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THE FUNCTIONAL ANALYTIC SETTING OF HUM PART II: THE MINDLIN-TIMOSHENKO PLATE MODEL

Michael Pedersen
Department of Mathematics
Technical University of Denmark
Kgs. Lyngby, DK 2800, DENMARK
e-mail: M.Pedersen@mat.dtu.dk

Abstract: The question of controllability for partial differential evolution equations of hyperbolic and parabolic nature has been studied intensively over the last decade, motivated and inspired by numerous applications in science and technology. In the first part of this paper, see [6], we introduced the functional analytic setting of HUM – The Hilbert Uniqueness Method – due to J.L. Lions. In Part II we will now apply the functional analytic methods to control the full Mindlin-Timoshenko plate system from the boundary.

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1. Introduction

The approach as developed in Part I [6], is a "constructive" but essentially functional analytic based method for solving the control problem. It clarifies the necessary steps from the original HUM method, but in applications it causes exactly the same problems as the former and the question of controllability boils ultimatively down to whether the adjoint system is observable from Γ_0, the controlled part of the boundary Γ, or not. The formulation of the modern approach is basically a rediscovery of the formulation of the control problem as a minimization problem, which is well-known from linear systems theory and finite dimensional control theory. But in order to apply this formulation we have

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to rely on very advanced solvability results for the actual PDE systems. But systems theory tells us exactly which properties to look for and then we must return to PDE theory in order to prove these. And the fundamental properties are observability inequalities for our adjoint systems that will ensure that the operator $\Phi S(t)(\Phi^*\Phi)^{-1}$, mapping initial data to control input, is well-defined.

There is a vast literature on the Mindlin-Timoshenko plate model. A detailed derivation of the model is performed in Pedersen [7], where the variational formulation of the model that we will use in this paper is presented. Since we will typically be working on spatial domains with corners, we will rely on regularity results from Grisvard [2], [3], and [4]. We will follow the program of Lagnese and Lions, [5], and use the functional analytic setting to achive similar results.

2. The Classical Hilbert Uniqueness Method

Let us recall the control system (with the standard notation from [6])

$$\begin{cases} \mathbf{C}u_{tt} = \mathcal{A}u & \text{in } Q\,, \\ \mathcal{B}u = \kappa & \text{on } \Sigma_0\,, \\ u = 0 & \text{on } \Sigma_1\,, \\ (u(0), u_t(0)) = (u^0, u^1) & \text{in } \Omega\,. \end{cases} \tag{1}$$

For initial data $(u^0, u^1) \in \mathcal{H}' = H \times V'$ and $\kappa \in (L^2(0,T;[H^{\frac{1}{2}}(\Gamma_0)]^3))'$ we have that the control system has a unique solution $(u, u_t) \in C([0,T];\mathcal{H}')$ that depends continuously on the data.

Now given $T > 0$ and (u^0, u^1), we wish to find a control κ on Σ_0 such that

$$u(\cdot, T) = u_t(\cdot, T) = 0. \tag{2}$$

Due to the hyperbolic nature of the system the speed of propagation is finite, hence exact controllability is only possible if T is large enough, moreover T depends of the geometry of Ω and Γ_0.

Next, the adjoint system

$$\begin{cases} \mathbf{C}v_{tt} = \mathcal{A}v & \text{in } Q\,, \\ \mathcal{B}v = 0 & \text{on } \Sigma_0\,, \\ v = 0 & \text{on } \Sigma_1\,, \\ (v(0), v_t(0)) = (v^0, v^1) & \text{in } \Omega\,, \end{cases} \tag{3}$$

admits a unique solution for $(v^0, v^1) \in \mathcal{H}$ and the mapping $(v^0, v^1) \to (v, v_t) : \mathcal{H} \to C([0,T], \mathcal{H})$ is linear and continuous.

Let v be the solution to the adjoint system with initial data (v^0, v^1), and define according to [6] the observability operator Φ as the operator which solves the adjoint system with initial data $(v^0, v^1) \in \mathcal{H}$ and apply the complementary boundary operator to the solution

$$\Phi : \mathcal{H} \to (L^2(0,T; [H^{\frac{1}{2}}(\Gamma_0)]^3))' , \tag{4}$$

$$\Phi(v^0, v^1) = -\mathcal{C}v; \tag{5}$$

the operator is well-defined since $\mathcal{C}v \in (L^2(0,T; [H^{\frac{1}{2}}(\Gamma_0)]^3))'$.

Finally we consider the backwards system

$$\begin{cases} \mathbf{C}\hat{u}_{tt} = \mathcal{A}\hat{u} & \text{in } Q\,, \\ \mathcal{B}\hat{u} = \kappa & \text{on } \Sigma_0\,, \\ \hat{u} = 0 & \text{on } \Sigma_1\,, \\ (\hat{u}(T), \hat{u}_t(T)) = (0,0) & \text{in } \Omega\,, \end{cases} \tag{6}$$

and we define the operator

$$\Psi : (L^2(0,T; [H^{\frac{1}{2}}(\Gamma_0)]^3))' \to \mathcal{H}' \tag{7}$$

by

$$\Psi\kappa = (\hat{u}(0), \hat{u}_t(0)), \tag{8}$$

which, under these assumptions, is well-defined. We can now introduce the Λ operator from [6]

$$\Lambda = \Psi\Phi : \mathcal{H} \to \mathcal{H}' ,$$

$$\Lambda(v^0, v^1) = (\hat{u}(0), \hat{u}_t(0))\,. \tag{9}$$

From the construction of the adjoint and backward systems we see that if

$$\Lambda(v^0, v^1) = (u^0, u^1)\,, \tag{10}$$

then $u(T) = u_t(T) = 0$ and the control system is exactly controllable with $(u^0, u^1) \in \mathcal{H}'$ and control $\kappa = -\mathcal{C}v$, this is possible in general when Λ is an isomorphism from $\mathcal{H}$ onto $\mathcal{H}'$. In order to make this more explicit we can make the following considerations: If $u(T) = u_t(T) = 0$ for $t = T$, we can reduce the fundamental equality from [6] to

$$\langle (u^0, u^1), (v^0, v^1) \rangle_{\mathcal{H}', \mathcal{H}} = -\int_0^T \int_{\Gamma_0} \kappa(t) \cdot v(t) d\Gamma dt \tag{11}$$

using (10) and for our particular choice of control $\kappa = -\mathcal{C}v = -v|_{\Sigma_0}$ we get

$$\langle \Lambda(v^0, v^1), (v^0, v^1)\rangle_{\mathcal{H}',\mathcal{H}} = \int_0^T \int_{\Gamma_0} v^2 d\Gamma dt\,.$$

Then, if for T large enough,

$$\int_0^T \int_{\Gamma_0} v^2 d\Gamma dt \tag{12}$$

defines a norm on $\mathcal{H}$ equivalent to the usual norm, then Λ is an isomorphism from $\mathcal{H}$ onto $\mathcal{H}'$ from Riesz' Representation Theorem. If we can show that (12) defines a norm on $\mathcal{H}$ then the control system is exactly controllable for initial data in $\mathcal{H}'$. Unfortunately (12) does not define a norm on $\mathcal{H}$, but (12) has the structure of the L^2 norm on Γ_0, therefore the obvious idea is to define a space that is forced to be contained in $L^2(\Gamma_0)$. This is exactly the space $\mathcal{F}_0$ defined in (13), endoved with the seminorm (12). We will now use some of the newer results developed in [6] to derive the observability of the adjoint system. The variational approach for the heat and wave equations is developed by Zuazua in [8].

3. Controlling $(u^0, u^1) \in \mathcal{F}_0{}'$

In the following section we will assume that $\|\cdot\|_{\mathcal{F}_0}$ defines a norm on $\mathcal{F}_0$; recall that

$$\mathcal{F}_0 = \{(v^0, v^1) \in \mathcal{H}' \mid v|_{\Sigma_0} \in [L^2(\Sigma_0)]^3\}\,, \tag{13}$$

endoved with the (semi) norm

$$\|(v^0, v^1)\|^2_{\mathcal{F}_0} = \int_0^T \int_{\Gamma_0} v^2 d\Gamma dt. \tag{14}$$

In Section 5 we will investigate conditions under which this is the case. As already mentioned, both the time T will have to be large enough and the geometry of Γ_0 have to be appropriate, i.e. satisfy what is now known as the *Geometric Control Condition* due to Bardos, Lebeau and Rauch [1]. We will comment on this later in Section 5.

Following [6] we start by modifying the operators Φ and Ψ accordingly:

$$\Phi : \mathcal{F}_0 \to [L^2(\Sigma_0)]^3\,,$$

$$\Phi(v^0, v^1) = -\mathcal{C}v = -v|_{\Sigma_0} ,$$

and (recall (6))

$$\Psi : [L^2(\Sigma_0)]^3 \to \mathcal{F}_0{}' ,$$
$$\Psi\kappa = (\hat{u}(0), \hat{u}_t(0)) .$$

We see that the Λ-operator (9) now becomes

$$\Lambda = \Psi\Phi : \mathcal{F}_0 \to \mathcal{F}_0{}' ,$$
$$\Lambda(v^0, v^1) = (\hat{u}(0), \hat{u}_t(0)). \tag{15}$$

First we present a necessary and sufficient condition for the control system (1) to be exactly controllable.

Theorem 1. *The initial data $(u^0, u^1) \in \mathcal{F}_0{}'$ of the control system is controllable to zero if and only if there exists $\kappa \in [L^2(\Sigma_0)]^3$ such that*

$$\langle (u^0, u^1), (v^0, v^1) \rangle_{\mathcal{F}_0{}', \mathcal{F}_0} + \int_0^T \int_{\Gamma_0} \kappa \cdot v d\Gamma dt = 0 \tag{16}$$

for all initial data $(v^0, v^1) \in \mathcal{F}_0$ with corresponding solution v of the adjoint system.

Proof. (16) simply follows from [6]. □

Furthermore we have

Theorem 2. *Assume that the control system is exactly controllable, then Φ and Ψ are adjoint operators.*

Proof. Using the definition of Ψ and (16) we have

$$\langle \Psi\kappa, (v^0, v^1) \rangle_{\mathcal{F}_0{}', \mathcal{F}_0} = \langle (u^0, u^1), (v^0, v^1) \rangle_{\mathcal{F}_0{}', \mathcal{F}_0}$$
$$= -\int_0^T \int_{\Gamma_0} \kappa \cdot v d\Gamma dt = (\kappa \mid \Phi(v^0, v^1))_{[L^2(\Sigma_0)]^3} ,$$

where we have used that the control system is exactly controllable, such that $(\hat{u}(0), \hat{u}_t(0)) = (u^0, u^1)$. □

Thus

$$\Lambda = \Phi^*\Phi : \mathcal{F}_0 \to \mathcal{F}_0{}' ,$$

$$\Lambda(v^0, v^1) = (u^0, u^1)\,, \tag{17}$$

where Φ^* is the adjoint of Φ.

Next, we define the functional $\mathcal{J}_0 : \mathcal{F}_0 \to \mathbb{R}$ as

$$\mathcal{J}_0(v^0, v^1) = \frac{1}{2}\int_0^T \int_{\Gamma_0} v^2 d\Gamma dt - \langle (u^0, u^1), (v^0, v^1)\rangle_{\mathcal{F}_0{}', \mathcal{F}_0}.$$

Obviously there is a connection between the critical points of $\mathcal{J}_0$ and (16) as seen in the following theorem.

Theorem 3. *Let $(u^0, u^1) \in \mathcal{F}_0{}'$ and assume that $(\check{v}^0, \check{v}^1) \in \mathcal{F}_0$ is a minimizer of $\mathcal{J}_0$, then*

$$\kappa = \Phi(\check{v}^0, \check{v}^1)$$

is a control which steers (u^0, u^1) to zero at time T.

Proof. From the assumption that $(\check{v}^0, \check{v}^1)$ is a minimizer of $\mathcal{J}_0$ we have

$$\begin{aligned}
0 &= \lim_{h\to 0}\frac{1}{h}(\mathcal{J}_0((\check{v}^0, \check{v}^1) + h(v^0, v^1)) - \mathcal{J}_0(\check{v}^0, \check{v}^1)) \\
&= \lim_{h\to 0}\frac{1}{h}(\frac{1}{2}\int_0^T \int_{\Gamma_0} (\Phi(\check{v}^0, \check{v}^1) + h\Phi(v^0, v^1))^2 d\Gamma dt \\
&- \langle (u^0, u^1), (\check{v}^0, \check{v}^1) + h(v^0, v^1)\rangle_{\mathcal{F}_0{}', \mathcal{F}_0} \\
&- \frac{1}{2}\int_0^T \int_{\Gamma_0} \Phi(\check{v}^0, \check{v}^1)^2 d\Gamma dt + \langle (u^0, u^1), (\check{v}^0, \check{v}^1)\rangle_{\mathcal{F}_0{}', \mathcal{F}_0} \\
&= \int_0^T \int_{\Gamma_0} \Phi(\check{v}^0, \check{v}^1)\Phi(v^0, v^1) d\Gamma dt - \langle (u^0, u^1), (v^0, v^1)\rangle_{\mathcal{F}_0{}', \mathcal{F}_0}\,,
\end{aligned}$$

for all $(v^0, v^1) \in \mathcal{F}_0$.

Using Theorem 1 we see that $\kappa = \Phi(\check{v}^0, \check{v}^1)$ is a control that steers the initial data (u^0, u^1) to zero at time T. □

3.1. The Observability Inequality

Definition 3.1. The adjoint system is said to be observable in time T if there is a constant $C > 0$ such that

$$C\|(v^0, v^1)\|^2_{\mathcal{H}'} \le \int_0^T \int_{\Gamma_0} |\Phi(v^0, v^1)|^2 d\Gamma dt \tag{18}$$

for all $(v^0, v^1) \in \mathcal{F}_0$.

We will refer to (18) as the observation (or observability) inequality, it is the fundamental inequality stating that the initial data are uniquely determined by the observed quantity $\int_0^T \int_{\Gamma_0} \Phi(v^0, v^1)^2 d\Gamma dt$.

Remark 4. Note how $\|(v^0, v^1)\|_{\mathcal{F}_0}$ is chosen as $\int_0^T \int_{\Gamma_0} \Phi(v^0, v^1)^2 d\Gamma dt$, so if we can prove the observability inequality then $\|(\cdot, \cdot)\|_{\mathcal{F}_0}$ is a norm on $\mathcal{F}_0$, since the opposite inequality already holds – independent of T. This is done in Section 5.

We recall that if $\mathcal{J}_0$ is strictly convex, lower semi-continuous and coercive, then $\mathcal{J}_0$ attains a minimum, i.e. there exists a unique $(\check{v}^0, \check{v}^1) \in \mathcal{F}_0$ such that

$$\mathcal{J}_0(\check{v}^0, \check{v}^1) < \mathcal{J}_0(v^0, v^1)$$

for all $(v^0, v^1) \in \mathcal{F}_0$.

Theorem 5. *Let $(u^0, u^1) \in \mathcal{F}_0{}'$ and suppose that the adjoint system is observable in time T. Then $\mathcal{J}_0$ has a unique minimum $(\check{v}^0, \check{v}^1) \in \mathcal{F}_0$.*

Proof. We follow the proof from [6]. It is easily seen that $\mathcal{J}_0$ is continuous. Coercivity here means

$$\lim_{\|(v^0,v^1)\|_{\mathcal{H}'} \to \infty} \mathcal{J}_0(v^0, v^1) = \infty$$

and can be verified by the following estimate: Using the Cauchy-Schwartz inequality on $\mathcal{J}_0$ gives

$$\begin{aligned} \mathcal{J}_0(v^0, v^1) &\geq \frac{1}{2} \int_0^T \int_{\Gamma_0} \Phi(v^0, v^1)^2 d\Gamma dt - |\langle (u^0, u^1), (v^0, v^1) \rangle_{\mathcal{F}_0{}', \mathcal{F}_0}| \\ &\geq \frac{1}{2} \int_0^T \int_{\Gamma_0} \Phi(v^0, v^1)^2 d\Gamma dt - \|(u^0, u^1)\|_{\mathcal{F}_0{}'} \|(v^0, v^1)\|_{\mathcal{F}_0}, \end{aligned}$$

using the observation inequality (18) we get

$$\mathcal{J}_0(v^0, v^1) \geq \frac{C}{2} \|(v^0, v^1)\|_{\mathcal{H}'}^2 - \|(u^0, u^1)\|_{\mathcal{F}_0{}'} \|(v^0, v^1)\|_{\mathcal{H}'}$$

for some $C > 0$, thus $\mathcal{J}_0$ is coercive.

$\mathcal{J}_0$ is strictly convex if

$$\mathcal{J}_0(\lambda(v^0, v^1) + (1 - \lambda)(\eta^0, \eta^1)) < \lambda \mathcal{J}_0(v^0, v^1) + (1 - \lambda) \mathcal{J}_0(\eta^0, \eta^1)$$

holds for all $(v^0, v^1), (\eta^0, \eta^1) \in \mathcal{F}_0$ and $\lambda \in]0, 1[$. By calculating the LHS we get

$$\mathcal{J}_0(\lambda(v^0, v^1) + (1-\lambda)(\eta^0, \eta^1)) = \frac{1}{2}\int_0^T \int_{\Gamma_0} \lambda^2 \Phi(v^0, v^1)^2$$
$$+2\lambda(1-\lambda)\Phi(v^0, v^1)\Phi(\eta^0, \eta^1) + (1-\lambda)^2\Phi(\eta^0, \eta^1)^2 d\Gamma dt$$
$$-\langle (u^0, u^1), \lambda(v^0, v^1) + (1-\lambda)(\eta^0, \eta^1)\rangle_{\mathcal{F}_0{}', \mathcal{F}_0}$$
$$= \lambda \mathcal{J}_0(v^0, v^1) + (1-\lambda)\mathcal{J}_0(\eta^0, \eta^1)$$
$$-\frac{\lambda(1-\lambda)}{2}\int_0^T \int_{\Gamma_0} (\Phi(v^0, v^1) - \Phi(\eta^0, \eta^1))^2 d\Gamma dt.$$

From the observability inequality we have

$$\int_0^T \int_{\Gamma_0} (\Phi(v^0, v^1) - \Phi(\eta^0, \eta^1))^2 d\Gamma dt \geq C\|(v^0, v^1) - (\eta^0, \eta^1)\|^2_{\mathcal{F}_0},$$

hence for all $(v^0, v^1) \neq (\eta^0, \eta^1)$ we have

$$\mathcal{J}_0(\lambda(v^0, v^1) + (1-\lambda)(\eta^0, \eta^1)) < \lambda \mathcal{J}_0(v^0, v^1) + (1-\lambda)\mathcal{J}_0(\eta^0, \eta^1). \qquad \square$$

Corollary 6. *If the adjoint system is observable in time T then the control system is exactly controllable in time T.*

3.2. Minimal $[L^2(\Sigma_0)]^3$-norm

Since the control $\kappa = \Phi(\check{v}^0, \check{v}^1)$ is obtained by minimizing $\mathcal{J}_0$ it is natural to ask if κ has minimal $[L^2(\Sigma_0)]^3$-norm among all other controls in $[L^2(\Sigma_0)]^3$, this is indeed the case. If we let $\zeta \in [L^2(\Sigma_0)]^3$ be a control which also drives the initial data $(u^0, u^1) \in \mathcal{F}_0{}'$ to zero in time T, then from Theorem 1 we have for all $(v^0, v^1) \in \mathcal{F}_0$ that

$$\langle (u^0, u^1), (v^0, v^1)\rangle_{\mathcal{F}_0{}', \mathcal{F}_0} + \int_0^T \int_{\Gamma_0} \zeta \cdot v d\Gamma dt = 0\,,$$

where v is a solution to the adjoint system. Hence in particular for the minimizer $(\check{v}^0, \check{v}^1)$ of $\mathcal{J}_0$ we have

$$\langle (u^0, u^1), (\check{v}^0, \check{v}^1)\rangle_{\mathcal{F}_0{}', \mathcal{F}_0} + \int_0^T \int_{\Gamma_0} \zeta \cdot \check{v} d\Gamma dt = 0. \qquad (19)$$

The $[L^2(\Sigma_0)]^3$-norm of κ is (by an application of (16))

$$\|\kappa\|^2_{[L^2(\Sigma_0)]^3} = \int_0^T \int_{\Gamma_0} \Phi(\check{v}^0, \check{v}^1)^2 d\Gamma dt = -\langle (u^0, u^1), (\check{v}^0, \check{v}^1)\rangle_{\mathcal{F}_0{}', \mathcal{F}_0}. \qquad (20)$$

Combining (19) and (20) gives

$$\begin{aligned}\|\kappa\|^2_{[L^2(\Sigma_0)]^3} = \int_0^T \int_{\Gamma_0} \zeta \cdot \check{v} d\Gamma dt \leq \|\zeta\|_{[L^2(\Sigma_0)]^3} \|\check{v}\|_{[L^2(\Sigma_0)]^3} \\ = \|\zeta\|_{[L^2(\Sigma_0)]^3} \|\kappa\|_{[L^2(\Sigma_0)]^3} \,, \qquad (21)\end{aligned}$$

so κ is of minimal $[L^2(\Sigma_0)]^3$-norm. This property of a control of the form $\kappa = \Phi(\check{v}^0, \check{v}^1)$ is important in applications since the norm of the control is the amount of force and momentum that need to be applied to the boundary of the plate in order to control the plate to rest. Inequality (21) ensures that the control $\kappa = \Phi(\check{v}^0, \check{v}^1)$ is the most efficient choice (in $[L^2(\Sigma_0)]^3$).

4. Controlling $(u^0, u^1) \in \mathcal{F}_1{}'$

We will now basically imitate the program from the previous section, but now with initial data $(u^0, u^1) \in \mathcal{F}_1{}'$ instead of $\mathcal{F}_0{}'$, also the control κ has a slightly different form. So in the following section we will assume that $\| \cdot \|_{\mathcal{F}_1}$ defines a norm on $\mathcal{F}_1$, recall that

$$\mathcal{F}_1 = \{(v^0, v^1) \in \mathcal{H} \mid v_t|_{\Sigma_0} \in [L^2(\Sigma_0)]^3\}$$

endoved with the (semi) norm

$$\|(v^0, v^1)\|^2_{\mathcal{F}_1} = \int_0^T \int_{\Gamma_0} v_t^2 d\Gamma dt.$$

Suppose the control $\kappa = \hat{\kappa}_t$, where $\hat{\kappa} \in [L^2(\Sigma_0)]^3$ and the time derivative is taken in the sense of distributions, this choice is possible due to the results from [6]. Now set $\hat{\kappa} = v_t|_{\Sigma_0}$, note that by the construction of $\mathcal{F}_1$ we have $v_t|_{\Sigma_0} \in [L^2(\Sigma_0)]^3$. Thus we will choose our control as $\kappa = \hat{\kappa}_t = v_{tt}|_{\Sigma_0} \in (H^1(0, T; [L^2(\Gamma_0)]^3))'$, where we define v_{tt} by

$$\int_0^T \int_{\Gamma_0} v_{tt} \eta d\Gamma dt = -\int_0^T \int_{\Gamma_0} v_t \eta_t d\Gamma dt \qquad (22)$$

for all $\eta \in H^1(0, T; [L^2(\Gamma_0)]^3)$.

Using this control we will modify the operators Φ and Ψ accordingly:

$$\begin{aligned}\Phi : \mathcal{F}_1 \to [L^2(\Sigma_0)]^3 \,, \\ \Phi(v^0, v^1) = v_t|_{\Sigma_0} \,,\end{aligned}$$

and

$$\Psi : [L^2(\Sigma_0)]^3 \to \mathcal{F}_1{}' \,,$$
$$\Psi\hat{\kappa} = (\hat{u}(0), \hat{u}_t(0)) \,.$$

Thus we see that the Λ operator is now (cf. (17))

$$\Lambda = \Psi\Phi : \mathcal{F}_1 \to \mathcal{F}_1{}' \,,$$
$$\Lambda(v^0, v^1) = (\hat{u}(0), \hat{u}_t(0)) \,.$$

A necessary and sufficient condition for exact controllability for the control system (1) is

Theorem 7. *The initial data $(u^0, u^1) \in \mathcal{F}_1{}'$ of the control system is controllable to zero if and only if there exists $\hat{\kappa} \in [L^2(\Sigma_0)]^3$ such that*

$$\langle (u^0, u^1), (v^0, v^1) \rangle_{\mathcal{F}_1{}', \mathcal{F}_1} - \int_0^T \int_{\Gamma_0} \hat{\kappa} \cdot v_t d\Gamma dt = 0 \tag{23}$$

for all initial data $(v^0, v^1) \in \mathcal{F}_1$ giving the solution v of the adjoint system.

Proof. From [6] we have

$$\langle (u^0, u^1), (v^0, v^1) \rangle_{\mathcal{F}_1{}', \mathcal{F}_1} + \int_0^T \int_{\Gamma_0} \kappa \cdot v d\Gamma dt = 0 \,, \tag{24}$$

since $\kappa = \hat{\kappa}_t$ we can make the calculation

$$\begin{aligned} \langle (u^0, u^1), (v^0, v^1) \rangle_{\mathcal{F}_1{}', \mathcal{F}_1} &= - \int_0^T \int_{\Gamma_0} \kappa \cdot v d\Gamma dt \\ &= - \int_0^T \int_{\Gamma_0} \hat{\kappa}_t \cdot v d\Gamma dt = \int_0^T \int_{\Gamma_0} \hat{\kappa} \cdot v_t d\Gamma dt. \quad \square \end{aligned}$$

Again we have adjoint relations between Φ and Ψ when the control system is exactly controllable

$$\begin{aligned} &\langle \Psi\hat{\kappa}, (v^0, v^1) \rangle_{\mathcal{F}_1{}', \mathcal{F}_1} = \langle (u^0, u^1), (v^0, v^1) \rangle_{\mathcal{F}_1{}', \mathcal{F}_1} \\ &= \int_0^T \int_{\Gamma_0} \hat{\kappa} \cdot v_t d\Gamma dt = (\hat{\kappa} \mid \Phi(v^0, v^1))_{[L^2(\Sigma_0)]^3}. \end{aligned}$$

Thus

$$\Lambda = \Phi^* \Phi : \mathcal{F}_1 \to \mathcal{F}_1{}' \,,$$

$$\Lambda(v^0, v^1) = (u^0, u^1)\,, \tag{25}$$

where Φ^* is the adjoint of Φ.

We now define the functional $\mathcal{J}_1 : \mathcal{F}_1 \to \mathbb{R}$ as

$$\mathcal{J}_1(v^0, v^1) = \frac{1}{2}\int_0^T \int_{\Gamma_0} v_t^2 d\Gamma dt - \langle (u^0, u^1), (v^0, v^1)\rangle_{\mathcal{F}_1{}', \mathcal{F}_1}$$

and use it to find the control $\hat{\kappa}$.

Theorem 8. *Let $(u^0, u^1) \in \mathcal{F}_1{}'$ and assume that $(\check{v}^0, \check{v}^1) \in \mathcal{F}_1$ is a minimizer of $\mathcal{J}_1$. Then*

$$\hat{\kappa} = \Phi(\check{v}^0, \check{v}^1)$$

and $\kappa = \hat{\kappa}_t$ is a control which steers (u^0, u^1) to zero at time T.

Proof. $\mathcal{J}_1$ has a minimum at $(\check{v}^0, \check{v}^1)$, thus

$$\begin{aligned}
0 &= \lim_{h\to 0} \frac{1}{h}(\mathcal{J}_1((\check{v}^0, \check{v}^1) + h(v^0, v^1)) - \mathcal{J}_1(\check{v}^0, \check{v}^1)) \\
&= \lim_{h\to 0} \frac{1}{h}(\frac{1}{2}\int_0^T \int_{\Gamma_0} (\Phi(\check{v}^0, \check{v}^1) + h\Phi(v^0, v^1))^2 d\Gamma dt \\
&- \langle (u^0, u^1), (\check{v}^0, \check{v}^1) + h(v^0, v^1)\rangle_{\mathcal{F}_1{}', \mathcal{F}_1} \\
&- \frac{1}{2}\int_0^T \int_{\Gamma_0} \Phi(\check{v}^0, \check{v}^1)^2 d\Gamma dt + \langle (u^0, u^1), (\check{v}^0, \check{v}^1)\rangle_{\mathcal{F}_1{}', \mathcal{F}_1} \\
&= \int_0^T \int_{\Gamma_0} \Phi(\check{v}^0, \check{v}^1)\Phi(v^0, v^1) d\Gamma dt - \langle (u^0, u^1), (v^0, v^1)\rangle_{\mathcal{F}_1{}', \mathcal{F}_1},
\end{aligned}$$

for all $(v^0, v^1) \in \mathcal{F}_1$.

Using Theorem 7 we get $\hat{\kappa} = \Phi(\check{v}^0, \check{v}^1)$ and $\kappa = \hat{\kappa}_t$ is a control, that steers the initial data (u^0, u^1) to zero at time T. □

4.1. The Observability Inequality

As for the definition of observability (Definition 3.1) we have the same notion when $(v^0, v^1) \in \mathcal{F}_1$, thus instead of (18) we have observability in time T if

$$C\|(v^0, v^1)\|_{\mathcal{H}}^2 \leq \int_0^T \int_{\Gamma_0} |\Phi(v^0, v^1)|^2 d\Gamma dt \tag{26}$$

for all $(v^0, v^1) \in \mathcal{F}_1$. We can now imitate the calculations from before.

Remark 9. Again we see that $\|(v^0, v^1)\|_{\mathcal{F}_1}$ is chosen as

$$\int_0^T \int_{\Gamma_0} \Phi(v^0, v^1)^2 d\Gamma dt.$$

In Section 5 we investigate when the assumption that $\|(v^0, v^1)\|_{\mathcal{F}_1}$ is in fact a norm on $\mathcal{F}_1$ holds for some $T > 0$.

Theorem 10. *Let $(u^0, u^1) \in \mathcal{F}_1{}'$ and suppose that the adjoint system is observable in time T. Then $\mathcal{J}_1$ has a unique minimizer $(\check{v}^0, \check{v}^1) \in \mathcal{F}_1$.*

Proof. It is easily seen that $\mathcal{J}_1$ is continuous. Coercivity, i.e.

$$\lim_{\|(v^0,v^1)\|_{\mathcal{H}} \to \infty} \mathcal{J}_1(v^0, v^1) = \infty$$

is again verified by the following estimate

$$\begin{aligned} \mathcal{J}_1(v^0, v^1) &\geq \frac{1}{2} \int_0^T \int_{\Gamma_0} \Phi(v^0, v^1)^2 d\Gamma dt - |\langle (u^0, u^1), (v^0, v^1) \rangle_{\mathcal{F}_1{}', \mathcal{F}_1}| \\ &\geq \frac{C}{2} \|(v^0, v^1)\|_{\mathcal{H}}^2 - \|(u^0, u^1)\|_{\mathcal{F}_1{}'} \|(v^0, v^1)\|_{\mathcal{F}_1}, \end{aligned}$$

where we have used the observation inequality (26), thus $\mathcal{J}_1$ is coercive.

$\mathcal{J}_1$ is strictly convex if

$$\mathcal{J}_1(\lambda(v^0, v^1) + (1-\lambda)(\eta^0, \eta^1)) < \lambda \mathcal{J}_1(v^0, v^1) + (1-\lambda)\mathcal{J}_1(\eta^0, \eta^1)$$

holds for all $(v^0, v^1), (\eta^0, \eta^1) \in \mathcal{F}_1$ and $\lambda \in (0,1)$. By calculating the LHS we get

$$\begin{aligned} \mathcal{J}_1(\lambda(v^0, v^1) + (1-\lambda)(\eta^0, \eta^1)) &= \frac{1}{2} \int_0^T \int_{\Gamma_0} \lambda^2 \Phi(v^0, v^1)^2 \\ +2\lambda(1-\lambda)\Phi(v^0, v^1)\Phi(\eta^0, \eta^1) &+ (1-\lambda)^2 \Phi(\eta^0, \eta^1)^2 d\Gamma dt \\ -\langle (u^0, u^1), \lambda(v^0, v^1) &+ (1-\lambda)(\eta^0, \eta^1) \rangle_{\mathcal{F}_1{}', \mathcal{F}_1} \\ &= \lambda \mathcal{J}_1(v^0, v^1) + (1-\lambda)\mathcal{J}_1(\eta^0, \eta^1) \\ -\frac{\lambda(1-\lambda)}{2} \int_0^T \int_{\Gamma_0} &(\Phi(v^0, v^1) - \Phi(\eta^0, \eta^1))^2 d\Gamma dt \end{aligned}$$

and using the observability inequality we have

$$\int_0^T \int_{\Gamma_0} (\Phi(v^0, v^1) - \Phi(\eta^0, \eta^1))^2 d\Gamma dt \geq C \|(v^0, v^1) - (\eta^0, \eta^1)\|_{\mathcal{H}}^2.$$

Thus

$$\mathcal{J}_1(\lambda(v^0, v^1) + (1-\lambda)(\eta^0, \eta^1)) < \lambda\mathcal{J}_1(v^0, v^1) + (1-\lambda)\mathcal{J}_1(\eta^0, \eta^1)$$

for all $(v^0, v^1) \neq (\eta^0, \eta^1)$. □

Corollary 11. *If the adjoint system* (3) *is observable in time* T *then the control system is exact controllable in time* T.

4.2. Minimal $[L^2(\Sigma_0)]^3$-Norm

Let $\zeta \in [L^2(\Sigma_0)]^3$ be a control which also drives the initial data $(u^0, u^1) \in \mathcal{F}_1{}'$ to zero in time T, then

$$\langle (u^0, u^1), (v^0, v^1)\rangle_{\mathcal{F}_1{}', \mathcal{F}_1} - \int_0^T \int_{\Gamma_0} \zeta \cdot v_t d\Gamma dt = 0$$

for all $(v^0, v^1) \in \mathcal{F}_1$ (Theorem 7). In particular for the minimizer $(\check{v}^0, \check{v}^1)$ of $\mathcal{J}_1$ we have

$$\langle (u^0, u^1), (\check{v}^0, \check{v}^1)\rangle_{\mathcal{F}_1{}', \mathcal{F}_1} - \int_0^T \int_{\Gamma_0} \zeta \cdot \check{v}_t d\Gamma dt = 0. \tag{27}$$

The $[L^2(\Sigma_0)]^3$-norm of $\hat{\kappa}$ is after an application of (23)

$$\|\hat{\kappa}\|^2_{[L^2(\Sigma_0)]^3} = \int_0^T \int_{\Gamma_0} \Phi(\check{v}^0, \check{v}^1)^2 d\Gamma dt = \langle (u^0, u^1), (\check{v}^0, \check{v}^1)\rangle_{\mathcal{F}_1{}', \mathcal{F}_1}\,. \tag{28}$$

Combining (27) and (28) gives

$$\begin{aligned}\|\hat{\kappa}\|^2_{[L^2(\Sigma_0)]^3} = \int_0^T \int_{\Gamma_0} \zeta \cdot \check{v}_t d\Gamma dt &\leq \|\zeta\|_{[L^2(\Sigma_0)]^3} \|\check{v}_t\|_{[L^2(\Sigma_0)]^3} \\ &= \|\zeta\|_{[L^2(\Sigma_0)]^3} \|\hat{\kappa}\|_{[L^2(\Sigma_0)]^3}\,, \end{aligned} \tag{29}$$

so $\hat{\kappa}$ is of minimal $[L^2(\Sigma_0)]^3$-norm.

5. Proof of the Observability Inequalities

We cannot simply choose any geometry of Γ_0 and Γ_1, so in order to obtain the necessary estimates which enables us to show that $\|\cdot\|_{\mathcal{F}_0}$ and $\|\cdot\|_{\mathcal{F}_1}$ define norms on $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively. We will choose Γ_0 and Γ_1 in the following way: For a fixed $(x_0, y_0) \in \mathbb{R}^2$ we define the vector field

$$m(x, y) = (x, y) - (x_0, y_0),$$

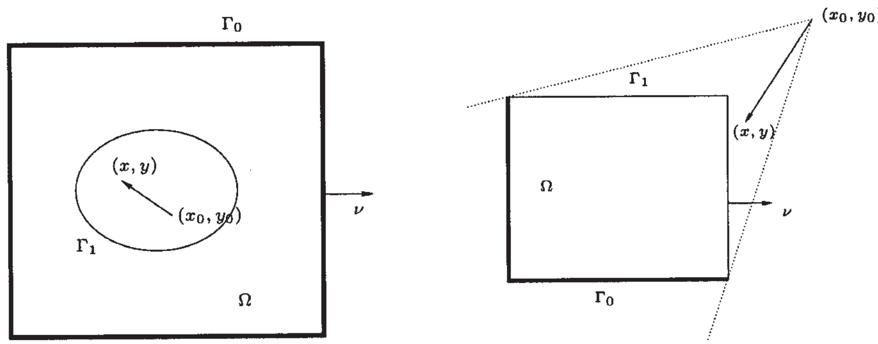

Figure 1: Two possible configurations of $\Gamma = \Gamma_0 \cup \Gamma_1$. To the left we have a plate with a hole, (x_0, y_0) is chosen inside this hole such that $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$. On the right (x_0, y_0) is chosen such that $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 \neq \emptyset$.

and then we assume that $\Gamma = \Gamma_0 \cup \Gamma_1$ is divided such that

$$m \cdot n \geq 0 \qquad \text{on } \Gamma_0\,, \tag{30}$$

$$m \cdot n \leq 0 \qquad \text{on } \Gamma_1\,. \tag{31}$$

Furthermore we assume that Γ satisfies:

(i) $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and $\Gamma \in C^{1,1}$ except at a finite number of corners.

or

(ii) $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 \neq \emptyset$ and $\Gamma \in C^{1,1}$ except at a finite number of corners, furthermore each line segment must be contained entirely in either Γ_0 or Γ_1 and the line segments carrying different boundary conditions must meet in a strictly convex angle.

Two examples of the configurations is shown in Figure 1.

These assumptions will assure that the solution to the adjoint system will satisfy

$$v \in C([0,T]; [H^s(\Omega)]^3 \cap V)\,, \quad v_t \in C([0,T]; V)\,, \quad v_{tt} \in C([0,T]; H)$$

where $\frac{3}{2} < s \leq 2$.

For later convenience we introduce

$$J_0 = \frac{1}{2} \int_0^T \int_{\Gamma_1} (m \cdot n) \Big[D\Big(-\frac{1+\mu}{2} (n_1 \psi_n + n_2 \varphi_n)^2 - \frac{1-\mu}{2} (\psi_n^2 + \varphi_n^2) \Big) - K w_n^2 \Big] d\Gamma dt, \tag{32}$$

$$J_1 = \frac{1}{2}\int_0^T \int_{\Gamma_0} (m\cdot n)\Big[D\left(\psi_x^2 + \varphi_y^2 + 2\mu\psi_x\varphi_y + \frac{1-\mu}{2}(\psi_y+\varphi_x)^2\right) + K((w_x+\psi)^2 + (w_y+\varphi)^2)\Big] d\Gamma dt, \quad (33)$$

$$J_2 = \frac{\rho h}{2}\int_0^T \int_{\Gamma_0} (m\cdot n)\left[\frac{h^2}{12}\psi_t^2 + \frac{h^2}{12}\varphi_t^2 + w_t^2\right] d\Gamma dt, \quad (34)$$

$$Y_1 = c(v_t, m\cdot\nabla v)\Big|_0^T, \quad (35)$$

$$Y_2 = \frac{\rho h^3}{12}\int_\Omega \psi\psi_t + \varphi\varphi_t dxdy\Big|_0^T, \quad (36)$$

$$Y_3 = \rho h\int_\Omega w w_t dxdy\Big|_0^T. \quad (37)$$

We are now ready to prove that $\|\cdot\|_{\mathcal{F}_0}$ and $\|\cdot\|_{\mathcal{F}_1}$ indeed defines norms, we will do so using multiplier techniques essentially due to Komornik. The proof given here is inspired by Lagnese and Lions [5].

Lemma 12. *Let $v \in [H^s(\Omega)]^3$ for some $\frac{3}{2} < s \le 2$, then*

$$Y_1 + \int_0^T c(v_t, v_t)dt - K\int_0^T \int_\Omega \psi(w_x+\psi) + \varphi(w_y+\varphi)dxdydt = J_2 - J_1 - J_0. \quad (38)$$

Proof. Let $v \in [H^s(\Omega)]^3$ be a solution to the adjoint system. In Green's formula we will use $m\cdot\nabla v \in [H^{s-1}(\Omega)]^3$ as a multiplier, here $m\cdot\nabla v$ equals the vector $(m\cdot\nabla\psi, m\cdot\nabla\varphi, m\cdot\nabla w)$. We find that

$$\int_0^T \int_\Omega \mathcal{A}v(m\cdot\nabla v)dxdydt = -\int_0^T a(v, m\cdot\nabla v)dt + \int_{\Gamma_1} \mathcal{B}v(m\cdot\nabla v)d\Gamma dt\,, \quad (39)$$

where we have used $\mathcal{B}v = 0$ on Γ_0. Note that the integral on the LHS is well defined since $\mathcal{A}v \in [H^{s-2}(\Omega)]^3$, and since $s-1 < 2-s < \frac{1}{2}$ we have that $m\cdot\nabla v \in [H_0^{2-s}(\Omega)]^3$ and due to the continuity in t of v, v_t and v_{tt}, the integral is simply considered as a duality product.

We now treat each term in (39) separately.

First we consider $\int_0^T\int_\Omega \mathcal{A}v(m\cdot\nabla v)dxdydt$, and from the first equation in the adjoint system we get

$$\int_0^T \int_\Omega \mathcal{A}v(m \cdot \nabla v)dxdydt = \int_0^T \int_\Omega \mathbf{C}v_{tt}(m \cdot \nabla v)dxdydt$$
$$= \int_0^T \int_\Omega c(v_{tt}, m \cdot \nabla v)dxdydt. \quad (40)$$

Now a useful identity is

$$\begin{aligned}\int_0^T \int_\Omega & \psi_{tt}(m \cdot \nabla \psi)dxdydt \\ &= \int_\Omega \psi_t(m \cdot \nabla \psi)dxdy\Big|_0^T - \int_0^T \int_\Omega \psi_t(m \cdot \nabla \psi_t)dxdydt \\ &= \int_\Omega \psi_t(m \cdot \nabla \psi)dxdy\Big|_0^T - \int_0^T \int_\Omega \frac{1}{2}\operatorname{div}(m\psi_t^2) - \psi_t^2 dxdydt \\ &= \int_\Omega \psi_t(m \cdot \nabla \psi)dxdy\Big|_0^T - \frac{1}{2}\int_0^T \int_{\Gamma_0} (m \cdot n)\psi_t^2 d\Gamma dt + \int_0^T \int_\Omega \psi_t^2 dxdydt\,,\end{aligned}$$

where we have used that $v_t = 0$ on Γ_1. We can now rewrite (40) as

$$\int_0^T c(v_{tt}, m \cdot \nabla v)dt = Y_1 - J_2 + \int_0^T c(v_t, v_t)dt\,, \quad (41)$$

where Y_1 and J_2 are given by (35) and (34), respectively.

For the term $a(v, m \cdot \nabla v)$ in (39) we will use the identity

$$\begin{aligned}\int_\Omega \psi_x(m \cdot \nabla \psi)_x dxdy &= \int_\Omega \psi_x(\psi_x + m_1\psi_{xx} + m_2\psi_{xy})dxdy \\ &= \int_\Omega \psi_x^2 + \frac{1}{2}m_1(\psi_x^2)_x + \frac{1}{2}m_1(\psi_x^2)_y dxdy = \frac{1}{2}\int_\Omega \operatorname{div}(m\psi_x^2)dxdy\,.\end{aligned}$$

Here ψ can of course be interchanged with φ or w and we could have differentiated with respect to y instead. Furthermore we have the identity

$$\begin{aligned}&\int_\Omega \varphi_y(m \cdot \nabla \psi)_x + \psi_x(m \cdot \nabla \psi)_y dxdy \\ &= \int_\Omega \varphi_y(m_1\psi_{xx} + \psi_x + m_2\psi_{yx}) + \psi_x(m_1\varphi_{xy} + \varphi_y + m_2\varphi_{yy})dxdy \\ &= \int_\Omega (m_1\psi_x\varphi_y)_x + (m_2\psi_x\phi_y)_y dxdy \\ &= \int_\Omega \operatorname{div}(m\psi_x\varphi_y)dxdy.\end{aligned}$$

From these identities and the divergence theorem we get

$$\begin{aligned}
&a_0(\psi,\varphi,m\cdot\nabla\psi,m\cdot\nabla\phi)\\
=&D\int_\Omega \psi_x(m\cdot\nabla\psi)_x+\varphi_y(m\cdot\nabla\varphi)_y+\mu\varphi_y(m\cdot\nabla\psi)_x\\
&+\mu\psi_x(m\cdot\nabla\varphi)_y+\frac{1-\mu}{2}(\psi_y+\varphi_x)((m\cdot\nabla\psi)_y+(m\cdot\nabla\varphi)_x)dxdy\\
=&\frac{D}{2}\int_\Omega \operatorname{div}(m\psi_x^2+m\varphi_y^2+2\mu m\psi_x\varphi_y+\frac{1-\mu}{2}m(\psi_y+\varphi_x)^2)dxdy\\
=&\frac{D}{2}\int_\Gamma (m\cdot n)\left[\psi_x^2+\varphi_y^2+2\mu\psi_x\varphi_y+\frac{1-\mu}{2}(\psi_y+\varphi_x)^2\right]d\Gamma, \qquad (42)
\end{aligned}$$

and

$$\begin{aligned}
a_1(v,m\cdot\nabla v)=&\int_\Omega (w_x+\psi)((m\cdot\nabla w)_x+m\cdot\nabla\psi)\\
&+(w_y+\varphi)((m\cdot\nabla w)_y+m\cdot\nabla\varphi)dxdy\\
=&\int_\Omega \frac{1}{2}\operatorname{div}\left(m(w_x+\psi)^2+m(w_y+\varphi)^2\right)\\
&-(\psi(w_x+\psi)+\varphi(w_y+\varphi))\,dxdy\\
=&\frac{1}{2}\int_\Gamma (m\cdot n)\left((m(w_x+\psi)^2+(w_y+\varphi)^2\right)d\Gamma\\
&-\int_\Omega \psi(w_x+\psi)+\varphi(w_y+\varphi)dxdy. \qquad (43)
\end{aligned}$$

We now split the boundary integrals appearing in (42) and (43) into integrals over Γ_0 and Γ_1 and including the t-integration we find:

$$\begin{aligned}
&\int_0^T a(v,m\cdot\nabla v)dt\\
&\quad=\frac{1}{2}\int_0^T\int_{\Gamma_1}(m\cdot n)\Big[D\left(\psi_x^2+\varphi_y^2+2\mu\psi_x\varphi_y+\frac{1-\mu}{2}(\psi_y+\varphi_x)^2\right)\\
&\quad+K(w_x^2+w_y^2)\Big]d\Gamma dt+J_1-K\int_0^T\int_\Omega \psi(w_x+\psi)+\varphi(w_y+\varphi)dxdydt\,,
\end{aligned}$$

where J_1 is given by (33).

Since $\psi=\varphi=w=0$ on Γ_1 we have

$$\nabla\psi=n\psi_n,\qquad \nabla\varphi=n\varphi_n,\qquad \nabla w=nw_n,\qquad \text{on } \Gamma_1.$$

Using this we rewrite $\int a(v, m\cdot v)dt$ as

$$\int_0^T a(v,m\cdot v)dt = \frac{1}{2}$$

$$\int_0^T \int_{\Gamma_1} (m\cdot n)\Big[D\left(n_1^2\psi_n^2 + n_2^2\varphi_n^2 + 2\mu n_1 n_2\psi_n\varphi_n + \frac{1-\mu}{2}(n_2\psi_n + n_1\varphi_n)^2\right)$$
$$+ Kw_n^2\Big]d\Gamma dt + J_1 - K\int_0^T \int_\Omega \psi(w_x+\psi) + \varphi(w_y+\varphi)dxdydt$$
$$= \frac{1}{2}\int_0^T \int_{\Gamma_1} (m\cdot n)\Big[D\left(\frac{1+\mu}{2}(n_1\psi_n + n_2\varphi_n)^2 + \frac{1-\mu}{2}(\psi_n^2+\varphi_n^2)\right)$$
$$+ Kw_n^2\Big]d\Gamma dt + J_1 - K\int_0^T \int_\Omega \psi(w_x+\psi) + \varphi(w_y+\varphi)dxdydt. \quad (44)$$

Finally, we rewrite the domain integral of the last term in (39) as follows:

$$\int_{\Gamma_1} \mathcal{B}v(m\cdot\nabla v)d\Gamma = \int_{\Gamma_1} \mathcal{B}v(m\cdot n)v_n d\Gamma$$
$$= \int_{\Gamma_1} (m\cdot n)\Big[D\Big(n_1^2\psi_n^2 + n_2^2\varphi_n^2 + 2\mu n_1 n_2 \psi_n\varphi_n$$
$$+ \frac{1-\mu}{2}(n_2^2\psi_n^2 + 2n_1n_2\psi_n\varphi_n + n_1^2\varphi_n^2)\Big) + Kw_n^2\Big]d\Gamma$$
$$= \int_{\Gamma_1} (m\cdot n)\Big[D\left((n_1\psi_n + n_2\varphi_n)^2 - 2n_1n_2\psi_n\varphi_n + 2\mu n_1n_2\psi_n\varphi_n\right.$$
$$\left.+ \frac{1-\mu}{2}((n_2\psi_n - n_1\varphi_n)^2 + 4n_1n_2\psi_n\varphi_n)\right) + Kw_n^2\Big]d\Gamma$$
$$= \int_{\Gamma_1} (m\cdot n)\Big[D\left((n_1\psi_n + n_2\varphi_n)^2\right.$$
$$\left.+ \frac{1-\mu}{2}(n_2\psi_n - n_1\varphi_n)^2\right) + Kw_n^2\Big]d\Gamma. \quad (45)$$

Inserting (40), (41), (44) and (45) back into (39) yields

$$Y_1 + \int_0^T c(v_t,v_t)dt - \int_0^T \int_\Omega \psi(w_x+\psi) + \varphi(w_y+\varphi)dxdydt$$
$$= J_2 - J_1 - \frac{1}{2}\int_0^T \int_{\Gamma_1} (m\cdot n)\Big[D\left(\frac{1+\mu}{2}(n_1\psi_n + n_2\varphi_n)^2 + \frac{1-\mu}{2}(\psi_n^2+\varphi_n^2)\right)$$
$$+ Kw_n^2\Big]d\Gamma dt + \int_0^T \int_{\Gamma_1} (m\cdot n)\Big[D\left((n_1\psi_n + n_2\varphi_n)^2\right.$$

$$+ \frac{1-\mu}{2}(n_2\psi_n - n_1\varphi_n)^2\Big) + Kw_n^2\Big]d\Gamma dt. \quad (46)$$

Now, since

$$\frac{1}{2}\int_0^T\int_{\Gamma_1}(m\cdot n)\left[D\left(\frac{1+\mu}{2}(n_1\psi_n+n_2\varphi_n)^2+\frac{1-\mu}{2}(\psi_n^2+\varphi_n^2)\right)+Kw_n^2\right]d\Gamma dt$$
$$-\int_0^T\int_{\Gamma_1}(m\cdot n)\left[D\left((n_1\psi_n+n_2\varphi_n)^2+\frac{1-\mu}{2}(n_2\psi_n-n_1\varphi_n)^2\right)+Kw_n^2\right]d\Gamma dt$$
$$=\frac{1}{2}\int_0^T\int_{\Gamma_1}(m\cdot n)\Big[D\Big(\frac{-3+\mu}{2}(n_1\psi_n+n_2\varphi_n)^2+\frac{1-\mu}{2}(\psi_n^2+\varphi_n^2)$$
$$+(1-\mu)(n_1\psi_n+n_2\varphi_n)^2-(1-\mu)(\psi_n^2+\varphi_n^2)\Big)-Kw_n^2\Big]d\Gamma dt = J_0\,,$$

where J_0 is given by (32), (46) can be written

$$Y_1+\int_0^T c(v_t,v_t)dt - K\int_0^T\int_\Omega \psi(w_x+\psi)+\varphi(w_y+\varphi)dxdydt$$
$$= J_2 - J_1 - J_0. \quad \square$$

Lemma 13. *Let $v\in[H^s(\Omega)]^3$ for some $\frac{3}{2}<s\le 2$ and let $\varepsilon>0$. Then*

$$Y_1+(1-\varepsilon)Y_2+\varepsilon Y_3+\varepsilon\int_0^T c(v_t,v_t)dt+(1-\varepsilon)\int_0^T a_0(\psi,\varphi,\psi,\varphi)dt$$
$$+(1-2\varepsilon)\rho h\int_0^T\int_\Omega w_t^2dxdydt$$
$$+\varepsilon K\int_0^T\int_\Omega w_x^2+w_y^2-\psi^2-\varphi^2dxdydt = J_2-J_1-J_0\,. \quad (47)$$

Proof. Using the functions $\tilde\psi=\psi$, $\tilde\varphi=\varphi$ and $\tilde w=0$ in Green's formula we get

$$\frac{\rho h^3}{12}\int_\Omega \psi\psi_{tt}+\varphi\varphi_{tt}dxdy$$
$$=-a_0(\psi,\varphi,\psi,\varphi)-K\int_\Omega\psi(w_x+\psi)+\varphi(w_y+\varphi)dxdy\,,$$

where we again have interpreted the LHS of Green's formula as a duality between $[H^{s-2}(\Omega)]^3$ and $[H_0^{2-s}(\Omega)]^3$. Integration from 0 to T followed by an integration by parts on the LHS of the result yields

$$\frac{\rho h^3}{12}\int_0^T\int_\Omega \psi_t^2+\varphi_t^2 dxdy - Y_2 = \int_0^T a_0(\psi,\varphi,\psi,\varphi)dt + K\int_0^T\int_\Omega \psi(w_x+\psi)+\varphi(w_y+\varphi)dxdydt\,, \quad (48)$$

where Y_2 is given by (36).

On the other hand, an application of Green's formula with the functions $\tilde{\psi}=\tilde{\varphi}=0$ and $\tilde{w}=w$ gives

$$\rho h\int_\Omega ww_{tt}dxdy = -K\int_\Omega w_x(\psi+w_x)+w_y(w_y+\varphi)dxdy.$$

If we integrate from 0 to T and then apply integration by parts on the LHS, we get

$$\rho h\int_0^T\int_\Omega w_t^2 dxdydt - Y_3 = K\int_0^T\int_\Omega w_x(\psi+w_x)+w_y(w_y+\varphi)dxdydt\,, \quad (49)$$

where Y_3 is given by (37).

Now, let $\varepsilon>0$ and multiply (48) with $(1-\varepsilon)$ and (49) with ε and adding the results to (38) gives

$$Y_1+(1-\varepsilon)Y_2+\varepsilon Y_3+\varepsilon\int_0^T c(v_t,v_t)dt+(1-\varepsilon)\int_0^T a_0(\psi,\varphi,\psi,\varphi)dt + (1-2\varepsilon)\rho h\int_0^T\int_\Omega w_t^2 dxdydt + \varepsilon K\int_0^T\int_\Omega w_x^2+w_y^2-\psi^2-\varphi^2 dxdydt = J_2-J_1-J_0. \quad \square$$

Theorem 14. *Let $(v^0,v^1)\in\mathcal{F}_1$. Then there is a $T_0>0$ and a constant $c_1>0$ such that we have the observability inequality*

$$c_1(T-T_0)\|(v^0,v^1)\|_{\mathcal{H}}^2 \le \|(v^0,v^1)\|_{\mathcal{F}_1}^2 \quad (50)$$

for $T>T_0$.

Proof. We will use (47) to make the estimate (50), but first we estimate $a_1(v,v)$:

$$a_1(v,v)=\int_\Omega (w_x+\psi)^2+(w_y+\varphi)^2 dxdy$$

$$\leq \int_\Omega w_x^2 + w_y^2 + \psi^2 + \varphi^2 + 2(|w_x||\psi| + |w_y||\varphi|)dxdy$$
$$\leq 2\int_\Omega (w_x^2 + w_y^2 + \psi^2 + \varphi^2)dxdy.$$

We apply the above estimate to the last term on the LHS of (47) as follows:

$$\varepsilon K \int_0^T \int_\Omega w_x^2 + w_y^2 - \psi^2 - \varphi^2 dxdydt \geq \frac{\varepsilon K}{2} \int_0^T a_1(v,v)dt - 2\varepsilon K \int_0^T \int_\Omega \psi^2 + \varphi^2 dxdydt\,. \quad (51)$$

From Poincare's inequality (since $\Gamma_1 \neq \emptyset$) there exists a constant $\lambda_0 > 0$ such that

$$\|\psi\|^2_{L^2(\Omega)} \leq \lambda_0 \|\nabla\psi\|^2_{L^2(\Omega)} \qquad \text{for all } \psi \in H^1_{\Gamma_1}(\Omega).$$

Thus we can estimate on the last term of (51)

$$2\varepsilon K \int_0^T \|\psi\|^2_{L^2(\Omega)} + \|\varphi\|^2_{L^2(\Omega)} dt \leq 2\varepsilon K\lambda_0 \int_0^T \|\nabla\psi\|^2_{L^2(\Omega)} + \|\nabla\varphi\|^2_{L^2(\Omega)} dt$$
$$\leq 2\varepsilon K\lambda_0 \int_0^T \|\psi\|^2_{H^1(\Omega)} + \|\varphi\|^2_{H^1(\Omega)} dt \leq 2\varepsilon K \frac{\lambda_0}{\alpha_0} \int_0^T a_0(\psi,\varphi,\psi,\varphi)dt\,, \quad (52)$$

where the last estimate follows from [6]. Inserting (52) into (51) gives

$$\varepsilon K \int_0^T \int_\Omega w_x^2 + w_y^2 - \psi^2 - \varphi^2 dxdydt$$
$$\geq \frac{\varepsilon K}{2} \int_0^T a_1(v,v)dt - 2\varepsilon K \frac{\lambda_0}{\alpha_0} \int_0^T a_0(\psi,\varphi,\psi,\varphi)dt.$$

Using the above estimate in (47) we find

$$Y_1 + (1-\varepsilon)Y_2 + \varepsilon Y_3 + \varepsilon \int_0^T c(v_t,v_t)dt + \left((1-\varepsilon) - 2\varepsilon K\frac{\lambda_0}{\alpha_0}\right) \int_0^T a_0(\psi,\varphi,\psi,\varphi)dt + K\frac{\varepsilon}{2}\int_0^T a_1(v,v)dt + (1-2\varepsilon)\rho h \int_0^T \int_\Omega w_t^2 dxdydt \leq J_2 - J_1 - J_0. \quad (53)$$

Now, if we choose ε such that

$$0 < \varepsilon \leq \frac{1}{2(1 + K\frac{\lambda_0}{\alpha_0})} \Leftrightarrow \varepsilon \leq (1-\varepsilon) - 2\varepsilon K \frac{\lambda_0}{\alpha_0},$$

then the last term on the LHS of (53) is positive, thus it can be discarded. Then (53) becomes

$$\begin{aligned}&Y_1+(1-\varepsilon)Y_2+\varepsilon Y_3\\&+\varepsilon\int_0^T(c(v_t,v_t)+a_0(\psi,\varphi,\psi,\varphi)+\frac{K}{2}a_1(v,v))dt\le J_2-J_1-J_0.\end{aligned}$$

Since we have homogeneous boundary conditions, the energy E_0 of the system is constant, and using the expression for the energy from [7] we get

$$\begin{aligned}Y_1+(1-\varepsilon)Y_2+\varepsilon Y_3+\varepsilon\int_0^T(E_0+\frac{1}{2}(c(v_t,v_t)+a_0(\psi,\varphi,\psi,\varphi)))dt\\\le J_2-J_1-J_0.\end{aligned}$$

$c(v_t,v_t)$ and $a_0(\psi,\varphi,\psi,\varphi)$ are both nonnegative, hence

$$Y_1+(1-\varepsilon)Y_2+\varepsilon Y_3+\varepsilon TE_0\le J_2-J_1-J_0. \tag{54}$$

We have (since $\Gamma_0\neq\emptyset$) furthermore

$$|Y_1+(1-\varepsilon)Y_2+\varepsilon Y_3|\le\varepsilon T_0E_0$$

for some $T_0>0$, and using this in (54) we get

$$\varepsilon(T-T_0)E_0\le J_2-J_1-J_0. \tag{55}$$

We now turn to the RHS of the above; by the assumption (31) J_0 is obvious nonnegative, so it can be omitted. J_1 is estimated as follows

$$\begin{aligned}J_1=&\frac{1}{2}\int_0^T\int_{\Gamma_0}(m\cdot n)\Big[D\left(\psi_x^2+\varphi_y^2+2\mu\psi_x\varphi_y+\frac{1-\mu}{2}(\psi_y+\varphi_x)^2\right)\\&+K((w_x+\psi)^2+(w_y+\varphi)^2)\Big]d\Gamma dt\\\ge&\frac{1}{2}\int_0^T\int_{\Gamma_0}(m\cdot n)\Big[D\left(\psi_x^2+\varphi_y^2-2\mu|\psi_x||\varphi_y|+\frac{1-\mu}{2}(\psi_y+\varphi_x)^2\right)\\&+K((w_x+\psi)^2+(w_y+\varphi)^2)\Big]d\Gamma dt\\\ge&\frac{1}{2}\int_0^T\int_{\Gamma_0}(m\cdot n)\Big[D\left((1-\mu)(\psi_x^2+\varphi_y^2)+\frac{1-\mu}{2}(\psi_y+\varphi_x)^2\right)\\&+K((w_x+\psi)^2+(w_y+\varphi)^2)\Big]d\Gamma dt\,,\end{aligned}$$

hence J_1 is nonnegative. From (55) we then have

$$\varepsilon(T - T_0)E_0 \leq J_2 \,. \tag{56}$$

The energy E_0 is equivalent to the usual norm on $\mathcal{H}$ and since $m \cdot n \geq 0$ on Γ_0, J_2 is equivalent to $\|(\cdot,\cdot)\|_{\mathcal{F}_1}$, hence there exists a constant $c_1 > 0$ such that

$$c_1(T - T_0)\|(v^0, v^1)\|^2_{\mathcal{H}} \leq \|(v^0, v^1)\|^2_{\mathcal{F}_1}$$

for $T > T_0$. □

Theorem 15. *Let $(v^0, v^1) \in \mathcal{F}_0$. Then there is a $T_0 > 0$ and a constant $c_0 > 0$ such that we have the observability inequality*

$$c_0(T - T_0)\|(v^0, v^1)\|^2_{\mathcal{H}'} \leq \|(v^0, v^1)\|^2_{\mathcal{F}_0}$$

for $T > T_0$.

Proof. Recall from [7] the isomorphisms $C_c : H \to H$ and $A_a : V \to V'$ defined by

$$(C_c v \mid u)_H = c(v, u) \qquad \text{for all } u, v \in H \,,$$
$$\langle A_a v, u\rangle_{V' \times V} = a(v, u) \qquad \text{for all } u, v \in V \,.$$

We will denote by $\hat{C}_c$ the extension of C_c to V'.

Assume v is the solution to the adjoint system with initial data $(v^0, v^1) \in \mathcal{H}'$, and define $\hat{v}^1 = (-A_a\hat{C}_c)^{-1}v^1$. Then let $\hat{v}$ be the solution to tha adjoint system with initial data $(\hat{v}^1, v^0) \in \mathcal{H}$. We see that $v = \hat{v}_t$ by construction.

For initial data in $\mathcal{H}'$ the 'energy' functional equivalent to the $\mathcal{H}'$-norm is

$$\hat{E}(t) = \frac{1}{2}(c(v, v) + a(\hat{v}, \hat{v}))$$

which is a constant, $\hat{E}_0$.

If we return to (56), i.e.

$$\varepsilon(T - T_0)E_0 \leq J_2 \,,$$

we see that since $(\hat{v}^1, v^0) \in \mathcal{H}$ we can simply write this as

$$\varepsilon(T - T_0)\hat{E}_0 \leq \int_0^T \int_{\Gamma_0} \left[\frac{h^2}{12}\psi^2 + \frac{h^2}{12}\varphi^2 + w^2\right] d\Gamma dt \,,$$

hence there is a constant c_0 such that

$$c_0(T - T_0)\|(v^0, v^1)\|^2_{\mathcal{H}'} \leq \|(v^0, v^1)\|^2_{\mathcal{F}_0}. \qquad \square$$

We have now verified that the injections

$$\mathcal{H} \hookrightarrow \mathcal{F}_0 \hookrightarrow \mathcal{H}', \qquad D(\bar{A}) \hookrightarrow \mathcal{F}_1 \hookrightarrow \mathcal{H},$$
$$\mathcal{H} \hookrightarrow \mathcal{F}_0{}' \hookrightarrow \mathcal{H}', \qquad \mathcal{H}' \hookrightarrow \mathcal{F}_1{}' \hookrightarrow D(\bar{A}),$$

are continuous. We see by the construction of $\mathcal{F}_0$ and $\mathcal{F}_1$ and their respective norms $\|(\cdot,\cdot)\|_{\mathcal{F}_0}$ and $\|(\cdot,\cdot)\|_{\mathcal{F}_1}$ that for $(v^0, v^1) \in \mathcal{F}_0$ we have

$$\|(v^0, v^1)\|_{\mathcal{F}_0} \leq M_0\|(v^0, v^1)\|_{\mathcal{H}'}$$

for some $M_0 > 0$, and for $(v^0, v^1) \in \mathcal{F}_1$ we have

$$\|(v^0, v^1)\|_{\mathcal{F}_1} \leq M_1\|(v^0, v^1)\|_{\mathcal{H}}$$

for some $M_1 > 0$. Then for $T > T_0$, $\|(\cdot,\cdot)\|_{\mathcal{F}_0}$ and $\|(\cdot,\cdot)\|_{\mathcal{F}_1}$ indeed define norms on $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively. We have then justified the calculations done in the first sections of this paper, and we have the following.

Corollary 16. *Let $\Gamma = \Gamma_0 \cup \Gamma_1$ satisfy the assumptions in the beginning of Section 5. Then there is a control $\kappa = -v|_{\Sigma_0}$ and a $T_0 > 0$ such that for $T > T_0$ the control system* (1) *with initial data $(u^0, u^1) \in \mathcal{F}_0{}'$ $(\supset \mathcal{H})$ is exactly controllable.*

Corollary 17. *Let $\Gamma = \Gamma_0 \cup \Gamma_1$ satisfy the assumptions in the beginning of Section 5. Then there is a control $\kappa = v_{tt}|_{\Sigma_0}$ and a $T_0 > 0$ such that for $T > T_0$ the control system* (1) *with initial data $(u^0, u^1) \in \mathcal{F}_1{}'$ $(\supset \mathcal{H}')$ is exactly controllable.*

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