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ON THE COHOMOLOGY OF CERTAIN RANK TWO VECTOR BUNDLES ON HIRZEBRUCH SURFACES

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we give a cohomological characterization of extensions of certain line bundles on a Hirzebruch surface F_e, $e > 0$.

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1. The Statements

Let F_e, $e > 0$, denote the Hizebruch surface with a section with self-intersection $-e$. For any $L \in \mathrm{Pic}(F_e)$ and any vector bundle E on F_e we will say that E has property β (resp. α) with respect to L if $h^1(F_e, E \otimes L^{\otimes m}) = 0$ for all $m \in \mathbb{Z}$ (resp. for all $m \in \mathbb{Z}$ such that $h^0(F_e, E \otimes L^{\otimes m}) \neq 0$). We think that property α is nicer for reasonable L. We take as a basis of $\mathrm{Pic}(F_e) \cong \mathbb{Z}^2$ a fiber f of the ruling $\pi : F_e \to \mathbf{P}^1$ and the section h with negative self-intersection. Thus $h^2 = -e$, $h{\cdot}f = 1$ and $f^2 = 0$. we have $\omega_{F_e} \cong \mathcal{O}_{F_e}(-2h-(e+2)f)$. $\mathcal{O}_{F_e}(\alpha h+\beta f)$ is spanned (resp. ample) if and only if $\alpha \geq 0$ and $\beta \geq \alpha e$ (resp. $\alpha > 0$ and $\beta > e\alpha$). Hence every ample line bundle on F_e is spanned. The Leray spectral sequence of π and Serre duality give $h^1(F_e, \mathcal{O}_{F_e}(\gamma h + \delta f)) = 0$ if and only if either $\gamma \geq 0$ and $\delta \geq e\gamma - 1$ or $\gamma = -1$ or $\gamma \leq -2$ and $-\delta - e - 2 \geq e(-\gamma - 2) - 1$ (i.e. $\delta \leq e\gamma + e - 1$). In Section 2 we will prove the following results.

Theorem 1. *Fix integers $x > e > 0$. Let E be a rank 2 torsion free sheaf on F_e, $e > 0$. Set $R := \mathcal{O}_{F_e}(h + xf)$ and $\mathcal{O}_{F_e}(uh + vf) := c_1(E)$. Assume $h^1(F_e, E \otimes R^{\otimes z}) = 0$ for all $z \in \mathbb{Z}$. Let c be the first integer such*

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that $h^0(F_e, E \otimes R^{\otimes c}) \neq 0$. *Then either* E *is an extension of two line bundles or there is an integer* $b \geq 0$ *and a locally complete intersection* $Z \subset F_e$ *with* $0 < z := \text{length}(Z) = (2v + 4cx - 2b - 2x - u - 2c + 1)(u + 2c)/2$, $u + 2c \geq 0$, $v + 2cx - b \geq -1$, *and* E *fits in an exact sequence*

$$0 \to \mathcal{O}_{F_e}(bf) \to E \otimes R^{\otimes c} \to \mathcal{I}_Z((u + 2c)h + (v + 2xc - b)f) \to 0\,. \qquad (1)$$

Theorem 2. *Fix integers* $x_1 > e$ *and* $x > e > 0$ *such that* $x_1 \neq x$. *Let* E *be a rank 2 torsion free sheaf on* F_e, $e > 0$. *Set* $R := \mathcal{O}_{F_e}(h + xf)$ *and* $R_1 := \mathcal{O}_{F_e}(h + x_1 f)$. *Let* c *be the first integer such that* $h^0(F_e, E \otimes R^{\otimes c}) \neq 0$. *Assume* $h^1(F_e, E \otimes R^{\otimes z}) = 0$ *for all* $z \in \mathbb{Z}$ *and* $h^1(F_e, E \otimes R^{\otimes c} \otimes R_1^*) = 0$. *Then* E *is locally free and it is an extension of 2 line bundles.*

Theorem 3. *There is an indecomposable rank 2 vector bundle* E *on* F_e, $e > 0$, *such that* E *is an extension of two line bundles and* $h^1(F_e, E \otimes H^{\otimes z}) = 0$ *for all* $z \in \mathbb{Z}$ *and all line bundles* $H := \mathcal{O}_{F_e}(\alpha h + \beta f)$ *with* $\alpha > 0$ *and* $\beta \geq e\alpha$.

We work over an algebraically closed field $\mathbb{K}$.

2. The Proofs

Remark 1. Let X be a smooth and connected projective surface and $R \in \text{Pic}(X)$ such that $|R|$ contains the sum of an effective divisor and of an ample divisor. Let E be a torsion free sheaf on X such that $h^1(X, E \otimes R^{\otimes t}) = 0$ for infinitely many negative integers t. Consider the exact sequence

$$0 \to E \to E^{**} \to E^{**}/E \to 0\,.$$

Since $h^0(X, E^{**} \otimes R^{\otimes t}) = 0$ for $t \ll 0$, E is locally free.

Lemma 1. *Set* $L := \mathcal{O}_{F_e}(ah + bf)$, $e > 0$, *and* $R := \mathcal{O}_{F_e}(h + xf)$, $x > e > 0$. $h^1(F_e, L \otimes R^{\otimes z}) = 0$ *for all* $z \in \mathbb{Z}$ *if and only if either* $a \geq 0$ *and* $ea - 1 \geq b \leq (a + 2)x - e - 1$ *or* $a < 0$ *and* $b \geq ax - 1$.

Proof. Fix $a' \leq -2$. By Serre duality we have $h^1(F_e, \mathcal{O}_{F_e}(a'h + b'f)) = h^1(F_e, \mathcal{O}_{F_e}((-a' - 2)h + (-b' - 2 - e)f))$. Hence $h^1(F_e, \mathcal{O}_{F_e}(a'h + b'f)) = 0$ if and only if $(-b' - 2 - e) \geq e(-a' - 2) - 1$, i.e. if and only if $b' \leq ea' + e - 1$. If $a \geq 0$ and $b \leq ea - 2$, then $h^1(F_e, L) \neq 0$. Now assume $a \geq 0$ and $b \geq ea - 1$. We have $h^1(F_e, L \otimes R^{\otimes z}) = 0$ for all $z \geq -a - 1$. Take any $z \leq -a - 2$. We get $h^1(F_e, L \otimes R^{\otimes z}) = 0$ if and only if $b + zx \leq ea + ez + e - 1$. Since $x > e$, the inequality $b + zx \leq ea + ez + e - 1$ is satisfied for all $z \leq -a - 2$ if and only if it is satisfied for the integer $z := -a - 2$, i.e. if and only if $b \leq (a + 2)x - e - 1$, Now assume $a < 0$. If $z = -a - 1$, then $h^1(F_e, L \otimes R^{\otimes z}) = 0$. Now fix any $z \leq -a-2$. We have $h^1(F_e, L \otimes R^{\otimes z}) = 0$ if and only if $b + zx \leq ea + ez + e - 1$. Since $x > e$, $b + zx \leq ea + ez + e - 1$ for all $z \leq -a-2$ if and only this is true for $z = -a-2$, i.e. if and only if $b \leq ax + 2x - ea - e - 1$. Since $x > e$, as above we get that $h^1(F_e, L \otimes R^{\otimes z}) = 0$ for all $z \geq -a$ if and only if this equality is true for $z := -a$, i.e. if and only if $b \geq ax - 1$. □

Lemma 2. *Set $L := \mathcal{O}_{F_e}(ah + bf)$ and $H := \mathcal{O}_{F_e}(\alpha h + \beta f)$, $e > 0$, $\alpha > 0$ and $\beta \geq e\alpha$. $h^1(F_e, L \otimes H^{\otimes z}) = 0$ for all $z \in \mathbb{Z}$ if either $a \geq 0$ and $ea - 1 \geq b \leq ae + e - 1$ or $a < 0$ and $b \geq ae - 1$.*

Proof. First assume $a \geq 0$. Obviously, $h^1(F_e, L \otimes H^{\otimes z}) = 0$ if $z \geq 0$ or if $a + \alpha z = -1$ or if $a + \alpha z \leq -2$ and $b + \beta z \leq ea + e\alpha z + e - 1$. The latter inequality is satisfied, because in this subcase $z < 0$, $e\alpha \leq \beta$ and $b \leq ea + e - 1$. Now assume $a < 0$. If $z \geq 0$ and $a + \alpha z = -1$, then $h^1(F_e, L \otimes H^{\otimes z}) = 0$. If $a + \alpha z \geq 0$, then $z > 0$ and hence $b + \beta z \geq b + e\alpha \geq ea - 1 + e\alpha$, i.e. $h^1(F_e, L \otimes H^{\otimes z}) = 0$. If $z \geq 0$ and $a + \alpha z \leq -2$, then $b + \beta z \geq ae - 1 + ez\alpha$ and hence $h^1(F_e, L \otimes H^{\otimes z}) = 0$. Now assume $z < 0$ and $a + \alpha z \leq -2$. In this subcase $h^1(F_e, L \otimes H^{\otimes z}) = h^1(F_e, \mathcal{O}_{F_e}((-z\alpha - a - 2)h + (-z\beta - b - e - 2)f)$. Hence $h^1(F_e, L \otimes H^{\otimes z}) = 0$ if and only if $-z\beta - b - e - 2 \geq -ze\alpha - ea - 2e - 1$, i.e. if and only if $(-z)(\beta - e\alpha) + ea + e - 1 \geq b$. The last inequality is true by our assumptions. □

Remark 2. Fix integers $x > e > 0$ and $r \geq 2$. Set $R := \mathcal{O}_{F_e}(h + xf)$. Let $L_i := \mathcal{O}_{F_e}(a_i h + b_i f)$, $1 \leq i \leq z$, such that $h^1(F_e, L_i \otimes R^{\otimes z}) = 0$ for all $z \in \mathbb{Z}$ and all $1 \leq i \leq r$. Let E_2 denote any extension of L_1 be L_2. If $r \geq 3$ take as E_r any extension of E_{r-1} by L_r. Oviously, $h^1(F_e, E_r \otimes R^{\otimes z}) = 0$ for all $z \in \mathbb{Z}$. There are many vector bundles as above which are not direct sums of line bundles (for certain integers a_i, b_i, $1 \leq i \leq r$. Take for instance $r = 2$. We have $\mathrm{Hom}(L_1, L_2) = \mathcal{O}_{F_e}((a_2 - a_1)h + (b_2 - b_1)f) = 0$. Hence $h^1(F_e, \mathrm{Hom}(L_1, L_2)) > 0$ if and only if either $a_2 \geq a_1$ and $b_2 - b_1 \leq (a_2 - a_1) - 2$ or $a_2 \leq a_1 - 2$ and $b_2 - b_1 \leq e(a_2 - a_1) + e - 1$. Since $h > 0$, $h^1(F_e, \mathcal{O}_{F_e}(2h)) > 0$ and hence there is a non-trival extension F of $\mathcal{O}_{F_e}$ by $\mathcal{O}_{F_e}(2h)$. For any such H and any ample line bundle H on F_e, F is not H-semistable and $\mathcal{O}_{F_e}(2h)$ is its unique line subbundle with maximal H-slope. Since F comes from a non-trivial extension, the existence of such line subbundle implies that F is indecomposable. Fix an integer $a \leq -4$, set $L_1 := \mathcal{O}_{F_e}(ah)$, $L_2 := \mathcal{O}_{F_e}((a+2)h)$ and take $E_2 := F \otimes L_1$.

Remark 3. Fix a rank 2 vector bundle E on F_e, $e > 0$, and $L_i, M_i \in \mathrm{Pic}(F_e)$, $i = 1, 2$. Set $\mathcal{O}_{F_e}(uh + vf) := c_1(E)$, $\mathcal{O}_{F_e}(ah + bf) := L_1$ and $\mathcal{O}_{F_e}(ch + df) := M_1$. Assume that E is an extension of L_2 by L_1 and an extension of M_2

by M_1. Hence $c_1(E) = L_1 \otimes L_2 = M_1 \otimes M_2$. Hence $L_2 \cong \mathcal{O}_{F_e}((u-a)h+(v-b)f)$ and $M_2 \cong \mathcal{O}_{F_e}((u-c)h+(v-d)f)$. Since $c_2(E) = L_1 \cdot L_2 = m_1 \cdot M_2$, we get $a(v-b)+(u-a)b-ea(u-a) = c(v-d)+(u-c)d-ec(u-c)$. Since the injective maps $i_1 : L_1 \to F$ and $i_2 : M_1 \to F$ have locally free cokernels, none of them may factor through the other, unless they coincide. If $F \ncong L_1 \oplus L_2$, then $h^1(F_e, \mathcal{O}_{F_e}((2a-u)h+(2b-v)f) \neq 0$, i.e either $2a-u \geq 0$ and $2b-v \leq e(2a-u)-2$ or $2a-u \leq -2$ and $v-2b-e-2 \leq e(u-2a-2)-2$. Assume $h^1(F_e, F \otimes R^{\otimes z}) = 0$ for all $z \in \mathbb{Z}$. First assume $a \geq u-a$. Hence $h^0(F_e, L_2 \otimes R^{\otimes(-a-2)}) = 0$. From the exact sequence

$$0 \to L_1 \otimes R^{\otimes z} \to F \otimes R^{\otimes z} \to L_2 \otimes R^{\otimes z} \to 0 \tag{2}$$

for $z = -a-2$ we get $h^1(F_e, L_1 \otimes R^{\otimes(-a-2)}) = 0$, i.e. $h^1(F_e, \mathcal{O}_{F_e}(-2h+(b-ax-2x)f) = 0$, i.e. $b-ax-2x \leq e+1$. Similarly, $d-cx-2x \leq e+1$. From (2) taking $z = a-u-1$ we get $h^1(F_e, \mathcal{O}_{F_e}((2a-u-1)h+(b+ax-ux-x)f) = 0$, i.e. either $2a \geq u+1$ and $b+ax-ux-1 \geq e(2a-u-1)-1$ or $2a = u$ or $2a < u$ and $ux+x-ax-b-e-2 \geq e(u-2a-1)-1$. Similarly, either $2c \geq u+1$ and $d+cx-ux-1 \geq e(2c-u-1)$ or $2c = u$ or $2c < u$ and $ux+x-cx-d-e-2 \geq e(u-2c-1)-1$. First assume $a > u-a$. Let $H := \mathcal{O}_{F_e}(h+zf)$ be an ample line bundle. Taking $z \gg 0$ we get $L_1 \cdot H > L_2 \cdot H$. Hence F is not H-semistable and L_1 is the unique H-destabilizing line subbundle of F. We saw that $b+ax-ux-x \geq e(2a-u-1)-1$. If $c > u-c$, we get $L_1 \cong M_1$ and hence $L_2 \cong M_2$. If $c \leq u-c$, then the non-zero map $i_2(M_1) \to L_2$ shows that either $F \cong L_1 \oplus L_2$, $M_1 \cong L_2$ and $M_2 \cong L_1$ or $c < u-a$.

Proof of Theorems 1 and 2. Assume $h^1(F_e, E \otimes R^{\otimes z}) = 0$ for all $z \in \mathbb{Z}$. Since R is ample, Remark 1 gives that E is locally free. Hence from now on we assume that E is locally free. Fix any $\sigma \in H^0(F_e, E \otimes R^{\otimes c}) \backslash \{0\}$ and let A denote the divisorial part of the zero-locus of σ. Hence A is a non-negative divisor and σ induces an exact sequence

$$0 \to \mathcal{O}_{F_e}(A) \to E \otimes R^{\otimes c} \to \mathcal{I}_Z((u+2c)h+(v+2xc)f)(-A) \to 0 \tag{3}$$

with Z a zero-dimensional locally complete intersection subscheme of F_e. Set $\mathcal{O}_{F_e}(ah+bf) := \mathcal{O}_{F_e}(A)$ and $z := \text{length}(Z)$. If $z = 0$, then E is an extension of 2 line bundles. Hence from now on we assume $z > 0$. The minimality of c and the effectiveness of A give $a \geq 0$, $b \geq 0$ and either $b \leq x-1$ or $a = 0$. Since $h^0(F_e, \mathcal{I}_Z(((u+2c)h+(v+2xc)f)(-A) \otimes R^{\otimes t}) = 0$ for $t \ll 0$ and $h^1(F_e, E \otimes R^{\otimes t}) = 0$ for $t \ll 0$, (3) gives $h^1(F_e, \mathcal{O}_{F_e}(A) \otimes R^{\otimes t}) = 0$ for $t \ll 0$. As in the proof of Lemma 1 we get $b \leq ax-1$. Notice that $h^i(F_e, \mathcal{O}_{F_e}(A) \otimes R^{\otimes(-a-1)}) = 0$ for $i = 0,1,2$. Hence the assumption $h^1(F_e, E \otimes R^{\otimes(c-a-1)}) = 0$

and the minimality of c give $h^i(F_e, \mathcal{I}_Z(((u+2c-a-1-a)h+(v+2xc-2ax-b)f)) = 0$ for $i = 0, 1$. Since $h^1(Z, T) = 0$ for any coherent sheaf T on Z, the equality $h^1(F_e, \mathcal{I}_Z(((u+2c-a-1-a)h+(v+2xc-2ax-b)f) = 0$ gives $z \le h^0(F_e, \mathcal{O}_{F_e}(u+2c-a-1-a)h+(v+2cx-2ax--xb)f))$ and

$$h^1(F_e, \mathcal{O}_{F_e}(u+2c-a-1-a)h+(v+2cx-2xa-x-b)f)) = 0 \qquad (4)$$

The equality $h^0(F_e, \mathcal{I}_Z(((u+2c-a-1-a)h+(v+2cx-2xa-x-b)f)) = 0$ gives $z \ge h^0(F_e, \mathcal{O}_{F_e}(u+2c-a-1-a)h+(v+2cx-2a-b)f))$. Hence $z = h^0(F_e, \mathcal{O}_{F_e}(u+2c-a-1-a)h+(v+2cx-2ax-x-b)f))$.Since we assumed $z > 0$, $u+2c-2a-1 \ge 0$ and $v+2cx-2ax-b \ge 0$. Hence if $z > 0$, then (4) is equivalent to the inequality $v+2cx-2ax-x-b \ge e(u+2c-2a-1)-1$, while (if the latter inequality is satisfied) the equality $z = h^0(F_e, \mathcal{O}_{F_e}(u+2c-a-1-a)h+(v+2cx-2ax-x-b)f))$ is equivalent to the equality $z = \sum_{j=0}^{u+2c-2a-1} v+2cx-2ax-x-b-je$, i.e. $z = (2v+4cx-4ax-2x-b-e(u+2c-2a-1)(u+2c-2a)/2$. Now assume $z = 0$. Hence either $u+2c-a-1 < 0$ or $v+2cx-2ax-x-b < 0$. Since $h^2(F_e, R^{\otimes t}(A)) = 0$ for all $t \ge -a-1$, we also get $h^1(F_e, \mathcal{O}_{F_e}(((u+2c-a-t)h+(v+2xc-2ax-tx-b)f) = 0$ for all $t \ge -a-1$. If $u+2c-2a-1 \ge 0$, then taking $t = -a-1$ we get $v+2xc-2axx-2x-b \ge -1$. If $u+2c-a-1 \le -2$, then take $t := -u-2c+a-1$; we get $(v+2xc-2ax-(-u-2c+a-1)x-b) \le -e-1$, i.e. $v+4xc-ax+ux+x+e+1-b \le 0$. In this case E is an extension of two line bundles.

(a) Here we will check that if $z > 0$, then $a \le 1$. Assume $z > 0$ and $a \ge 2$. We know that $z = h^0(F_e, \mathcal{O}_{F_e}(u+2c-a-1-a)h+(v+2cx-2ax-x-b)f))$ and $h^1(F_e, \mathcal{I}_Z((u+2c-a-1-a)h+(v+2cx-ax-x-b)f)) = 0$. Since E is locally free and $\omega_{F_e} \cong \mathcal{O}_{F_e}(-2h-(e+2)f)$, (3) implies that Z satisfies the Cayley-Bacarach condition with respect to the line bundle $\mathcal{O}_{F_e}((u+2c-2-2a)h+(v+2cx-2b-e-2)f)$ and in particular $h^1(F_e, \mathcal{I}_Z((u+2c-2-2a)h+(v+2cx-2b-e-2)f)) > 0$ ([1], Theorem 1.4). Since $a \ge 2$, $b \le x-1$ and $x \ge e+1$, the line bundle $\mathcal{O}_{F_e}(h+(ax+x-b-e-2)f)$ is spanned and with no higher cohomologyy. Hence $h^1(F_e, \mathcal{I}_Z((u+2c-a-1-a)h+(v+2cx-ax-x-b)f)) = 0$ implies $h^1(F_e, \mathcal{I}_Z((u+2c-2-2a)h+(v+2cx-2b-e-2)f)) = 0$, contradiction.

(b) Now assume $a = 1$ and $z > 0$. As in part (a) we get a contradiction if $2x-b-e-2 \ge 0$. Since $x > e$ and $b > x$, the inequality $2x-b-e-2 \ge 0$ is satisfied, unless $b = x-1 = e$. Now assume $b = x-1 = e$. We know that $u+2c-5 \ge 0$ and that $h^1(F_e, \mathcal{I}_Z((u+2c-5)h+(v+2ce+2c-5e-2)f)) = 0$. Since $\omega_{F_e} \otimes c_1(E)(-2A) \cong \mathcal{O}_{F_e}((u+2c-4)h+(v+2ce+2c-2e-e-2)$, again we got a contradiction using the Cayley-Bacharach condition. Hence we the proof of Theorem 1 is over.

(c) Now assume $h^1(F_e, E \otimes R^{\otimes c} \otimes R_1^*) = 0$. By tensoring (3) with R_1^* we get $z = (2v + 4cx - 2b - 2x_1 - u - 2c + 1)(u + 2c)/2$. Since $x_1 \neq x$, this is absurd. □

Proof of Theorem 3. Apply Lemmas 1 and 2 and Remark 2. □

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