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A SIXTH ORDER LINEAR MULTISTEP METHOD FOR THE DIRECT SOLUTION OF $y'' = f(x, y, y')$

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Abstract: A linear multistep method (LMM)with continuous coefficients is considered and directly applied to solve second order initial value problems (IVPs). The continuous method is used to obtain Multiple Finite Difference Methods (MFDMs) (each of order 6) which are combined as simultaneous numerical integrators to provide a direct solution to IVPs over sub-intervals which do not overlap. The convergence of the MFDMs is discussed by conveniently representing the MFDMs as a block method and verifying that the block method is zero-stable and consistent. The superiority of the MFDMs over the methods in Awoyemi [2], [3] is established numerically.

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1. Introduction

The second-order ordinary differential equation of the form

$$y'' = f(x, y, y'), \tag{1}$$
$$y(a) = y_0, \quad y'(a) = \delta_0 \,,$$

where f is a continuous function, is conventionally solved by first reducing it to a system of first-order differential equations and then applying the various methods available for solving systems of first order IVPs. This approach is

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extensively discussed in the literature and in this paper we cite just a few notable ones such as Lambert [14], [16], Brugnano and Trigiante [4], Onumanyi et al [19], [18], Fatunla [7], Jennings [13] and Jator [12]. Although there has been tremendous success with this approach, it has certain draw backs. For instance, computer programs associated with the methods are often complicated especially when incorporating subroutines to supply the starting values for the methods resulting in longer computer time and more computational work. In addition Vigo-Aguiar and Ramos [24] stated that these methods do not utilize additional information associated with a specific ordinary differential equation, such as the oscillatory nature of the solution.

Considerable attention has been devoted to the development of various methods for solving $y'' = f(x, y), \quad y(a) = y_0, \quad y'(a) = \delta_0$ directly without first reducing it to a system of first order differential equations. For instance, Twizell and Khaliq [22], Yusuph and Onumanyi [25], Simos [20], Fatunla [6], Henrici [10], and Lambert [15], [16].

Several methods have also been proposed in the literature for solving (1) directly without first reducing it to an equivalent first-order system. For instance, Hairer and Wanner [9] proposed Nystrom type methods and stated order conditions for determining the parameters of the methods. Other methods of the Runge-Kutta type are due to Chawla and Sharma [5]. Methods of the LMM type have been considered by Vigo-Aguiar and Ramos [23] and Awoyemi [2], [3]. In [23], variable stepsize multistep schemes based on the Falkner method were developed and directly applied to (1) in a predictor corrector (PC) mode. In [2] and [3] LMMs were proposed and also implemented in a predictor − corrector mode using the Taylor series algorithm to supply the starting values. Although, the implementation of the methods in a PC mode yielded good accuracy, the procedure is more costly to implement. For instance, PC subroutines are very complicated to write, since they require special techniques for supplying the starting values and for varying the step-size, which lead to longer computer time and more human effort. Our method is cheaper to implement, since it is self-starting and therefore does not share these drawbacks.

Recently, Jator and Li [11] proposed an order 5 method that was implemented without the need for either predictors or starting values from other methods. In this paper, we develop an order 6 method which is applied as a block method. We also show that the block method is zero-stable and consistent, hence the method converges. The method is derived through interpolation and collocation, see Lie and Norsett [17], Atkinson [1], Onumanyi et al [18]. We take advantage of this approach by exploring the link between the the finite difference methods (FDMs) and the k-step multistep collocation (MC) procedure, which are two important global methods which have been used with piecewise continuous approximate solutions of ordinary differential equations (ODEs) Gladwell and Sayers [8].

The paper is organized as follows. In Section 2, we derive an approximation $\overline{y}(x)$ for the exact solution $y(x)$ which is continuous using the matrix inversion approach. Section 3 is devoted to the specification of the method and how the MFDMs are obtained. The analysis and implementation of the MFDMs are discussed in Section 4. Numerical examples are given in Section 5 to show the efficiency of the MFDMs. Finally, the conclusion of the paper is discussed in Section 6.

2. The Derivation of the Method

In this section, we use the interpolation and collocation procedures to characterize the LMM method that is of interest to us by choosing the right number of interpolation points (r) and the right number of collocation points (s). The process leads to a system of $(r+s)$ equations involving $(r+s)$ unknown coefficients, which are determined by the matrix inversion approach. The formula is much easier to derive using the matrix inversion approach (see [25]) rather than using the purely algebraic approach. It is worth noting that LMMs have been widely used to provide the numerical solution of first order systems of IVPs. In this paper, we propose a LMM that is applied directly to (1) without first reducing it to a system of first order ODEs. Although the proposed LMM can be obtained as a finite difference method with constant coefficients as in the conventional fashion, it has more advantages when initially derived and expressed with continuous coefficients for the direct solution of (1). Thus, we approximate the exact solution $y(x)$ by seeking the continuous method $\overline{y}(x)$ of the form

$$\overline{y}(x) = \sum_{j=0}^{r-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{s-1} \beta_j(x) f_{n+j}\,, \tag{2}$$

where $x \in [a,b]$ and the following notations are introduced. The positive integer $k \geq 2$ denotes the step number of the method (2), which is applied directly to provide the solution to (1). In this light, we seek a solution on

$$\pi_N : a = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < ... < x_N = b\,,$$
$$h = x_{n+1} - x_n, \quad n = 0, 1, \ldots, N\,,$$

where π_N is a partition of $[a,b]$ and h is the constant step-size of the partition of π_N. The number of interpolation points r and the number of distinct collocation points s are chosen to satisfy $2 \le r \le k$, and $0 < s \le k+1$ respectively. We then construct a k-step multistep collocation method of the form (2) by imposing the following conditions.

$$\overline{y}\,(x_{n+j}) = y_{n+j}, j = 0, 1, 2, \ldots, r-1\,, \tag{3}$$

$$\overline{y}\,''\,(x_{n+j}) = f_{n+j}, j = 0, 1, 2, \ldots, s-1\,. \tag{4}$$

Equations (3) and (4) lead to a system of $(r+s)$ equations and $(r+s)$ unknown coefficients to be determined. In order to solve this system, we require that the linear k-step method (2) be defined by the assumed polynomial basis functions

$$\alpha_j(x) = \sum_{j=0}^{r-1} \alpha_{i+1,j} P_i(x); j \epsilon \{0, 1, \ldots, r-1\}\,,$$

and

$$\beta_j(x) = \sum_{j=0}^{s-1} h^2 \beta_{i+1,j} P_i(x); j \epsilon \{0, 1, \ldots, s-1\}\,,$$

where the constants $\alpha_{i+1,j}$ and $h^2\beta_{i+1,j}$, $j = \{0, 1, \ldots, r+s-1\}$ are undetermined elements of the $(r+s) \times (r+s)$ matrix E, given by

$$E = \begin{pmatrix} \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,r-1} & h^2\beta_{1,0} & \cdots & h^2\beta_{1,s-1} \\ \alpha_{2,0} & \alpha_{2,1} & \cdots & \alpha_{2,r-1} & h^2\beta_{2,0} & \cdots & h^2\beta_{2,s-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha_{r+s,0} & \alpha_{r+s,1} & \cdots & \alpha_{r+s,r-1} & h^2\beta_{r+s,0} & \cdots & h^2\beta_{r+s,s-1} \end{pmatrix}.$$

We also define the interpolation/collocation matrix W as

$$W = \begin{pmatrix} P_0(x_n) & \cdots & P_{r+s-1}(x_n) \\ P_0(x_{n+1}) & \cdots & P_{r+s-1}(x_{n+1}) \\ \vdots & & \vdots \\ P_0(x_{n+r-1}) & \cdots & P_{r+s-1}(x_{n+r-1}) \\ P''_0(x_n) & \cdots & P''_{r+s-1}(x_n) \\ P''_0(x_{n+1}) & \cdots & P''_{r+s-1}(x_{n+1}) \\ \vdots & & \vdots \\ P''_0(x_{n+s-1}) & \cdots & P''_{r+s-1}(x_{n+s-1}) \end{pmatrix}.$$

We consider further notations by defining the following vectors:

$\bigwedge = (y_n, y_{n+1}, \ldots, y_{n+r-1}, f_n, f_{n+1}, \ldots, f_{n+s-1})^T$,

$\Upsilon(x) = (P_0(x), P_1(x), \ldots, P_{r+s-1}(x))^T$, where T denotes the transpose of the vectors.

The collocation points are selected from the extended set Ω, where

$$\Omega = \{x_n, \dots, x_{n+k}\} \bigcup \{x_{n+k-1}, x_{n+k}\}.$$

Theorem 2.1. *Let $\overline{y}(x)$ satisfy conditions (3) and (4), then for the non-singular matrix W, and the matrix of undetermined coefficients E the following hold:*

I. $\overline{y}(x) = \bigwedge^T \left(W^{-1}\right)^T \Upsilon(x)$.

II. $E = W^{-1}$.

Proof. The proof is the same as that given in [25] for deriving methods for $y'' = f(x,y)$. □

In the next section, we discuss how the MFDMs are obtained from $\overline{y}(x)$.

3. Specification of the Method

Our method is obtained from part (I) of Theorem 2.1 after some manipulation and expressed in the form (2) given by

$$\overline{y}(x) = \sum_{j=0}^{r-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{s-1} \beta_j(x) f_{n+j}\,, \tag{5}$$

and the formula for the derivative is expressed as

$$\overline{y}\,'(x) = \frac{1}{h} \sum_{j=0}^{r-1} \alpha'_j(x) y_{n+j} + h \sum_{j=0}^{s-1} \beta'_j(x) f_{n+j}\,, \tag{6}$$

with the following specifications: $r = 2$, $s = 6$, $k = 5$, $\Upsilon_i(x) = x^i$, $i = 0, 1, \dots, 7$. For convenience, we express $\alpha_j(x)$ and $\beta_j(x)$ as functions of z where $z = (x - x_{n+4})/h$ in what follows:

$$\alpha_0(z) = -(3+z)\,, \quad \alpha_1(z) = (4+z),$$

$$\beta_0(z) = \frac{1}{10080}(2184 + 674z + 84z^3 + 35z^4 - 21z^5 - 14z^6 - 2z^7),$$

$$\beta_1(z) = \frac{1}{10080}(28308 + 10021z - 560z^3 - 210z^4 + 147z^5 + 84z^6 + 10z^7)),$$

$$\beta_2(z) = \frac{1}{5040}(8904 + 4210z + 840z^3 + 245z^4 - 231z^5 - 98z^6 - 10z^7),$$

$$\beta_3(z) = \frac{1}{5040}(5964 + 6227z - 1680z^3 - 70z^4 + 357z^5 + 112z^6 + 10z^7),$$

$$\beta_4(z) = \frac{1}{10080}(168 + 3818z + 5040z^2 + 1820z^3 - 525z^4 - 525z^5 - 126z^6 - 10z^7),$$

$$\beta_5(z) = \frac{1}{10080}(84 - 107z + 336z^3 + 350z^4 + 147z^5 + 28z^6 + 2z^7).$$

The coefficients $\alpha'_j(x)$ and $\beta'_j(x)$ are easily obtained by differentiating $\alpha_j(x)$ and $\beta_j(x)$.

The MFDMs are obtained by evaluating (5) at $x = \{x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}\}$ to obtain the following methods

$$y_{n+2} = -y_n + 2y_{n+1} + \frac{h^2}{240}(18f_n + 209f_{n+1} + 4f_{n+2} + 14f_{n+3} - 6f_{n+4} + f_{n+5})\,, \tag{7}$$

$$y_{n+3} = -2y_n + 3y_{n+1} + \frac{h^2}{240}(35f_n + 442f_{n+1} + 202f_{n+2} + 52f_{n+3} - 13f_{n+4} + 2f_{n+5})\,, \tag{8}$$

$$y_{n+4} = -3y_n + 4y_{n+1} + \frac{h^2}{120}(26f_n + 337f_{n+1} + 212f_{n+2} + 142f_{n+3} + 2f_{n+4} + f_{n+5})\,, \tag{9}$$

$$y_{n+5} = -4y_n + 5y_{n+1} + \frac{h^2}{24}(7f_n + 90f_{n+1} + 66f_{n+2} + 52f_{n+3} + 23f_{n+4} + 2f_{n+5})\,. \tag{10}$$

Additional Equations are obtained from the derivative by imposing that

$$\overline{y'}(x) = \delta(x), \quad \overline{y'}(a) = \delta_0\,,$$

where $\delta(x)$ is a continuous function. In particular, to start the initial value problem for $n = 0$, we obtain the following equation from $\overline{y'}(a) = \delta_0$.

$$h\delta_0 = -y_0 + y_1 + \frac{h^2}{10080}(-2462f_0 - 4315f_1 + 3044f_2 - 1882f_3 + 682f_4 - 107f_5)\,. \tag{11}$$

It is worth noting that the derivatives are provided by $\delta(x_{n+\tau}) = \delta_{n+\tau}$, $\tau = 1, \ldots, 5$ as follows:

$$h\delta_{n+1} = -y_n + y_{n+1} + \frac{h^2}{10080}(863f_n + 5674f_{n+1} - 2542f_{n+2} + 1492f_{n+3} - 529f_{n+4} + 82f_{n+5})\,,$$

$$h\delta_{n+2} = -y_n + y_{n+1} + \frac{h^2}{10080}(674f_n$$

$$+ \quad 10133f_{n+1} + 4612f_{n+2} - 314f_{n+3} + 10f_{n+4} + 5f_{n+5})\,,$$

$$h\delta_{n+3} = -y_n + y_{n+1} + \frac{h^2}{10080}(751f_n$$

$$+ \quad 9482f_{n+1} + 10226f_{n+2} + 5300f_{n+3} - 641f_{n+4} + 82f_{n+5})\,,$$

$$h\delta_{n+4} = -y_n + y_{n+1} + \frac{h^2}{10080}(674f_n$$

$$+ \quad 10021f_{n+1} + 8420f_{n+2} + 12454f_{n+3} + 3818f_{n+4} - 107f_{n+5})\,,$$

$$h\delta_{n+5} = -y_n + y_{n+1} + \frac{h^2}{10080}(863f_n$$

$$+ \quad 8810f_{n+1} + 11794f_{n+2} + 6868f_{n+3} + 13807f_{n+4} + 3218f_{n+5})\,.$$

Remark 3.1. We emphasize that the method (10) is the main method that can be implemented conventionally in the PC mode in which case predictors and starting values will be demanded. Therefore, the implementation strategy is less economical than the one we have adopted in this paper, in terms of both computational work and computer time.

4. Analysis and Implementation of the Method

Following Fatunla [6] and Lambert [14] we define the local truncation error associated with the conventional form of (2) to be the linear difference operator

$$L[y(x);h] = \sum_{j=0}^{k}\{\alpha_j y(x+jh) - h^2\beta_j y''(x+jh)\}\,. \tag{12}$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (12) as a Taylor series about the point x to obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 hy' + \ldots, +C_q h^q y^q(x) + \ldots, \tag{13}$$

where the constant coefficients C_q , $q = 0, 1, \ldots$ are given as follows:

$$C_0 = \sum_{j=0}^{k}\alpha_j\,,$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j\,,$$

$$\vdots$$

$$C_q = \frac{1}{q!}[\sum_{j=1}^{k} j^q \alpha_j - q(q-1)\sum_{j=1}^{k} j^{q-2}\beta_j].$$

According to Henrici [10], we say that the method (5) has order p if

$$C_0 = C_1 = \ldots = C_p = C_{p+1} = 0, \quad C_{p+2} \neq 0\,.$$

Our calculations reveal that the methods (7) to (11) have order $p = 6$ and error constants given by the vector $C_8 = (\frac{-221}{60480}, \frac{-137}{20160}, \frac{-19}{2016}, \frac{-95}{6048}, \frac{-199}{24192})^T$.

In order to analyze the methods for zero-stability, we normalize (7) to (11) and write them as a block method given by the matrix difference equation

$$A^0 Y_{\mu+1} = A^1 Y_\mu + h^2[B^0 F_{\mu+1} + B^1 F_\mu] + hC^1\Delta_\mu\,, \tag{14}$$

where $Y_{\mu+1} = (y_{n+1}, \ldots, y_{n+5})^T$, $Y_\mu = (y_{n-4}, \ldots, y_n)^T$, $F_{\mu+1} = (f_{n+1}, \ldots, f_{n+5})^T$, $F_\mu = (f_{n-4}, \ldots, f_n)^T$, $\Delta_\mu = (\delta_{n-4}, \ldots, \delta_n)^T$, $\mu = 0, 1, \ldots$ and $n = 0, 5, \ldots$ and the matrices A^0, A^1, B^0, B^1,and C^1 are defined as follows:

A^0 is an identity matrix of dimension 5,

$$A^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B^0 = \begin{pmatrix} 863/2016 & -761/2520 & 941/5040 & -341/5040 & 107/10080 \\ 544/315 & -37/63 & 136/315 & -101/630 & 8/315 \\ 3501/1120 & -9/140 & 87/112 & 85/168 & 9/224 \\ 1424/315 & 176/315 & 608/315 & -16/63 & 16/315 \\ 11875/2016 & 625/504 & 3125/1008 & 625/1008 & 275/2016 \end{pmatrix},$$

$$B^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1231/5040 \\ 0 & 0 & 0 & 0 & 71/126 \\ 0 & 0 & 0 & 0 & 123/140 \\ 0 & 0 & 0 & 0 & 1609/1260 \\ 0 & 0 & 0 & 0 & 1525/1008 \end{pmatrix}, \; C^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

It is worth noting that zero-stability is concerned with the stability of the difference system in the limit as h tends to zero. Thus, as $h \to 0$, the method (14) tends to the difference system

$$A^0 Y_{\mu+1} - A^1 Y_\mu = 0\,,$$

whose first characteristic polynomial $\rho(R)$ is given by

$$\rho(R) = \det(RA^0 - A^1) = R^4(R-1)\,. \tag{15}$$

Following Fatunla [6], the block method (14) is zero-stable, since from (15), $\rho(R) = 0$ satisfy $|R_j| \leq 1$, $j = 1, \dots, k$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 2. The block method (14) is consistent as it has order $p > 1$. According to Henrici [10], we can safely assert the convergence of the block method (14).

It is vital to note that the main method given by (10) can be used as a numerical integrator directly and singly in the conventional way on overlapping sub-intervals. However, our method is implemented more efficiently by combining methods (7) to (11), each of order six with relatively small error constants, as simultaneous integrators for IVPs without looking for any other methods to provide the starting values. We proceed by explicitly obtaining initial conditions at $x_{n+5}, n = 0, 5, \dots, N-5$ using the computed values $\overline{y}(x_{n+5}) = y_{n+5}$ and $\delta(x_{n+5}) = \delta_{n+5}$ over sub-intervals $[x_0, x_5], \dots, [x_{N-5}, x_N]$ which do not overlap (see [25]). For instance, $n = 0$, $\mu = 0$, $(y_1, \dots, y_5)^T$ are simultaneously obtained over the sub-interval $[x_0, x_5]$, as y_0 is known from the IVP, for $n = 5$, $\mu = 1$, $(y_6, \dots, y_{10})^T$ are simultaneously obtained over the sub-interval $[x_5, x_{10}]$, as y_5 is known form the previous block, and so on. Hence, the sub-intervals do not over-lap and the solutions obtained in this manner are more accurate that those obtained in the conventional way.

5. Numerical Examples

In this section, we have tested the performance our method on 4 problems by considering two non-linear IVPs, (Examples 5.1 and 5.2), the "almost" periodic orbit problem solved by Stiefel and Bettis [21] (Example 5.3) and a mildly stiff IVP (Example 5.4). For each example we find absolute errors of the approximate solution in π_N, where N is chosen to be divisible by k. All computations were carried out using our written *Mathematica* code in *Mathematica 5.2* (see Tables 1 to 6).

Example 5.1. We consider the non-linear IVP which was also solved by Awoyemi [2], [3]

$$\begin{aligned} &2yy'' - (y')^2 + 4y^2 = 0, \quad y(\frac{\pi}{6}) = \frac{1}{4}, \quad y'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}, \\ &\text{Exact}: y(x) = (\sin x)^2 . \end{aligned}$$

In this example, our method of order $p = 6$ is compared with the methods in Awoyemi [2], [3], each of order $p = 6$ and $p = 8$ respectively. Hence, the methods are comparable in both computational demands and the level of accuracy at the mesh points. In the area of computational work, our method is self-starting, while the methods in Awoyemi [2], [3] involve the use of a predictor and the Taylor series algorithm to supply the starting values, which increase the computational cost. Thus, the computational cost is lower in our method than the methods in Awoyemi [2], [3]. In terms of accuracy, our method performs better than those given in Awoyemi [2], [3], despite the fact that we used a lager step-size $h = 0.049213$. Therefore, for this example, our method is clearly superior. Our method is also highly efficient since it involves less computational work and yields highly accurate results. The details of the numerical results at some selected points are given in Table 1. The x-values are given to one decimal place.

x	Awoyemi Order $p=6$ $h=0.003125$	Awoyemi and Kayode Order $p=8$ $h=0.003125$	Our Method Order $p=6$ $h=0.049213$
1.1	4.69215×10^{-7}	4.16328×10^{-7}	2.80474×10^{-10}
1.2	4.08029×10^{-7}	4.58667×10^{-7}	2.79504×10^{-10}
1.3	2.28974×10^{-7}	4.09282×10^{-7}	2.14906×10^{-10}
1.4	0.81287×10^{-7}	2.62955×10^{-7}	0.54975×10^{-10}
1.5	5.24472×10^{-7}	0.45539×10^{-7}	1.15455×10^{-10}
1.6	10.8974×10^{-7}	4.80548×10^{-7}	4.48252×10^{-10}
1.7	17.5373×10^{-7}	10.3225×10^{-7}	7.79692×10^{-10}
1.8	24.8148×10^{-7}	16.7850×10^{-7}	11.8405×10^{-10}
1.9	32.2842×10^{-7}	23.8575×10^{-7}	16.3181×10^{-10}
2.0	39.4302×10^{-7}	31.1084×10^{-7}	20.5676×10^{-10}

Table 1: Absolute errors, $|y(x) - y|$, for example 5.1, where $y(x) = (\sin x)^2$

Example 5.2. We consider the non-linear IVP which was also solved by Awoyemi [2], [3] for the step-size $h = 0.003125$.

$$y'' - x(y')^2 = 0, \quad y(0) = 1, \quad y'(0) = 1/2\,,$$
$$\text{Exact}:\ y(x) = 1 + \frac{1}{2}\ln((2+x)/(2-x))\,.$$

Except in the isolated case $x = 0.1$ in favor of the method in [3], it is observed that our method performs better than those given in Awoyemi [2] and [3] despite the fact that we used a lager step-size $h = 0.05$. Hence, for this example, our method is clearly superior. The details of the numerical results at some selected points are given in Table 2.

x	Awoyemi Order $p=6$ $h=0.003125$	Awoyemi and Kayode Order $p=8$ $h=0.003125$	Our Method Order $p=6$ $h=0.05$
0.1	0.26075×10^{-9}	0.66391×10^{-13}	0.71629×10^{-11}
0.2	1.98167×10^{-9}	0.20012×10^{-9}	$0.15091. \times 10^{-10}$
0.3	6.50741×10^{-9}	1.72007×10^{-9}	0.45286×10^{-10}
0.4	15.5924×10^{-9}	5.89464×10^{-9}	1.08084×10^{-10}
0.5	31.5045×10^{-9}	14.4347×10^{-9}	1.78186×10^{-10}
0.6	56.3746×10^{-9}	41.8664×10^{-9}	4.44344×10^{-10}
0.7	96.1640×10^{-9}	53.1096×10^{-9}	7.44460×10^{-10}
0.8	156.868×10^{-9}	91.1317×10^{-9}	15.0098×10^{-10}
0.9	248.698×10^{-9}	149.242×10^{-9}	37.5797×10^{-10}
1.0	387.984×10^{-9}	237.189×10^{-9}	47.4108×10^{-10}

Table 2: Absolute errors, $|y(x) - y|$, for Example 5.2, where $y(x) = 1 + \frac{1}{2}\ln((2+x)/(2-x))$

Example 5.3. In this example, we test the performance of our method on the nearly periodic IVP which was earlier studied by Stiefel and Bettis [21] and also solved by Lambert and Watson [15]

$$y'' + y = 0.001e^{ix} = 0, \quad y(0) = 1, \quad y'(0) = 0.9995i\,,$$

which has the equivalent form

$$u'' + u = 0.001\cos(x), \quad u(0) = 1, \quad u'(0) = 0\,, \tag{16}$$

$$v'' + v = 0.001\sin(x), \quad v(0) = 0, \quad v'(0) = 0.9995\,, \tag{17}$$

with the following theoretical solution:

$$y(x) = u(x) + iv(x),\ u, v \in R, \quad u(x) = \cos(x) + 0.0005x\sin(x),$$

$$v(x) = \sin(x) - 0.0005x\cos(x).$$

which represents motion on a perturbation of a circular orbit in the complex plane in which the point $y(x)$ spirals slowly outwards so that the distance of this point from the center of the orbit at time x is given by $\gamma(x) = [u^2(x) + v^2(x)]^{\frac{1}{2}}$.

The system given by (16) and (17) was solved numerically to obtain the values of u and v using the MFDMs, for $h = \pi/4, \pi/5, \pi/6, \pi/9, \pi/12$, in the range $[0, 40\pi]$, which corresponds to 20 orbits of the point $y(x)$.

The corresponding values of y and γ are computed as follows:

$$y = u + iv, \quad \gamma = [u^2 + v^2]^{\frac{1}{2}}.$$

The results for γ at $x = 40\pi$ were obtained using the MFDMs and compared with the related results of Lambert and Watson [15] as given in Table 3. It obvious from Table 3 that all the solutions generated by our method spiral outward for all step lengths in agreement with the theoretical solution as well as the symmetric scheme given in [15]. However, the first three values of the Störmer-Cowell scheme spiral inward.

The errors in the computed values of y and γ are computed as follows:

$$ER(y) \equiv |y(x) - y| = [(u(x) - u)^2 + (v(x) - v)^2]^{\frac{1}{2}},$$
$$ER(\gamma) \equiv |\gamma(x) - \gamma| = |[u(x)^2 + v(x)^2]^{\frac{1}{2}} - [u^2 + v^2]^{\frac{1}{2}}|.$$

The values $ER(\gamma)$ and $ER(y)$ are displayed in Tables 4 and 5. Related results obtained by Lambert and Watson [15] are also reproduced in Tables 4 and 5. It is seen that our method performs better than the Störmer-Cowell scheme except in the isolated case for $h = \pi/12$ for $ER(y)$. However, our method performs better for $h = \pi/4, \pi/5$, while the the symmetric scheme in [15] performs better for $h = \pi/6, \pi/9, \pi/12$. Thus, for this example, our method is superior to the Störmer-Cowell scheme and competitive with the symmetric scheme in [15].

h	Störmer-Cowell Order $p = 6$	Lambert and Watson Order $p = 6$	Our Method Order $p = 6$
$\pi/4$	0.965645	1.003067	1.002084
$\pi/5$	0.993734	1.002217	1.002117
$\pi/6$	0.999596	1.002047	1.002064
$\pi/9$	1.001829	1.001978	1.001984
$\pi/12$	1.001953	1.001973	1.001974

Table 3: Computed values of γ, $x = 40\pi$, $\gamma(x) = 1.001972$, for Example 5.3

h	Störmer-Cowell Order $p = 6$	Lambert and Watson Order $p = 6$	Our Method Order $p = 6$
$\pi/4$	3.6327×10^{-2}	1.0950×10^{-3}	1.1194×10^{-4}
$\pi/5$	8.2380×10^{-3}	2.4500×10^{-4}	1.4542×10^{-4}
$\pi/6$	2.3760×10^{-3}	7.5000×10^{-5}	9.1758×10^{-5}
$\pi/9$	1.4300×10^{-4}	6.0000×10^{-6}	1.2232×10^{-5}
$\pi/12$	1.9000×10^{-5}	1.0000×10^{-6}	2.4664×10^{-6}

Table 4: Absolute errors, $ER(\gamma) = |\gamma(x) - \gamma|$, $x = 40\pi$, $\gamma(x) = 1.001972$ for Example 5.3

h	Störmer-Cowell Order $p=6$	Lambert and Watson Order $p=6$	Our Method Order $p=6$
$\pi/4$	4.8014×10^{-2}	3.1272×10^{-2}	3.5515×10^{-3}
$\pi/5$	1.3136×10^{-2}	7.3000×10^{-3}	4.7938×10^{-3}
$\pi/6$	4.4940×10^{-3}	2.3030×10^{-3}	2.9390×10^{-3}
$\pi/9$	4.0500×10^{-4}	1.8800×10^{-4}	3.9177×10^{-4}
$\pi/12$	7.3000×10^{-5}	3.3000×10^{-5}	7.8996×10^{-5}

Table 5: Absolute errors, $ER(y) = |y(x) - y|$, $x = 40\pi$, $u(x) = 1$, $v(x) = -0.062832$ for Example 5.3

Example 5.4. We consider the mildly stiff IVP

$$y'' + 1001y' + 1000y = 0, \quad y(0) = 1, \quad y'(0) = -1\,,$$
$$\text{Exact}: \; y(x) = e^{-x}\,.$$

Although the numerical results for this problem were not compared with another method, the results were compared with the theoretical solution as shown in Table 6.

x	$y(x)$	y	$Error$
0.0	1.000000000000000000	1.000000000000000000	0.00000×10^{-12}
0.1	0.904837418035959473	0.904837418042946239	6.98677×10^{-12}
0.2	0.818730753077981887	0.818730753076979134	1.00275×10^{-12}
0.3	0.740818220681717853	0.740818220689576634	7.85878×10^{-12}
0.4	0.670320046035639371	0.670320046025161531	10.4778×10^{-12}
0.5	0.606530659712633379	0.606530659775854541	63.2212×10^{-12}
0.6	0.548811636094026500	0.548811636083975606	10.0508×10^{-12}
0.7	0.496585303791409593	0.496585303800772948	9.36336×10^{-12}
0.8	0.449328964117221563	0.449328964114573903	2.64766×10^{-12}
0.9	0.406569659740599131	0.406569659751278411	10.6793×10^{-12}
1.0	0.367879441171442334	0.367879441148169217	23.2731×10^{-12}

Table 6: Absolute errors, $|y(x) - y|$, for Example 5.4, where $y(x) = e^{-x}$

6. Conclusions

We have proposed a five-step LMM with continuous coefficients from which MFDMs were obtained and applied as simultaneous numerical integrators to $y'' = f(x, y, y')$ without first adapting the ODE to an equivalent first order system. The method is derived through the interpolation and collocation procedures by the matrix inverse approach. An essential ingredient in the method involves the way in which it is applied. For instance, we proceed by explicitly obtaining initial conditions at $x_{n+5}, n = 0, 5, ..., N-5$ using the computed values $\overline{y}(x_{n+5}) = y_{n+5}$ and $\delta(x_{n+5}) = \delta_{n+5}$ over sub-intervals $[x_0, x_5], [x_5, x_{10}], ..., [x_{N-5}, x_N]$ which do not overlap.

This implementation strategy is more efficient than those given in Awoyemi [2], [3], and Lambert and Watson [15] which are applied over overlapping intervals in the PC modes. It is clearly shown in the numerical experiments (see Tables 1 to 5) that our method of order $p = 6$ is superior to the methods in Awoyemi [2], [3], and the Störmer-Cowell method in [15], which have orders $p = 6$, $p = 8$, and $p = 6$, respectively. It is also shown in Tables 1 and 2 that our method performs better than the methods in Awoyemi [2], [3] despite the fact that we used larger step-sizes. Hence our method can be used to compute the solution on wider intervals and can also be used for moderately stiff problems (see Table 6). Our future research will be focused on developing a an accurate global error estimation strategy and an automatic code for its implementation. We also desire to apply the MFDMs to solve boundary value problems.

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