International Journal of Pure and Applied Mathematics
Volume 40 No. 3 2007, 367-372

EXISTENCE AND UNIQUENESS RESULTS FOR A CLASS OF NONLINEAR DIFFERENTIAL SYSTEMS

Rodica Luca
Department of Mathematics
"Gh. Asachi" Technical University
11 Bd. Carol I, Iaşi, 700506, ROMANIA
e-mail: rluca@math.tuiasi.ro

Abstract: We study the existence and uniqueness of the strong and weak solutions to a nonlinear differential system with second-order differences, subject to some extreme conditions and initial data.

AMS Subject Classification: 34G20, 35L55, 47H05, 39A12
Key Words: differential system, extreme conditions, maximal monotone operator, Cauchy problem, strong solution, weak solution

1. Introduction

Let H be a real Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$. We consider the nonlinear differential system with second-order differences

$$\begin{cases} u_j'(t) + \dfrac{v_{j+1}(t) - 2v_j(t) + v_{j-1}(t)}{h_j^2} + c_j A(u_j(t)) \ni f_j(t)\,, \\ v_j'(t) - \dfrac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{\overline{h}_j^2} + d_j B(v_j(t)) \ni g_j(t), \end{cases} \tag{S}$$

$0 < t < T$, $j = \overline{1,N}$, in H, with the extreme conditions

$$\begin{array}{l} \begin{pmatrix} u_1(t) - u_0(t) \\ -v_1(t) + v_0(t) \end{pmatrix} \in \alpha \begin{pmatrix} v_1(t) \\ u_1(t) \end{pmatrix}, \\ \begin{pmatrix} -u_{N+1}(t) + u_N(t) \\ v_{N+1}(t) - v_N(t) \end{pmatrix} \in \beta \begin{pmatrix} v_N(t) \\ u_N(t) \end{pmatrix}, \end{array} \tag{EC}$$

Received: August 20, 2007

for $0 < t < T$ and the initial data

$$u_j(0) = u_{j0},\ \ v_j(0) = v_{j0},\ \ j = \overline{1,N}, \tag{ID}$$

where $N \in I\!N$, $N \geq 2$, $T > 0$, $c_j,\ d_j,\ h_j,\ \overline{h}_j > 0$, for all $j = \overline{1,N}$, α, β and A, B are operators in H^2, respectively H, which satisfy some assumptions.

The above problem is a discrete version with respect to x (with $H = I\!R$) of the nonlinear system

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t}(t,x) + \dfrac{\partial^2 v}{\partial x^2}(t,x) + c(x)A(u(t,x)) \ni f(t,x)\,, \\ \dfrac{\partial v}{\partial t}(t,x) - \dfrac{\partial^2 u}{\partial x^2}(t,x) + d(x)B(v(t,x)) \ni g(t,x), \end{array} \right. \tag{S$_0$}$$

$$0 < x < 1,\ \ 0 < t < T,\ \text{ in } I\!R,$$

subject to boundary conditions

$$\left(\begin{array}{c} \dfrac{\partial u}{\partial x}(t,0) \\ -\dfrac{\partial v}{\partial x}(t,0) \end{array} \right) \in \alpha \left(\begin{array}{c} v(t,0) \\ u(t,0) \end{array} \right), \ \left(\begin{array}{c} -\dfrac{\partial u}{\partial x}(t,1) \\ \dfrac{\partial v}{\partial x}(t,1) \end{array} \right) \in \beta \left(\begin{array}{c} v(t,1) \\ u(t,1) \end{array} \right), \tag{BC$_0$}$$

for $0 < t < T$ and the initial data

$$u(0,x) = u_0(x),\ \ v(0,x) = v_0(x),\ \ 0 < x < 1. \tag{IC$_0$}$$

This problem and some generalizations of it (with higher-order partial derivatives, time dependent coefficients in $(\mathrm{S})_0$ or extra functions in $(\mathrm{BC})_0$) have been studied in Luca [6], Luca-Tudorache [9], Moroşanu et al [10]. The conditions $(\mathrm{BC})_0$ are general ones. By making suitable choices of α and β we deduce many classical boundary conditions.

In this paper we investigate the existence and uniqueness of the strong and weak solutions for the problem (S)+(EC)+(ID). In our proofs we use some results related to maximal monotone operators and nonlinear evolution equations in Hilbert spaces (see the monographs Barbu [2], Brezis [3], Ladde et al [4], Lakshmikantham et al [5]). For other differential and difference equations in abstract spaces we mention the papers Agarwal et al [1], Luca [7], Luca [8], Rousseau et al [11].

We present the assumptions that we use in the sequel:

(H1) The operators $A : D(A) \subset H \to H$, $B : D(B) \subset H \to H$ are maximal monotone, possibly multivalued.

(H2) The operators $\alpha : D(\alpha) \subset H^2 \to H^2$, $\beta : D(\beta) \subset H^2 \to H^2$ are maximal monotone, possibly multivalued.

(H3) i) The operators α and β are bounded on bounded sets.
ii) $(\mathrm{int}D(\alpha))\cap(D(B)\times D(A))\neq\emptyset$ and $(\mathrm{int}D(\beta))\cap(D(B)\times D(A))\neq\emptyset$.
(H4) The constants $h_j>0$, $\overline{h}_j>0$, for all $j=\overline{1,N}$.
(H5) The constants $c_j>0$, $d_j>0$, for all $j=\overline{1,N}$.

2. Existence and Uniqueness of Solutions

We express our problem (S)+(EC)+(ID) as a Cauchy problem in a certain Hilbert space, using some maximal monotone operators. We consider the Hilbert space $X=H^{2N}=\{(u_1,u_2,\dots,u_N,v_1,v_2,\dots,v_N)^T;\ u_j,\, v_j\in H,\ j=\overline{1,N}\}$ with the scalar product $\langle(u_1,\dots,u_N,v_1,\dots,v_N)^T,(\overline{u}_1,\dots,\overline{u}_N,\overline{v}_1,\dots,\overline{v}_N)^T\rangle_X=\sum_{j=1}^{N}h_j^2\langle u_j,\overline{u}_j\rangle+\sum_{j=1}^{N}\overline{h}_j^2\langle v_j,\overline{v}_j\rangle$ and the corresponding norm $\|\cdot\|_X$.

We introduce the operator $\mathcal{A}_1:D(\mathcal{A}_1)=X\to X$,

$$\mathcal{A}_1((u_1,u_2,\dots,u_N,v_1,v_2,\dots,v_N)^T)=\Bigg(\frac{v_2-2v_1}{h_1^2},\frac{v_3-2v_2+v_1}{h_2^2},\dots,\frac{v_N-2v_{N-1}+v_{N-2}}{h_{N-1}^2},\frac{-2v_N+v_{N-1}}{h_N^2},-\frac{u_2-2u_1}{\overline{h}_1^2},-\frac{u_3-2u_2+u_1}{\overline{h}_2^2},\dots,-\frac{u_N-2u_{N-1}+u_{N-2}}{\overline{h}_{N-1}^2},-\frac{-2u_N+u_{N-1}}{\overline{h}_N^2}\Bigg)^T,$$

and the operator $\mathcal{A}_2:D(\mathcal{A}_2)\subset X\to X$, $D(\mathcal{A}_2)=\{(u_1,\dots,u_N,v_1,\dots,v_N)^T,\ (v_1,u_1)^T\in D(\alpha),\ (v_N,u_N)^T\in D(\beta)\}$,

$$\mathcal{A}_2((u_1,\dots,u_N,v_1,\dots,v_N)^T)=\Bigg\{\Bigg(\frac{v_0}{h_1^2},0,\dots,0,\frac{v_{N+1}}{h_N^2},-\frac{u_0}{\overline{h}_1^2},0,\dots,0,-\frac{u_{N+1}}{\overline{h}_N^2}\Bigg)^T,$$

$$(u_1-u_0,-v_1+v_0)^T\in\alpha((v_1,u_1)^T),\ \ (-u_{N+1}+u_N,v_{N+1}-v_N)^T\in\beta((v_N,u_N)^T)\Bigg\}.$$

Lemma 1. *If the assumption* (H4) *hold, then the operator* $\mathcal{A}_1$ *is maximal monotone in* X.

Lemma 2. *If the assumptions* (H2) *and* (H4) *hold, then the operator* $\mathcal{A}_2$ *is maximal monotone in* X.

We now define the operator $\mathcal{A} : D(\mathcal{A}) = D(\mathcal{A}_2) \subset X \to X$, $\mathcal{A}(U) = \mathcal{A}_1(U) + \mathcal{A}_2(U)$.

Lemma 3. *If the assumptions* (H2) *and* (H4) *hold, then the operator* $\mathcal{A}$ *is maximal monotone in* X.

Next, we define the operator $\mathcal{B} : D(\mathcal{B}) \subset X \to X$, $D(\mathcal{B}) = D(A)^N \times D(B)^N$,

$$\mathcal{B}((u_1, u_2, \dots, u_N, v_1, v_2, \dots, v_N)^T) = \{(c_1\gamma_1, c_2\gamma_2, \dots, c_N\gamma_N, d_1\delta_1, d_2\delta_2, \dots, d_N\delta_N)^T,\ \ \gamma_i \in A(u_i),\ \ \delta_i \in B(v_i),\ \ i = \overline{1, N}\}.$$

Lemma 4. *If the assumptions* (H1), (H4) *and* (H5) *hold, then the operator* $\mathcal{B}$ *is maximal monotone in* X.

Theorem 1. *If the assumptions* (H1), (H2), ((H3)i) *or* (H3)ii)), (H4) *and* (H5) *hold, then the operator* $\mathcal{A} + \mathcal{B}$ *is maximal monotone.*

Using the operators $\mathcal{A}$ and $\mathcal{B}$ our problem can be written as the following Cauchy problem in the space X

$$\begin{cases} \dfrac{dU}{dt}(t) + \mathcal{A}(U(t)) + \mathcal{B}(U(t)) \ni F(t), \\ U(0) = U_0, \end{cases} \tag{P}$$

where $U = (u_1, \dots, u_N, v_1, \dots, v_N)^T$, $U_0 = (u_{10}, \dots, u_{N0}, v_{10}, \dots, v_{N0})^T$, $F = (f_1, \dots, f_N, g_1, \dots, g_N)^T$.

Theorem 2. *Assume that the assumptions* (H1), (H2), ((H3)i) *or* (H3)ii)), (H4) *and* (H5) *hold. If* $(v_{10}, u_{10})^T \in D(\alpha) \cap (D(B) \times D(A))$, $u_{j0} \in D(A)$, $v_{j0} \in D(B)$, *for all* $j = \overline{2, N-1}$, $(v_{N0}, u_{N0})^T \in D(\beta) \cap (D(B) \times D(A))$ *(that is* $U_0 \in D(\mathcal{A}) \cap D(\mathcal{B})$*),* $f_j,\ g_j \in W^{1,1}(0, T; H)$, $j = \overline{1, N}$, *then there exist unique functions* $u_j,\ v_j \in W^{1,\infty}(0, T; H)$, $j = \overline{1, N}$ $(v_1(t), u_1(t))^T \in D(\alpha) \cap (D(B) \times D(A))$, $(v_N(t), u_N(t))^T \in D(\beta) \cap (D(B) \times D(A))$, $u_j(t) \in D(A)$, $v_j(t) \in D(B)$, *for all* $j = \overline{2, N-1}$, *for all* $t \in [0, T]$, *that verify the system* (S) *and the extreme conditions* (EC) *for all* $t \in [0, T)$ *and the initial data* (ID). *Besides,* $u_j,\ v_j$, $j = \overline{1, N}$ *are everywhere differentiable from right in the topology of* H *and*

$$\frac{d^+ u_j}{dt}(t) = \left(f_j(t) - c_j A(u_j(t)) - \frac{v_{j+1}(t) - 2v_j(t) + v_{j-1}(t)}{h_j^2} \right)^0, \quad j = \overline{2, N-1},$$

$$\frac{d^+ v_j}{dt}(t) = \left(g_j(t) - d_j B(v_j(t)) + \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{\overline{h}_j^2} \right)^0,$$

$$j = \overline{2, N-1},$$

$$\begin{pmatrix} \frac{d^+u_1}{dt}(t) \\ \frac{d^+v_1}{dt}(t) \end{pmatrix} = \begin{pmatrix} f_1(t) - c_1 A(u_1(t)) - \frac{v_2(t) - 2v_1(t) + v_0(t)}{h_1^2} \\ g_1(t) - d_1 B(v_1(t)) + \frac{u_2(t) - 2u_1(t) + u_0(t)}{\overline{h}_1^2} \end{pmatrix}^0,$$

$$\begin{pmatrix} \frac{d^+u_N}{dt}(t) \frac{d^+v_N}{dt}(t) \end{pmatrix}$$
$$= \begin{pmatrix} f_N(t) - c_N A(u_N(t)) - \frac{v_{N+1}(t) - 2v_N(t) + v_{N-1}(t)}{h_N^2} \\ g_N(t) - d_N B(v_N(t)) + \frac{u_{N+1}(t) - 2u_N(t) + u_{N-1}(t)}{\overline{h}_N^2} \end{pmatrix}^0,$$

for all $t \in [0, T)$, *with* $(u_1(t) - u_0(t), -v_1(t) + v_0(t))^T \in \alpha((v_1(t), u_1(t))^T)$, $(-u_{N+1}(t) + u_N(t), v_{N+1}(t) - v_N(t))^T \in \beta((v_N(t), u_N(t))^T)$, *for all* $t \in [0, T)$.

Proof. By Theorem 1, the operator $\mathcal{A}+\mathcal{B}$ is maximal monotone in X. Using Barbu [2], Theorem 2.2, Corollary 2.1, Chapter III, we deduce that for $U_0 \in D(\mathcal{A}) \cap D(\mathcal{B})$ and $F \in W^{1,1}(0, T; X)$, the problem (P)≡ (S)+(EC)+(ID) has a unique strong solution $U = (u_1, \ldots, u_N, v_1, \ldots, v_N)^T \in W^{1,\infty}(0, T; X)$, $U(t) \in D(\mathcal{A}) \cap D(\mathcal{B})$, for all $t \in [0, T)$. We can conclude that $U(T) \in D(\mathcal{A}) \cap D(\mathcal{B})$, by extending correspondingly the functions f_j, g_j, $j = \overline{1, N}$ and by considering the equation $(\mathrm{P})_1$ in the interval $[0, T + \varepsilon]$, with $\varepsilon > 0$. The solution U is everywhere differentiable from right and $\frac{d^+U}{dt}(t) = (F(t) - \mathcal{A}(U(t)) - \mathcal{B}(U(t)))^0$, for all $t \in [0, T)$, that is we have the relations from the conclusion of the theorem. Moreover we have

$$\left\| \frac{d^+U}{dt}(t) \right\|_X \leq \|(F(0) - \mathcal{A}(U_0) - \mathcal{B}(U_0))^0\|_X + \int_0^t \left\| \frac{dF}{ds}(s) \right\|_X ds, \ \forall t \in [0, T).$$

The proof is completed. □

Remark. Under the assumptions of Theorem 2, if $U_0 \in \overline{D(\mathcal{A}) \cap D(\mathcal{B})}$ and $F \in L^1(0, T; X)$, then by Barbu [2], Corollary 2.2, Chapter III, we deduce that the problem (P) has a unique weak solution $U \in C([0, T]; X)$.

References

[1] R.P. Agarwal, D. O'Regan, V. Lakshmikantham, Discrete second order inclusions, *J. Difference Equ. Appl.*, **9** (2003), 879-885.

[2] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden (1976).

[3] H. Brezis, *Operateurs Maximaux Monotones et Semigroupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam (1973).

[4] S. Ladde, V. Lakshmikantham, A.S. Vatsalo, *Monotone, Iterative Techniques for Nonlinear Differential Equations*, Monographs, Advanced Texts and Surveys in Pure and Applied Mathematics, 27, Pitman (1985).

[5] V. Lakshmikantham, S. Leela, *Nonlinear Differential Equations in Abstract Spaces*, International Series in Nonlinear Mathematics: Theory, Methods and Applications, Volume 2, Pergamon Press (1981).

[6] R. Luca, Existence and uniqueness results for a higher order hyperbolic system, *Math. Reports* (*Stud. Cerc. Mat.*), **1**, No. 51 (1999), 551-571.

[7] R. Luca, Existence for a class of discrete hyperbolic problems, *Adv. Difference Equ.* (2006), Article ID 89260, 1-14.

[8] R. Luca, Existence and uniqueness for a nonlinear discrete hyperbolic system, *Nonlinear Anal., Theory Methods Appl.*, 67 (2007), 2433-2446.

[9] R. Luca-Tudorache, *Boundary Value Problems for Nonlinear Hyperbolic Systems and Applications*, Casa de Editura Venus, Iaşi (2003), In Romanian.

[10] G. Moroşanu, D. Petrovanu, Nonlinear boundary value problems for a class of hyperbolic partial differential systems, *Atti. Sem. Mat. Fis. Univ. Modena*, **34** (1985-86), 295-316.

[11] A. Rousseau, R. Temam, J. Tribbia, Boundary conditions for an ocean related system with a small parameter, In: *Nonlinear Partial Differential Equations and Related Analysis, Contemporary Mathematics*, **371** (2005), 231-263.