i^f(x) \right) \\
&= \varphi(x) \left(\lim_{k\to\infty} - \sum_{i=0}^{k} \alpha_{i+1} \eta_{i+1}^f(x) + \lim_{k\to\infty} \sum_{i=0}^{k} \alpha_i \eta_i^f(x) \right) \\
&= \varphi(x) \left(\lim_{k\to\infty} \sum_{i=0}^{k} \alpha_i \eta_i^f(x) - \alpha_i \eta_{i+1}^f(x) \right) \\
&= \varphi(x) \left(\lim_{k\to\infty} \eta(x) - \alpha_{k+1} \eta_{k+1}^f(x) \right) = \varphi(x)\eta(x).
\end{aligned}$$

By virtue of Lemma 1 this implies that $\overline{\psi}$ is a solution of inequality (4) which ends the proof of the theorem. $\square$

3. Necessary Condition

Now we consider assumptions which implies that solution of inequality (1) have to a form (7). For this purpose we must limited to some class of functions.

Definition 4. We denote by Φ the family od all continuous solutions $\varphi : I \longrightarrow \mathbf{R}$ of equation (5) such that

$$\varphi(x) > 0\,, \quad x \in I^\star.$$

Definition 5. We denote by Ψ_d the family of continuous functions ψ defined in I for which there exists a continuous solution $\varphi \in \Phi$ of equation (5) such that

$$\lim_{x\to 0^+} \frac{\psi(x)}{\varphi(x)} = d < \infty\,.$$

Remark 6. It is obvious (see [1]) that for example if $|g(0)| < 1$ then $\Phi \neq \emptyset$.

Theorem 7. *If (H_1) and (H_2) hold and $\psi \in \Psi_d$ is a solution of inequality (1) such that*

$$d = 0 \qquad \text{iff} \quad n \in 2\mathbf{N},$$

$$\psi(x) \neq 0 \qquad \text{for} \quad x \in I^\star,$$

the limit

$$s = \lim_{x\to 0^+} \frac{\psi[f^{n-1}(x)]}{\psi(x)} \tag{11}$$

exists and

$$0 < s < |g(0)|^{n-1}, \tag{12}$$

then there exists an $\{f\}$-decreasing, continuous function η, satisfying (8)-(10) and a continuous solution $\varphi \in \Phi$ of equation (5) such the function ψ given by (7) is a continuous solution to inequality (1).

Proof. Denote by $\overline{\psi}$ continuous solution of inequality (4) defined by formula (3). We have

$$\lim_{x\to 0^+} \frac{\overline{\psi}(x)}{\varphi(x)} = \lim_{x\to 0^+} \left(\frac{\psi[f^{n-1}(x)]}{G_{n-1}(x)\varphi(x)} - \frac{\psi(x)}{\varphi(x)} \right)$$
$$= \lim_{x\to 0^+} (-1)^n \frac{\psi[f^{n-1}(x)]}{\varphi[f^{n-1}(x)]} - \frac{\psi(x)}{\varphi(x)} = 0\,.$$

This implies that $\eta : I \longrightarrow \mathbf{R}$ given by

$$\eta(x) := \begin{cases} \frac{\overline{\psi}(x)}{\varphi(x)}\,, & x \in I^{\star}, \\ 0\,, & x = 0, \end{cases} \tag{13}$$

is a continuous, $\{f\}$-decreasing function such that (6) holds. Moreover, from (11) we obtain

$$\lim_{x\to 0^+} \frac{\overline{\psi}[f^{n-1}(x)]}{\overline{\psi}(x)} = \lim_{x\to 0^+} \frac{\psi[f^{n-1}(x)]}{\psi(x)} \left(\frac{\left(\frac{\psi[f^{2n-2}(x)]}{\psi[f^{n-1}(x)]}\right)\frac{1}{G_{n-1}(f^{n-1}(x))} - 1}{\left(\frac{\psi[f^{n-1}(x)]}{\psi(x)}\right)\frac{1}{G_{n-1}(x)} - 1} \right) = s\,. \tag{14}$$

By induction we get

$$\psi(x) = \frac{\psi[f^{k(n-1)}(x)]}{G_{k(n-1)}(x)} - \sum_{i=0}^{k-1} \frac{\overline{\psi}[f^{i(n-1)}(x)]}{G_{i(n-1)}(x)} \qquad \text{for} \quad k \in \mathbf{N}$$

and consequently, by (14):

$$\psi(x) = \frac{\psi[f^{k(n-1)}(x)]}{G_{k(n-1)}(x)} - \varphi(x)\sum_{i=0}^{k-1}(-1)^{i(n-1)}\eta[f^{i(n-1)}(x)]. \tag{15}$$

Let us observe that (6), (11), (12) imply

$$\lim_{k\to\infty} \left| \frac{\psi[f^{(k+1)(n-1)}(x)]}{G_{(k+1)(n-1)}(x)} \frac{G_{k(n-1)}(x)}{\psi[f^{k(n-1)}(x)]} \right| = \frac{s}{|g(0)|^{n-1}} < 1$$

and

$$\lim_{i\to\infty} \left| \frac{\eta[f^{(i+1)(n-1)}(x)]}{\eta[f^{i(n-1)}(x)]} \right| = \lim_{i\to\infty} \left| \frac{\overline{\psi}[f^{(i+1)(n-1)}(x)]G_{i(n-1)}(x)}{\overline{\psi}[f^{i(n-1)}(x)]G_{(i+1)(n-1)}(x)} \right| = \frac{s}{|g(0)|^{n-1}} < 1.$$

Then, by virtue of d'Alembert's criterion for sequences or series, respectively if $k \to \infty$ in (15) then we obtain formula (7). This ends the proof of theorem. ☐

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