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ON NEW SUBCLASS OF ANALYTIC UNIVALENT FUNCTION

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Abstract: In the present paper, the authors introduce and study a new subclass of normalized analytic functions in the open unit disk. The results in this paper generalize many earlier results in the literature.

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1. Introduction and Motivation

Let $\mathcal{A}$ be the class of functions f *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are *analytic* in the *open* unit disk

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

As usual, we denote by S the subclass of $\mathcal{A}$, consisting of functions which are also *univalent* in D. We recall here the definitions of the well-known classes of starlike function and convex function:

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$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > 0,\ z \in D \right\},$$
$$S^c = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} \right) > 0,\ z \in D \right\}.$$

Let w be a fixed point in D and

$$\mathcal{A}(w) = \left\{ f \in H(D) : f(w) = f'(w) - 1 = 0 \right\}.$$

In [11], Kanas and Ronning introduced the following classes

$$S_w = \{ f \in \mathcal{A}(w) : f \text{ is univalent in } D \},$$
$$ST_w = \left\{ f \in \mathcal{A}(w) : \operatorname{Re}\left(\frac{(z-w)\, f'(z)}{f(z)} \right) > 0,\ z \in D \right\}, \tag{2}$$
$$CV_w = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{(z-w)\, f''(z)}{f'(z)} \right) > 0,\ z \in D \right\}, \tag{3}$$

and these classes were extensively studied by Acu and Owa [1].

The class S_w^* is defined by geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class S_w^c is defined by the property that the image of any circular arc centered at w is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [8] and [9] for uniformly starlike and convex functions, except that in this case the point w is fixed.

The function $f(z)$ in S_w is said to be starlike of order β if and only if

$$\operatorname{Re}\left\{ \frac{(z-w)f'(z)}{f(z)} \right\} > \beta \quad (z \in D) \tag{4}$$

for some $\beta\,(0 \le \beta < 1)$. We denote by $ST_w(\beta)$ the class of all starlike functions of order β. Similarly, a function $f(z)$ in S_w is said to be convex of order β if and only if

$$\operatorname{Re}\left(1 + \frac{(z-w)\, f''(z)}{f'(z)} \right) > \beta \quad (z \in D) \tag{5}$$

for some $\beta\,(0 \le \beta < 1)$. We denote by $CV_w(\beta)$ the class of all convex functions of order β. We note that the class $ST_0(\beta)$ and various other subclasses of $ST_w(\beta)$ have been studied rather extensively by Nehari and Netanyahu [14], Acu and Owa [1], Clunie [5], Pommerenke [15, 16], Miller [12], Royster [17], and others (cf., e.g. Bajpai [18], Goel and Sohi [7], Mogra et al [13], Uralegaddi and Ganigi [19], Cho et al [4], Aouf [2], and Uralegaddi and Somantha [20] and [21]; see also Duren [6], pp. 29 and 137, and Srivastava and Owa [18]).

Let S_w denoted the subclass of $\mathcal{A}(w)$ consisting of the function of the form

$$f(z) = \frac{\alpha}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n \quad (a_n \geq 0)\,, \tag{6}$$

where $\alpha = \text{Res}\ (z, w)$ with $0 < \alpha \leq 1$.

For the function $f(z)$ in the class S_w, we define

$$I^0 f(z) = f(z),$$
$$I^1 f(z) = (z-w) f'(z) + \frac{2\alpha}{z-w},$$
$$I^2 f(z) = (z-w)\left(I^1 f(z)\right)' + \frac{2\alpha}{z-w},$$

and for $k = 1, 2, 3, ...$ we can write

$$I^k f(z) = (z-w)\left(I^{k-1} f(z)\right)' + \frac{2\alpha}{z-w} = \frac{\alpha}{z-w} + \sum_{n=1}^{\infty} n^k a_n (z-w)^n. \tag{7}$$

In the case $w = 0$ the differential operator I^k, was given by Frasin and Darus [10].

With the help of the differential operator I^k, we define the class $ST_w(k, \beta)$ as follows.

Definition 1. The function $f(z) \in S_w$ is said to be a member of the class $ST_w(k, \beta)$ if it satisfies

$$\left| \frac{(z-w)\left(I^k f(z)\right)'}{I^k f(z)} + 1 \right| < \left| \frac{(z-w)\left(I^k f(z)\right)'}{I^k f(z)} + 2\beta - 1 \right|$$

($k \in \mathbb{N}_0 = \mathbb{N} \cup 0$), for some β $(0 \leq \beta < 1)$ and for all z $(0 \leq z < 1)$ in D.

It is easy to check that $ST_w(0, \beta)$ is the class of starlike functions of order β and $ST_w(0, 0)$ gives the starlike functions for all D.

Let us write

$$S_W^*(k, \beta) = ST_w(k, \beta) \cap S_w\,, \tag{8}$$

where S_w is the class of functions of the form (6) that are analytic and univalent in D. In our paper we will consider the properties for the classes $ST_w(k, \beta)$ and $S_W^*(k, \beta)$.

2. Coefficient Estimates

Our first result provides a sufficient condition for a function, analytic in D, to be in $ST_w(k,\beta)$.

Theorem 1. *Let the function $f(z)$ be defined by (6) and β $(0 \le \beta < 1)$. If*

$$\sum_{n=1}^{\infty} n^k (n+\beta) |a_n| \le \alpha (1-\beta) \quad (k \in \mathbb{N}_0), \tag{9}$$

then $f(z) \in ST_w(k,\beta)$.

Proof. Suppose that (9) holds true for $0 \le \beta < 1$. Consider the expression

$$M(f,f') = \left| (z-w) \left(I^k f(z) \right)' + I^k f(z) \right| - \left| (z-w) \left(I^k f(z) \right)' + (2\beta - 1) I^k f(z) \right|,$$

then for $0 < |z-w| = r < 1$ we have

$$M(f,f') = \left| \sum_{n=1}^{\infty} n^k (n+1) a_n (z-w)^n \right| - \left| \frac{2\alpha(\beta-1)}{z-w} + \sum_{n=1}^{\infty} n^k (n+2\beta-1) a_n (z-w)^n \right|,$$

$$rM(f,f') \le \sum_{n=1}^{\infty} n^k (n+1) |a_n| r^{n+1} - \left(2\alpha(1-\beta) + \sum_{n=1}^{\infty} n^k \times (n+2\beta-1) |a_n| r^{n+1} \right) \le \sum_{n=1}^{\infty} 2n^k (n+\beta) |a_n| r^{n+1} - 2\alpha(1-\beta). \tag{10}$$

The inequality in (10) holds true for all r $(0 \le r < 1)$. Therefore, letting $r \to 1$ in (10), we obtain

$$M(f,f') \le \sum_{n=1}^{\infty} 2n^k (n+\beta) |a_n| - 2\alpha(1-\beta)$$

by the hypothesis (9). Hence it follows that

$$\left|\frac{(z-w)\left(I^k f(z)\right)'}{I^k f(z)}+1\right| < \left|\frac{(z-w)\left(I^k f(z)\right)'}{I^k f(z)}+2\beta-1\right|,$$

so that $f(z) \in ST_w(k,\beta)$. Hence the theorem is proved. □

Corollary 1. *Let $k=\beta=0$ in Theorem 1, then we have $\sum\limits_{n=1}^{\infty} n\,|a_n| \le \alpha$, therefore f is starlike univalent in all $z \in D$ with condition $(0 \le z < 1)$.*

Corollary 2. *Let $k=\beta=0$ and $\alpha=1$ in Theorem 1, then we have $\sum\limits_{n=1}^{\infty} n\,|a_n| \le 1$, therefore f is starlike univalent in D.*

Corollary 3. *Let $k=1$ and $\beta=0$ in Theorem 1, then we have $\sum\limits_{n=1}^{\infty} n^2\,|a_n| \le \alpha$, therefore f is convex univalent in all $z \in D$ with condition $(0 \le z < 1)$.*

Corollary 4. *Let $k=1$, $\beta=$ and $\alpha=1$ in Theorem 1, then we have $\sum\limits_{n=1}^{\infty} n^2\,|a_n| \le 1$, therefore f is convex univalent in D.*

Next we give a necessary and sufficient condition for a function $f \in S_w$ to be in the class $S^*_W(k,\beta)$.

Theorem 2. *Let the function $f(z)$ be defined by (6) and let $f(z) \in S_w$. Then $f(z) \in S^*_W(k,\beta)$ if and only if (9) is satisfied. The result (9) is sharp.*

Proof. In view of Theorem 1, it sufficient to show that the "only if" part. Assume that $f \in S^*_W(k,\beta)$. Then

$$\left|\frac{\frac{(z-w)\left(I^k f(z)\right)'}{I^k f(z)}+1}{\frac{(z-w)\left(I^k f(z)\right)'}{I^k f(z)}+2\beta-1}\right| = \left|\frac{\sum\limits_{n=1}^{\infty} n^k\,(n+1)\,a_n(z-w)^n}{\frac{2\alpha(1-\beta)}{z-w}-\sum\limits_{n=1}^{\infty} n^k\,(n+2\beta-1)\,a_n(z-w)^n}\right| < 1 \quad (z \in D)\,. \tag{11}$$

Since $\mathrm{Re}(z) \le |z|$ for all z, it follows from (11) that

$$\mathrm{Re}\left\{\frac{\sum\limits_{n=1}^{\infty} n^k\,(n+1)\,a_n(z-w)^n}{\frac{2\alpha(1-\beta)}{z-w}-\sum\limits_{n=1}^{\infty} n^k\,(n+2\beta-1)\,a_n(z-w)^n}\right\} < 1 \quad (z \in D)\,. \tag{12}$$

We now choose the values $f(z)$ on the real axis so that $\frac{(z-w)\left(I^k f(z)\right)'}{I^k f(z)}$ is real. Upon clearing the denominator in (12) and letting $(z-w) \to 1$ through real values, we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} n^k\,(n+1)\,a_n \leq 2\alpha(1-\beta) - \sum_{n=1}^{\infty} n^k\,(n+2\beta-1)\,a_n, \\ &\sum_{n=1}^{\infty} 2n^k\,(n+\beta)\,a_n \leq 2\alpha(1-\beta), \end{aligned} \tag{13}$$

which immediately yields the required condition (11).

Our assertion in Theorem 2 is sharp for functions of the form

$$f_n(z) = \frac{\alpha}{z-w} + \frac{\alpha(1-\beta)}{n^k\,(n+\beta)}(z-w)^n (n \geq 1;\ k \in \mathbb{N}_0)\,. \tag{14}$$

□

Corollary 5. *Let the function $f(z)$ be defined by (6) and let $f(z) \in S_w$. If $f \in S^*_W(k,\beta)$, then*

$$a_n \leq \frac{\alpha(1-\beta)}{n^k\,(n+\beta)}\,. \tag{15}$$

The result (15) is sharp for functions $f_n(z)$ given by (14).

A distortion property for functions in the class $S^*_W(k,\beta)$ is contained in the following theorem.

Theorem 3. *If the function $f(z)$ be defined by (6) is in the class $S^*_W(k,\beta)$, then for $0 < |z-w| = r < 1$ we have*

$$\frac{\alpha}{r} - \frac{\alpha(1-\beta)}{(1+\beta)}r \leq |f(z)| \leq \frac{\alpha}{r} + \frac{\alpha(1-\beta)}{(1+\beta)}r \tag{16}$$

with equality for

$$f_1(z) = \frac{\alpha}{z-w} + \frac{\alpha(1-\beta)}{(1+\beta)}(z-w) \quad (z = ir, r)$$

and

$$\frac{\alpha}{r^2} - \frac{\alpha(1-\beta)}{(1+\beta)} \leq \left|f'(z)\right| \leq \frac{\alpha}{r^2} + \frac{\alpha(1-\beta)}{(1+\beta)} \tag{17}$$

with equality for

$$f_1(z) = \frac{\alpha}{z-w} + \frac{\alpha(1-\beta)}{(1+\beta)}(z-w) \quad (z = \pm ir, \pm r)\,.$$

Proof. Since $f \in S_W^*(k,\beta)$, Theorem 2 readily yields the inequality

$$\sum_{n=1}^{\infty} a_n \leq \frac{\alpha(1-\beta)}{n^k\,(n+\beta)}. \tag{18}$$

Thus, for $0 < |z-w| = r < 1$, and making use of (18) we have

$$|f(z)| \leq \left|\frac{\alpha}{z-w}\right| + \sum_{n=1}^{\infty} a_n\,|z-w|^n \leq \frac{\alpha}{r} + r\sum_{n=1}^{\infty} a_n \leq \frac{\alpha}{r} + \frac{\alpha(1-\beta)}{(1+\beta)}r$$

and

$$|f(z)| \geq \left|\frac{\alpha}{z-w}\right| - \sum_{n=1}^{\infty} a_n\,|z-w|^n \geq \frac{\alpha}{r} - r\sum_{n=1}^{\infty} a_n \geq \frac{\alpha}{r} - \frac{\alpha(1-\beta)}{(1+\beta)}r\,.$$

Also from Theorem 2, it follows that

$$\sum_{n=1}^{\infty} na_n \leq \frac{\alpha(1-\beta)}{n^{k-1}\,(n+\beta)}\,. \tag{19}$$

Hence

$$\begin{aligned} \left|f'(z)\right| &\leq \left|\frac{\alpha}{(z-w)^2}\right| + \sum_{n=1}^{\infty} na_n\,|z-w|^{n-1} \\ &\leq \frac{\alpha}{r^2} + \sum_{n=1}^{\infty} na_n \leq \frac{\alpha}{r^2} + \frac{\alpha(1-\beta)}{(1+\beta)} \end{aligned} \tag{20}$$

and

$$\left|f'(z)\right| \geq \left|\frac{\alpha}{(z-w)^2}\right| - \sum_{n=1}^{\infty} na_n\,|z-w|^{n-1} \geq \frac{\alpha}{r^2} - \sum_{n=1}^{\infty} na_n \geq \frac{\alpha}{r^2} - \frac{\alpha(1-\beta)}{(1+\beta)}.$$

This completes the proof of Theorem 3. □

3. Radii of Starlikeness and Convexity

The radii of starlikeness and convexity for the class $S_W^*(k,\beta)$ is given by the following theorems.

Theorem 4. *If the function $f(z)$ defined by (6) is in the class $S_W^*(k,\beta)$, then $f(z)$ is starlike of order $\delta (0 \le \delta < 1)$ in $|z-w| < r_1$, where*

$$r_1 = r_1(k,\beta,\delta) = \inf \left\{ \frac{n^k\,(n+\beta)\,(1-\delta)}{(n+2-\delta)\,(1-\beta)} \right\}^{\frac{1}{n+1}}. \tag{21}$$

The result is sharp for the function $f_n(z)$ given by (14).

Proof. It is sufficient to prove that

$$\left| \frac{(z-w)f'(z)}{f(z)} + 1 \right| \tag{22}$$

for $|z-w| < r_1$. We have

$$\left| \frac{(z-w)f'(z)}{f(z)} + 1 \right| = \left| \frac{\sum\limits_{n=1}^{\infty} (n+1)\,a_n (z-w)^n}{\frac{\alpha}{z-w} + \sum\limits_{n=1}^{\infty} a_n (z-w)^n} \right|$$

$$= \left| \frac{\sum\limits_{n=1}^{\infty} (n+1)\,a_n (z-w)^{n+1}}{\alpha + \sum\limits_{n=1}^{\infty} a_n (z-w)^{n+1}} \right| \le \frac{\sum\limits_{n=1}^{\infty} (n+1)\,a_n\,|z-w|^{n+1}}{\alpha - \sum\limits_{n=1}^{\infty} a_n\,|z-w|^{n+1}}. \tag{23}$$

Hence (23) holds true

$$\sum_{n=1}^{\infty} (n+1)\,a_n\,|z-w|^{n+1} \le (1-\delta)\,\alpha - \sum_{n=1}^{\infty} a_n\,|z-w|^{n+1}, \tag{24}$$

or

$$\frac{\sum\limits_{n=1}^{\infty} (n+2-\delta)\,a_n\,|z-w|^{n+1}}{(1-\delta)\,\alpha} \le 1 \tag{25}$$

with the aid of (9), (25) is true if

$$\frac{\sum\limits_{n=1}^{\infty} (n+2-\delta)\,|z-w|^{n+1}}{(1-\delta)\,\alpha} \le \frac{n^k\,(n+\beta)}{\alpha(1-\beta)} \quad (n \ge 1). \tag{26}$$

Solving (26) for $|z-w|$, we obtain

$$|z-w| < \left\{ \frac{n^k (n+\beta)(1-\delta)}{(1-\beta)(n+2-\delta)} \right\}^{\frac{1}{n+1}} . \tag{27}$$

This completes the proof of Theorem 4. □

Theorem 5. *If the function $f(z)$ be defined by (6) is in the class $S_W^*(k,\beta)$ then $f(z)$ is convex of order $\delta (0 \leq \delta < 1)$ in $|z-w| < r_2$, where*

$$r_2 = r_2(k,\beta,\delta) = \inf \left\{ \frac{n^{k-1} (n+\beta)(1-\delta)}{(n+2-\delta)(1-\beta)} \right\}^{\frac{1}{n+1}} . \tag{28}$$

The result is sharp for the function $f_n(z)$ given by (14).

Proof. By using the technique employed in proof of Theorem 4, we can show that

$$\left| \frac{(z-w)f''(z)}{f'(z)} + 1 \right| \leq (1-\delta) \tag{29}$$

for $|z-w| < r_2$, with the aid of Theorem 1. Thus we have the assertion of Theorem 5. □

4. Convex Linear Combinations

Our next result involves linear combinations of several functions of the type (14).

Theorem 6. *Let*

$$f_0(z) = \frac{\alpha}{z-w} \tag{30}$$

and

$$f_n(z) = \frac{\alpha}{z-w} + \frac{\alpha(1-\beta)}{n^k (n+\beta)}(z-w)^n \qquad (n \geq 1;\ k \in \mathbb{N}_0). \tag{31}$$

Then $f(z) \in S_W^*(k,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)\,, \tag{32}$$

where $\lambda_n \geq 0$ and $\sum\limits_{n=0}^{\infty} \lambda_n = 1$.

Proof. From (30), (31), and (32), it is easily seen that

$$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z)=\frac{\alpha}{z-w}+\frac{\alpha(1-\beta)}{n^k\,(n+\beta)}\lambda_n(z-w)^n\,. \tag{33}$$

Since

$$\sum_{n=1}^{\infty}\frac{n^k\,(n+\beta)}{\alpha(1-\beta)}\lambda_n.\frac{\alpha(1-\beta)}{n^k\,(n+\beta)}=\sum_{n=1}^{\infty}\lambda_n=1-\lambda_0\le 1,$$

it follows from Theorem 2 that the function $f(z)\in S_W^*(k,\beta)$.

Conversely, let us suppose that $f(z)\in S_W^*(k,\beta)$. Since

$$a_n\le\frac{\alpha(1-\beta)}{n^k\,(n+\beta)}\quad(n\ge 1;\;k\in\mathbb{N}_0),$$

setting

$$\lambda_n=\frac{n^k\,(n+\beta)}{\alpha(1-\beta)}a_n\quad(n\ge 1;\;k\in\mathbb{N}_0)$$

and

$$\lambda_0=1-\sum_{n=1}^{\infty}\lambda_n,$$

it follows that $f(z)=\sum\limits_{n=0}^{\infty}\lambda_n f_n(z)$. This completes the proof of the theorem. □

Finally, we prove the following theorem.

Theorem 7. *The class $S_W^*(k,\beta)$ is closed under convex linear combination.*

Proof. Suppose that the function $f_1(z)$ and $f_2(z)$ defined by

$$f_j(z)=\frac{\alpha}{z-w}+\sum_{n=1}^{\infty}a_{n,j}(z-w)^n\quad(j=1,2;\;\;z\in D) \tag{34}$$

are in the class $S_W^*(k,\beta)$.

Setting

$$f(z)=\mu f_1(z)+(1-\mu)f_2(z)\quad(0\le\mu<1). \tag{35}$$

From (34) we can write

$$f(z)=\frac{\alpha}{z-w}+\sum_{n=1}^{\infty}\left\{\mu a_{n,1}+(1-\mu)\,a_{n,2}\right\}(z-w)^{n} \tag{36}$$

$((0\leq\mu<1);z\in D)$. Thus in view of Theorem 2, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\left[n^{k}\,(n+\beta)\right](\mu a_{n,1}+(1-\mu)\,a_{n,2})\\&\qquad=\mu\sum_{n=1}^{\infty}\left[n^{k}\,(n+\beta)\right]a_{n,1}+(1-\mu)\sum_{n=1}^{\infty}\left[n^{k}\,(n+\beta)\right]a_{n,2}\\&\qquad\qquad\leq\mu\alpha(1-\beta)+(1-\mu)\alpha(1-\beta)=\alpha(1-\beta)\end{aligned}$$

which shows that $f(z)\in S_{W}^{*}(k,\beta)$. Hence the theorem is proved. □

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