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A VARIATIONAL APPROXIMATION INHERITING A NON-DEGENERACY PROPERTY

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Abstract: We investigate an approximation to a variational functional with a singular term whose minimizers may have a free boundary. The approximation is considered in the sense of Γ-convergence, that is, a variational convergence. Our aim of this paper is to establish a criterion for the location of the free boundary in terms of the approximation. As a corollary, approximated minimizers are proved to inherit a non-degeneracy property in some weak sense. These results depend on a certain device of the choice of approximation of the singular term.

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1. Introduction

Alt and Caffarelli have established in a pioneering paper Alt et al [1] the regularity theory of free boundary for minimizers of the variational functional containing the characteristic function which is a singular term. They investigated the problem by treating directly the singular term. In contrast to their method, the problem was treated by the method of considering an approximated Euler-Lagrange equations (see Berestycki et al [7]), and the results were applied to the study of the corresponding time evolutional problems (Caffarelli

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et al [10], Caffarelli et al [9], Weiss [23]).

Being stimulated to their methods, the author treated in Yamaura [24] the same type of the functional with the non-linear quadratic form:

$$\mathbb{F}(u) = \int_\Omega \big(a^{ij}(u)\nabla_i u \nabla_j u + \chi_{u>0}\big) d\mathfrak{L}^n,$$

and constructed a Lipschitz continuous minimizer under some growth conditions of a^{ij}. Here $\chi_{u>0}$ is the characteristic function of the set where u is positive. The argument is based on the approximation obtained by applying the theory of Γ-convergence, which is a different point from the approach of Berestycki et al [7]. The general theory of Γ convergence was introduced by De Giorgi and Franzoni in De Giorgi [12], and the advantage is that the approximated function is not only a solution of PDE but also a minimizer of the approximated functionals, and both properties are available. In fact, we used in Yamaura [24] the minimality property in order to obtain the uniform Hölder estimate of approximated solutions with the aid of the theory of De Giorgi $\mathfrak{B}_2$-class, together with the property as a solution such as Harnack's inequality.

Through the argument of Yamaura [24], we consider in this paper the following question.

Question. Is it possible to determine the location of the free boundary of an $\mathbb{F}$-minimizer by using as a key the information for that of the approximated minimizers?

In trying to solve this question, a difficulty arises from the fact that approximated minimizers cannot satisfy the non-degeneracy property of the graph at the free boundary point, whereas the original minimizer must satisfy. This suggests us that it is not easy to determine the location of the free boundary by investigating only that of the approximated minimizer. To overcome this difficulty, we proceed our argument with the following two devices:

Firstly, it is natural that we take an approach of considering not a free boundary itself but an *approximated free boundary*. More precisely, let u_ε be a minimizer of the functional $\mathbb{F}_\varepsilon$ with $\chi_{u>0}$ term replaced by the approximated one whose approximation parameter is $\varepsilon > 0$. Then we define a set $F_{u_\varepsilon} := \Omega(0 < u_\varepsilon < \varepsilon) = \{x \in \Omega : 0 < u_\varepsilon(x) < \varepsilon\}$ which is called an "approximated free boundary". Secondly, we need a device for the way to approximate the characteristic function term. Namely, in this paper, we use the approximation by *piecewise linear* functions, although we used the approximation by C^∞-functions in the paper Yamaura [24]. This is an essential and important point of this paper (Remark 24).

Now let us state the main result of this paper. For simplicity here, we assume that u_ε converges as $\varepsilon \to 0$ uniformly to a certain minimizer u of $\mathbb{F}$, and we denote by $\partial\Omega(u > 0) = \Omega \cap \partial\{x \in \Omega : u(x) > 0\}$ the free boundary of u. We then give an answer to the question above from the measure-theoretic point of view in the following way (Theorem 22):

$$\partial\Omega(u > 0) \cap E = \emptyset \text{ is equivalent to } \mathcal{L}^n(E \cap \overline{F}_{u_\varepsilon}) = o(\varepsilon) \text{ as } \varepsilon \downarrow 0.$$

In general, it seems to be difficult to obtain an estimate of $||u_\varepsilon - u||_{\infty,\Omega}$ by using the parameter ε itself. Nevertheless, we can obtain the criterion which is expressed by explicitly using the approximation parameter ε. Moreover, as stated above, the approximated minimizer cannot satisfy the non-degeneracy property because it is differentiable at the free boundary point. However, it is shown as a corollary of the main result that the sequence of the approximated minimizers satisfies *a non-degeneracy property* in some weak sense (Corollary 23).

The device for the approximation, however, does not enable us to consider a single Euler-Lagrange equation in the whole of the domain, and for the sake, we cannot adopt directly the usual general theory of PDE such as Harnack's inequalities. As a result, it is difficult to obtain the Lipschitz continuity of minimizers. We recover the argument by further approximating the piecewise linear functions by smooth functions. This is carried out in Section 2 and Section 3. In Section 5, by a direct computation, we construct an example which satisfies the weak non-degeneracy property where we use the result of the computation of the first variation shown in Section 4. If we take C^1-function as an approximation of the characteristic function, then the weak non-degeneracy property does not necessarily hold, and we also construct an example of it.

We end this introduction by stating the references related to the theory of Γ-convergence. This theory is used for practical variational problems in Modica [21], Conti et al [11], Ambrosio et al [5], Ambrosio et al [4], and it is also known the application to the numerical experiments Baldo et al [6] and March [20]. Moreover, in Gobbino [16] a scheme of construction of gradient flow using the theory of Γ-convergence in the image-segmentation problem is introduced. Using this scheme, the author study a construction of gradient flow of Alt-Caffarelli type functional in one-dimension (Yamaura [25]).

2. Approximated Functionals and Uniform Lipschitz Continuous Minimizers

We define the characteristic function χ and its approximation of the two kinds.

Definition 1. (Approximation of Characteristic Function Term) Let $\chi : \mathbb{R} \to \{0, 1\}$ be the characteristic function of the interval $(0, \infty)$:

$$\chi(t) = \begin{cases} 0 & \text{for } t \leqq 0, \\ 1 & \text{for } t > 0. \end{cases}$$

(i) (Approximation by Piecewise Linear Function) For any positive constant ε,

$$\chi_\varepsilon(t) := \max\left(0,\ \min\left(1, \frac{t}{\varepsilon}\right)\right) \qquad \text{for } t \in \mathbb{R}.$$

(ii) (Approximation by Smooth Function) Let $\{\delta_k\}_{k=1}^{\infty}$ be a sequence such that $\frac{1}{2} > \delta_1 > \delta_2 > \cdots > 0$ and $\lim_{k\to\infty} \delta_k = 0$. Then for any positive constant ε and for each δ_k, we define

$$\chi_\varepsilon^{\delta_k}(t) := [\beta_\varepsilon^{\delta_k}]_{(\delta_k \varepsilon)}(t) \quad \text{ for } t \in \mathbb{R},$$

where $\beta_\varepsilon^{\delta_k}(t) = \max\big(0,\ \min(1, \frac{t}{\varepsilon} - \delta_k)\big)$ for $t \in \mathbb{R}$, and $[f]_\sigma$ for $f \in L^1_{\text{loc}}(\mathbb{R})$ and $\sigma > 0$ means σ-mollified function of f in the usual sense (see for instant Gilberg et al [17, p. 147, Section 7.2]).

We here sum up the properties of the functions $\chi_\varepsilon^{\delta_k}$ which follow directly from the definition.

Lemma 2. (Properties of $\chi_\varepsilon^{\delta_k}(\cdot)$) *Let $\chi_\varepsilon^{\delta_k}(\cdot)$ be as in Definition 1. Then the following properties hold:*

(i) $\chi_\varepsilon^{\delta_k} \in C^\infty(\mathbb{R})$,

(ii) $\chi_\varepsilon^{\delta_k} = 0$ *in* $(-\infty, 0]$, $\quad \chi_\varepsilon^{\delta_k} = 1$ *in* $[\varepsilon + \varepsilon\delta_k, +\infty)$, $\quad 0 \leqq \chi_\varepsilon^{\delta_k} \leqq 1$ *in* $\mathbb{R}$,

(iii) $0 \leqq (\chi_\varepsilon^{\delta_k})' \leqq \frac{1}{\varepsilon}$ *in* $\mathbb{R}$,

(iv) $\sup_{\mathbb{R}} |(\chi_\varepsilon^{\delta_k})''| < +\infty$,

(v) $\chi_\varepsilon^{\delta_k} \rightrightarrows \chi_\varepsilon$ *uniformly in* $\mathbb{R}$ *as* $k \to \infty$.

Proof. The assertion (i) and (ii) follow directly from the general property of mollifier. Let us show (iii). Since $\beta_\varepsilon^{\delta_k} \in L^1_{\text{loc}}(\mathbb{R})$ is differentiable in the weak sense and $(\beta_\varepsilon^{\delta_k})' \in L^\infty(\mathbb{R})$, we have from the property of mollifier Gilberg et al [17, Lemma 7.3],

$$(\chi_\varepsilon^{\delta_k})'(t) = \big([\beta_\varepsilon^{\delta_k}]_{\delta_k\varepsilon}\big)'(t) = [(\beta_\varepsilon^{\delta_k})']_{\delta_k\varepsilon}(t) \quad \text{for each } t \in \mathbb{R}.$$

Remarking that the inequalities $0 \leqq (\beta_\varepsilon^{\delta_k})' \leqq \frac{1}{\varepsilon}$ hold $\mathcal{L}^1$-almost everywhere in $\mathbb{R}$, we obtain the inequalities $0 \leqq (\chi_\varepsilon^{\delta_k})' \leqq \frac{1}{\varepsilon}$. By (i), $(\chi_\varepsilon^{\delta_k})'' \in C(\mathbb{R})$ and by (ii) $\operatorname{spt}(\chi_\varepsilon^{\delta_k})'' \subset [0, \varepsilon + 2\varepsilon\delta_k]$. Hence $(\chi_\varepsilon^{\delta_k})'' \in C_c(\mathbb{R})$, and in particular (iv) holds. In the same way as the proof of Gilberg et al [17, Lemma 7.1],

$$\begin{aligned}\sup_{\mathbb{R}} |\beta_\varepsilon^{\delta_k} - \chi_\varepsilon^{\delta_k}| &\leqq \sup_{x\in\mathbb{R}} \sup_{|z|\leqq 1} \Big|\beta_\varepsilon^{\delta_k}(x) - \beta_\varepsilon^{\delta_k}(x - \delta_k\varepsilon z)\Big| \\ &\leqq \big(\operatorname{Lip}_{\mathbb{R}} \beta_\varepsilon^{\delta_k}\big)\delta_k\varepsilon = \delta_k \to 0 \quad \text{ as } k \to \infty.\end{aligned}$$

If we notice the fact $\sup_{\mathbb{R}} |\beta_\varepsilon^{\delta_k} - \chi_\varepsilon| = \delta_k \to 0$ as $k \to \infty$, we establish (v). □

Let $\Omega \subset\subset \mathbb{R}^n$ $(n \geqq 2)$ be a bounded Lipschitz domain. Suppose that a function $\phi_0 \in W^{1,2}(\Omega)$ with $0 \leqq \phi_0 \leqq \sup_\Omega \phi_0 < \infty$ in Ω is given. Then we set

$$\mathcal{K}_{\phi_0} := \Big\{u \in W^{1,2}(\Omega) \;\Big|\; u = \phi_0 \text{ on } \partial\Omega,\ 0 \leqq u \leqq \sup_{\partial\Omega} \phi_0 \text{ in } \Omega\Big\}.$$

We define the real extended functional $\mathbb{F}$: $L^2(\Omega) \to [0, +\infty]$ as follows:

$$\mathbb{F}(u) = \begin{cases} \displaystyle\int_\Omega \big(a^{ij}(u)\nabla_i u \nabla_j u + \chi(u)\big) d\mathcal{L}^n & \text{if } u \in \mathcal{K}_{\phi_0}, \\ +\infty & \text{if } u \in L^2(\Omega) \setminus \mathcal{K}_{\phi_0}, \end{cases}$$

where $a^{ij} : \mathbb{R} \to \mathbb{R}$, $1 \leqq i, j \leqq n$, are $C^\infty(\mathbb{R})$-functions with the symmetricity for i, j: $a^{ij} = a^{ji}$ and the following properties:

$$\begin{cases} a^{ij}(0) = \delta_{ij}, \\ a^{ij}(t)\xi_i\xi_j \geqq \lambda|\xi|^2 \quad \text{for } \xi \in \mathbb{R}^n \quad \text{for some positive number } \lambda, \\ \dot{a}^{ij}(t)\xi_i\xi_j \geqq 0 \quad \text{for } \xi \in \mathbb{R}^n. \end{cases}$$

In the same manner as above, we define the real extended functionals $\mathbb{F}_\varepsilon^{\delta_k}$ and $\mathbb{F}_\varepsilon : L^2(\Omega) \to [0, +\infty]$ with χ replaced by $\chi_\varepsilon^{\delta_k}$ and χ_ε respectively. Throughout this paper, "$\mathbb{F}$-minimizer" means a minimizer of the functional $\mathbb{F}$ in the space $L^2(\Omega)$. Terminology "$\mathbb{F}_\varepsilon$-minimizer" and "$\mathbb{F}_\varepsilon^{\delta_k}$-minimizer" are also similarly understood. We use the constants defined as below:

$$\Lambda = \sum_{i,j=1}^n \sup_{|t|\leqq 2\sup_{\partial\Omega}\phi_0} |a^{ij}(t)|, \quad \tilde{\Lambda} = \sum_{i,j=1}^n \sup_{|t|\leqq 2\sup_{\partial\Omega}\phi_0} |\dot{a}^{ij}(t)|,$$

and

$$\tilde{\tilde{\Lambda}} = \sum_{i,j=1}^{n} \sup_{|t| \leqq 2 \sup_{\partial\Omega} \phi_0} |\ddot{a}^{ij}(t)|.$$

Now, we show without its proof the well-known lower semicontinuous inequality, which is used throughout of this paper.

Lemma 3. (Giaquinta [15, p. 18, Theorem 2.3]) *Let $b^{ij} : \mathbb{R} \to \mathbb{R}$, $1 \leqq i,j \leqq n$, be smooth functions with the symmetricity for i,j: $b^{ij} = b^{ji}$, and $b^{ij}(t)\xi_i\xi_j \geqq 0$ for any $t \in \mathbb{R}$ and for any $\xi \in \mathbb{R}^n$. For any $\{u_k\}_{k=1}^{\infty} \subset W^{1,2}(\Omega)$ and $u \in W^{1,2}(\Omega)$ satisfying*

(a) $u_k \to u$ $\mathcal{L}^n$-almost everywhere in Ω as $k \to \infty$,

(b) $\nabla u_k \to \nabla u$ weakly in $L^2(\Omega)$ as $k \to \infty$,

it holds that

$$\int_\Omega b^{ij}(u)\nabla_i u \nabla_j u\, \zeta d\mathcal{L}^n \leqq \varliminf_{k\to\infty} \int_\Omega b^{ij}(u_k)\nabla_i u_k \nabla_j u_k \zeta d\mathcal{L}^n$$

for any $\zeta \in L^\infty(\Omega)$ with $\zeta \geqq 0$ in Ω.

We start from the proof of the following convergence of the functionals.

Proposition 4. *For any fixed positive constant ε, it holds that*

$$\mathbb{F}_\varepsilon^{\delta_k} \to \mathbb{F}_\varepsilon \quad \text{as } k \to \infty \quad \text{in the sense of } \Gamma(L^2(\Omega)).$$

Proof. We directly show the definition of Γ-convergence.

— **LSC-Part.** We show that

$$\forall (u_k), u \subset L^2(\Omega) \quad \text{with } u_k \to u \text{ in } L^2(\Omega) \text{ as } k \to \infty,\ \mathbb{F}_\varepsilon(u) \leqq \varliminf_{k\to\infty} \mathbb{F}_\varepsilon^{\delta_k}(u_k). \tag{1}$$

Before showing this, remark that for any $\{u_k\}_{k=1}^{\infty}, u \in L^2(\Omega)$ with $u_k \to u$ in $L^2(\Omega)$ as $k \to \infty$,

$$\lim_{k\to\infty} \int_\Omega \chi_\varepsilon^{\delta_k}(u_k) d\mathcal{L}^n = \int_\Omega \chi_\varepsilon(u) d\mathcal{L}^n. \tag{2}$$

In fact, from (vi) of Lemma 2

$$\left| \int_\Omega \chi_\varepsilon^{\delta_k}(u_k) d\mathcal{L}^n - \int_\Omega \chi_\varepsilon(u) d\mathcal{L}^n \right| \leqq \int_\Omega |\chi_\varepsilon^{\delta_k}(u_k) - \chi_\varepsilon(u_k)| d\mathcal{L}^n + \int_\Omega |\chi_\varepsilon(u_k) - \chi_\varepsilon(u)| d\mathcal{L}^n$$

$$\leqq |\Omega| ||\chi_\varepsilon^{\delta_k} - \chi_\varepsilon||_{\infty,\mathbb{R}} + ||\chi_\varepsilon'||_{\infty,\mathbb{R}} \sqrt{|\Omega|}\, ||u_k - u||_{2,\Omega} \to 0 \quad \text{as } k \to \infty.$$

Let us prove (1). It is enough to show the case where $\varliminf_{k\to\infty} \mathbb{F}_\varepsilon^{\delta_k}(u_k) < \infty$. Extract a subsequence $\{u_{k_l}\}_{l=1}^\infty \subset \{u_k\}_{k=1}^\infty$ such that $\lim_{l\to\infty} \mathbb{F}_\varepsilon^{\delta_{k_l}}(u_{k_l}) = \varliminf_{k\to\infty} \mathbb{F}_\varepsilon^{\delta_k}(u_k)$. By choosing further subsequence if necessary, we may assume $\sup_{l\in\mathbb{N}} \mathbb{F}_\varepsilon^{\delta_{k_l}}(u_{k_l}) < \infty$. Therefore, in particular, it holds that $u_{k_l} \in \mathcal{K}_{\phi_0}$ for each $l \in \mathbb{N}$, and so

$$0 \leqq u_{k_l} \leqq \sup_\Omega \phi_0 \quad \text{in } \Omega \quad \text{for all } l \in \mathbb{N}. \tag{3}$$

From this and the uniform boundedness

$$\lambda \sup_{l\in\mathbb{N}} \int_\Omega |\nabla u_{k_l}|^2 d\mathcal{L}^n \leqq \sup_{l\in\mathbb{N}} \int_\Omega a^{ij}(u_{k_l}) \nabla_i u_{k_l} \nabla_j u_{k_l} d\mathcal{L}^n \leqq \sup_{l\in\mathbb{N}} \mathbb{F}_\varepsilon^{\delta_k}(u_{k_l}) < +\infty,$$

it turns out that the limit function u belongs to $W^{1,2}(\Omega)$, and we may assume that

$$u_{k_l} \to u \quad \text{in } L^2(\Omega) \text{ as } l \to \infty, \tag{4}$$

$$\nabla u_{k_l} \to \nabla u \quad \text{weak } L^2(\Omega) \text{ as } l \to \infty. \tag{5}$$

We in particular have from these convergence

$$u = \phi_0 \quad \text{on } L^2(\partial\Omega). \tag{6}$$

By (4), we may assume

$$u_{k_l} \to u \quad \mathcal{L}^n\text{-almost everywhere in } \Omega \text{ as } l \to \infty. \tag{7}$$

Hence, by (3), the inequalities $0 \leqq u \leqq \sup_\Omega \phi_0$ hold in Ω. So by (6), the limit function u turns out to belong to the function class $\mathcal{K}_{\phi_0}$. The convergence (5) and (7) enable us to apply Lemma 3 with $\zeta \equiv 1$ in Ω, so that

$$\int_\Omega a^{ij}(u) \nabla_i u \nabla_j u d\mathcal{L}^n \leqq \varliminf_{l\to\infty} \int_\Omega a^{ij}(u_{k_l}) \nabla_i u_{k_l} \nabla_j u_{k_l} d\mathcal{L}^n. \tag{8}$$

Thus, from (2) and (8), we obtain the desired lower semicontinuous inequality:

$$\begin{aligned}\mathbb{F}_\varepsilon(u) \leqq \varliminf_{l\to\infty} \int_\Omega a^{ij}(u_{k_l}) \nabla_i u_{k_l} \nabla_j u_{k_l} d\mathcal{L}^n + \lim_{l\to\infty} \int_\Omega \chi_\varepsilon^{\delta_{k_l}}(u_{k_l}) d\mathcal{L}^n \\ \leqq \lim_{l\to\infty} \mathbb{F}_\varepsilon^{\delta_{k_l}}(u_{k_l}) = \varliminf_{k\to\infty} \mathbb{F}_\varepsilon^{\delta_k}(u_k).\end{aligned}$$

— **USC-Part.** We show that

$$\begin{aligned}\forall u \in L^2(\Omega), \exists (u_k) \subset L^2(\Omega) \text{ with } u_k \to u \text{ in } L^2(\Omega) \text{ as } k \to \infty \\ \text{s.t. } \varlimsup_{k\to\infty} \mathbb{F}_\varepsilon^{\delta_k}(u_k) \leqq \mathbb{F}_\varepsilon(u).\end{aligned}$$

Since we have only to show the case where $\mathbb{F}_\varepsilon(u) < +\infty$, we may assume that $u \in W^{1,2}(\Omega)$, $u = \phi_0$ on $\partial\Omega$ and $0 \leqq u \leqq \sup_{\partial\Omega} \phi_0$ in Ω. Then it is sufficient to put $u_k = u$ in Ω ($\forall k \in \mathbb{N}$). In fact, by (2), we immediately obtain $\overline{\lim_{k\to\infty}} \mathbb{F}_\varepsilon^{\delta_k}(u_k) = \overline{\lim_{k\to\infty}} \mathbb{F}_\varepsilon^{\delta_k}(u) = \mathbb{F}_\varepsilon(u)$. □

Theorem 5. *There exists an $\mathbb{F}_\varepsilon^{\delta_k}$-minimizer. Let $u_\varepsilon^{\delta_k}$ be a minimizer. Then $u_\varepsilon^{\delta_k}$ belongs to $C^\infty(\Omega)$ and satisfies the Euler-Lagrange equation*

$$\nabla_j\big(a^{ij}(u_\varepsilon^{\delta_k})\nabla_j u_\varepsilon^{\delta_k}\big)= \frac{1}{2}\dot{a}^{ij}(u_\varepsilon^{\delta_k})\nabla_i u_\varepsilon^{\delta_k}\nabla_j u_\varepsilon^{\delta_k} + \frac{1}{2}(\chi_\varepsilon^{\delta_k})'(u_\varepsilon^{\delta_k}) = 0 \quad \textit{in } \Omega.$$

Proof. The existence of the minimizer is refer to Giaquinta [15, p. 18, Theorem 2.3]. The regularity of u is refer to Ladyzhenskaya et al [19, p. 284, Theorem 6.4]. □

Lemma 6. (Uniform Hölder Estimate) *Let $u_\varepsilon^{\delta_k}$ be an $\mathbb{F}_\varepsilon^{\delta_k}$-minimizer. Then $|u_\varepsilon^{\delta_k}(x) - u_\varepsilon^{\delta_k}(y)| \leqq H_U|x-y|^\alpha$ for $x, y \in U$, where $U \subset\subset \Omega$ be a subdomain and $H_U = H_U(n, \lambda, \Lambda, \sup_{\partial\Omega} \phi_0, \mathrm{dist}(U, \partial\Omega), U) > 0$, and $\alpha = \alpha(n, \lambda, \Lambda, \sup_{\partial\Omega} \phi_0)$ $\in (0, 1)$.*

Proof. Since χ_ε is non-decreasing function and $0 \leqq \chi_\varepsilon \leqq 1$ in $\mathbb{R}$, we can lead the conclusion with the aid of Yamaura [24, Theorem 2]. □

Lemma 7. (Uniform Gradient Bound) *For any $\mathbb{F}_\varepsilon^{\delta_k}$-minimizer $u_\varepsilon^{\delta_k}$ and for any subdomain $U \subset\subset \Omega$, there exists a positive constant σ_U such that $\varepsilon < \sigma_U$ implies $\sup_{k\in\mathbb{N}} \sup_U |\nabla u_\varepsilon^{\delta_k}| \leqq K_U$, where σ_U and K_U are positive constants depending only on $n, \lambda, \Lambda, \tilde{\Lambda}, \sup_{\partial\Omega} \phi_0$ and $\mathrm{dist}(U, \partial\Omega)$.*

Proof. Since the regularised function $\chi_{\frac{\varepsilon}{2}}^{\delta_k}$ satisfies

$$\begin{cases} \text{(i) } \chi_{\frac{\varepsilon}{2}}^{\delta_k} \in C^\infty(\mathbb{R}), \\ \text{(ii) } \chi_{\frac{\varepsilon}{2}}^{\delta_k} = 0 \text{ in } (-\infty, 0], \chi_{\frac{\varepsilon}{2}}^{\delta_k} = 1 \text{ in } [\varepsilon, +\infty), 0 \leqq \chi_{\frac{\varepsilon}{2}}^{\delta_k} \leqq 1 \text{ in } \mathbb{R}, \\ \text{(iii) } (\chi_{\frac{\varepsilon}{2}}^{\delta_k})' \geqq 0 \quad \text{in } \mathbb{R}, \\ \text{(iv) } |(\chi_{\frac{\varepsilon}{2}}^{\delta_k})'(t)| \leqq \dfrac{4}{\varepsilon} \quad \text{for any } t \in \mathbb{R}, \\ \text{(v) } \sup_{t\in\mathbb{R}} |(\chi_{\frac{\varepsilon}{2}}^{\delta_k})''(t)| < \infty, \end{cases}$$

we obtain the conclusion by adopting Yamaura [24, Theorem 3]. In particular, the restriction $\varepsilon < \sigma_U$ is followed from the restriction $\varepsilon < \varepsilon_0$ as in Yamaura [24, Theorem 11]. □

Let Ω_m ($m \in \mathbb{N}$) be a non-empty subdomain of Ω such that

$$\begin{cases} \Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \subset\subset \Omega, \\ \bigcup_{m=1}^{\infty} \Omega_m = \Omega. \end{cases}$$

Let σ_1 be a positive number Ω_{σ_1} obtained by applying Lemma 7 as $U = \Omega_1$. For $m \geqq 2$, we define $\sigma_m := \min(\sigma_{m-1}, \sigma_{\Omega_m})$ for $m \geqq 2$. We here note that $\sigma_1 \geqq \sigma_2 \geqq \sigma_3 \geqq \cdots > 0$.

Theorem 8. (A Construction of a Minimizer u_ε) *Let m be an integer, ε be a positive number less than σ_m, and $u_\varepsilon^{\delta_k}$ an $\mathbb{F}_\varepsilon^{\delta_k}$-minimizer. Then there exists a subsequence $(u_\varepsilon^{\delta_{k_l}}) \subset (u_\varepsilon^{\delta_k})$ and u_ε such that the following holds:*

I. (Global Result) $u_\varepsilon \in W^{1,2}(\Omega) \cap C^{0,\alpha}(\Omega)$ and

(a) *$u_\varepsilon^{\delta_{k_l}} \to u_\varepsilon$ locally uniformly in Ω as $l \to \infty$, and for any subdomain $U \subset\subset \Omega$, it holds that*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leqq H_U |x-y|^\alpha \quad \text{for } x, y \in U,$$

(b) *$u_\varepsilon^{\delta_{k_l}} \to u_\varepsilon$ in $L^2(\Omega)$ as $l \to \infty$,*

(c) *$\nabla u_\varepsilon^{\delta_{k_l}} \to \nabla u_\varepsilon$ weakly in $L^2(\Omega)$ as $l \to \infty$,*

(d) $||u_\varepsilon||_{W^{1,2}(\Omega)} \leqq |\Omega|^{\frac{1}{2}}\big(\sup_{\partial\Omega}\ \phi_0\big) + \Big\{\frac{1}{\lambda}\Big(\Lambda \int_\Omega |\nabla \phi_0|^2 d\mathcal{L}^n + |\Omega|\Big)\Big\}^{\frac{1}{2}}$,

(e) *u_ε is an $\mathbb{F}_\varepsilon$-minimizer,*

(f) $\lim_{l\to\infty} \mathbb{F}_\varepsilon^{\delta_{k_l}}(u_\varepsilon^{\delta_{k_l}}) = \mathbb{F}_\varepsilon(u_\varepsilon)$,

(g) *For any measurable subset $E \subset \Omega$,*

$$\lim_{l\to\infty} \int_E a^{ij}(u_\varepsilon^{\delta_{k_l}}) \nabla_i u_\varepsilon^{\delta_{k_l}} \nabla_j u_\varepsilon^{\delta_{k_l}} d\mathcal{L}^n = \int_E a^{ij}(u_\varepsilon) \nabla_i u_\varepsilon \nabla_j u_\varepsilon d\mathcal{L}^n.$$

II. (Local Result) $u_\varepsilon \in C^{0,1}(\Omega_m)$ and

(h) *$\nabla u_\varepsilon^{\delta_{k_l}} \to \nabla u_\varepsilon$ weakly $\star$ in $L^\infty(\Omega_m)$,*

(i) *$||\nabla u_\varepsilon||_{\infty,U} \leqq K_U$ for any domain $U \subset \Omega_m$.*

Here $\alpha \in (0,1)$ is a positive constant as in Lemma 6, and H_U and K_U are positive constants as in Lemma 6 and Lemma 7 respectively.

Proof. I. (a) Since $(u_\varepsilon^{\delta_k})$ are locally equi-continuous in Ω by Lemma 6, and $(u_\varepsilon^{\delta_k})$ are uniformly bounded in Ω because they are $\mathbb{F}_\varepsilon^{\delta_k}$-minimizer, applying Ascoli-Arzelà Theorem, we obtain the convergence in (a). The uniform Hölder estimate is followed by letting l to $+\infty$ in the estimate of Lemma 6.

Before showing (b)-(g), we remark the uniformly boundedness

$$\sup_{k\in\mathbb{N}} ||u_\varepsilon^{\delta_k}||_{W^{1,2}(\Omega)} < +\infty. \tag{9}$$

This fact is demonstrated by the inequality

$$||u_\varepsilon^{\delta_k}||_{W^{1,2}(\Omega)} \leqq |\Omega|^{\frac{1}{2}}\big(\sup_{\partial\Omega}\ \phi_0\big)+\Big\{\frac{1}{\lambda}\Big(\Lambda\int_\Omega |\nabla\phi_0|^2 d\mathcal{L}^n + |\Omega|\Big)\Big\}^{\frac{1}{2}}, \tag{10}$$

which is computed by taking ϕ_0 as the comparison function of the energy functional $\mathbb{F}_\varepsilon^{\delta_k}$.

(b) By (9), we can apply Rellich Compactness Theorem, and so we obtain (b).

(c) Since from (9) we in particular have $\sup_{k\in\mathbb{N}} ||\nabla u_\varepsilon^{\delta_k}||_{L^2(\Omega)} < +\infty$, (c) is followed by the weak compactness of $L^2(\Omega)$.

(d) From (a) and (c), we can know about the norm

$$||u_\varepsilon||_{W^{1,2}(\Omega)} \leqq \varliminf_{l\to\infty} ||u_\varepsilon^{\delta_{k_l}}||_{W^{1,2}(\Omega)}.$$

Thus we can lead (d) from (10).

(e) and (f) It is known from the general theory of Γ-convergence (see Yamaura [24, p. 1182, Corollary 1]).

(g) Let $E \subset \Omega$ be a measurable subset. Then by the convergence (b) and (c), we can apply Lemma 3 by putting $\zeta = \chi_E$ to have

$$\int_E a^{ij}(u_\varepsilon)\nabla_i u_\varepsilon \nabla_j u_\varepsilon d\mathcal{L}^n \leqq \varliminf_{l\to\infty}\int_E a^{ij}(u_\varepsilon^{\delta_{k_l}})\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j u_\varepsilon^{\delta_{k_l}} d\mathcal{L}^n. \tag{11}$$

So it is sufficient to show the upper semicontinuity inequality corresponding to (11). In the same way as the proof of (2) of Proposition 4, we have

$$\lim_{l\to\infty}\int_\Omega \chi_\varepsilon^{\delta_{k_l}}(u_\varepsilon^{\delta_{k_l}}) d\mathcal{L}^n = \int_\Omega \chi_\varepsilon(u_\varepsilon) d\mathcal{L}^n. \tag{12}$$

Hence from (f) and (11) with E replaced by $\Omega\setminus E$,

$$\begin{aligned}
&\int_\Omega a^{ij}(u_\varepsilon)\nabla_i u_\varepsilon \nabla_j u_\varepsilon d\mathcal{L}^n \\
&\geqq \varliminf_{l\to\infty}\int_{\Omega\setminus E} a^{ij}(u_\varepsilon^{\delta_{k_l}})\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j u_\varepsilon^{\delta_{k_l}} d\mathcal{L}^n + \varlimsup_{l\to\infty}\int_E a^{ij}(u_\varepsilon^{\delta_{k_l}})\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j u_\varepsilon^{\delta_{k_l}} d\mathcal{L}^n \\
&\qquad \geqq \int_{\Omega\setminus E} a^{ij}(u_\varepsilon)\nabla_i u_\varepsilon \nabla_j u_\varepsilon d\mathcal{L}^n + \varlimsup_{l\to\infty}\int_E a^{ij}(u_\varepsilon^{\delta_{k_l}})\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j u_\varepsilon^{\delta_{k_l}} d\mathcal{L}^n.
\end{aligned}$$

Therefore we obtain the desired upper semicontinuity inequality.

II. We use the uniform gradient bound:

$$\sup_{l\in\mathbb{N}}\sup_{\Omega_m}|\nabla u_\varepsilon^{\delta_{k_l}}| \leqq K_{\Omega_m}.$$

This fact itself follows from Lemma 7 by remarking $\varepsilon < \sigma_m$.

(h) From (2), by extracting further subsequence if necessary, we obtain (h).

(i) Owing to the convergence (h), L^∞-norm is lower semicontinuous, and hence we have (i). □

We next show the Γ-convergence $\mathbb{F}_\varepsilon \to \mathbb{F}$ as $\varepsilon \downarrow 0$. Different from the case of showing $\mathbb{F}_\varepsilon^{\delta_k} \to \mathbb{F}_\varepsilon$, as $k \to \infty$, this time it does not hold the uniform convergence $\chi_\varepsilon \rightrightarrows \chi$. For this reason, we must treat this term in the different way. The proof of the following lemma is refer to Yamaura [24, p. 1178, Lemma 2].

Theorem 9. (l.s.c. Property of χ-term) *Let $\varepsilon > 0$ and let $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a function such that $\eta_\varepsilon = 0$ in $(-\infty, 0]$, $\eta_\varepsilon = 1$ in $[\varepsilon, +\infty)$ and $0 \leqq \eta_\varepsilon \leqq 1$ in $\mathbb{R}$. Then the convergence $v_\varepsilon \to v$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ implies that the inequality*

$$\int_E \chi(v)d\mathcal{L}^n \leqq \varliminf_{\varepsilon\downarrow 0}\int_E \eta_\varepsilon(v_\varepsilon)d\mathcal{L}^n$$

holds for any measurable subset $E \subset \Omega$.

Proposition 10. *$\mathbb{F}_\varepsilon \to \mathbb{F}$ as $\varepsilon \downarrow 0$ in the sense of $\Gamma(L^2(\Omega))$-convergence.*

Proof. We only state the different point from the proof of Proposition 4. (LSC)-part can be proved by using Lemma 9. For (USC)-part, we have only to put $u_\varepsilon = u$ for $\varepsilon > 0$. In fact, since $\chi_\varepsilon \leqq \chi$, we have

$$\varlimsup_{\varepsilon\to 0}\int_\Omega \chi_\varepsilon(u)d\mathcal{L}^n \leqq \int_\Omega \chi(u)d\mathcal{L}^n. \qquad \square$$

Theorem 11. (A Construction of a Minimizer u) *For a positive number ε such that $\sigma_{m+1} \leqq \varepsilon < \sigma_m$ ($m \in \mathbb{N}$), let u_ε be an $\mathbb{F}_\varepsilon$-minimizer constructed as in Theorem 8. Then there exists a subsequence $\{u_{\varepsilon_l}\}_{l=1}^\infty \subset \{u_\varepsilon\}_{\varepsilon<\sigma_1}$ and $u \in W^{1,2}(\Omega) \cap C^{0,1}(\Omega)$ such that*

(a) *$u_{\varepsilon_l} \to u$ locally uniformly in Ω as $l \to \infty$,*

(b) *$u_{\varepsilon_l} \to u$ in $L^2(\Omega)$ as $l \to \infty$,*

(c) *$\nabla u_{\varepsilon_l} \to \nabla u$ weak $L^2(\Omega)$ as $l \to \infty$,*

(d) *u is an $\mathbb{F}$-minimizer,*

(e) *$\lim_{l\to\infty}\mathbb{F}_{\varepsilon_l}(u_{\varepsilon_l}) = \mathbb{F}(u)$,*

(f) *For any measurable subset* $E \subset \Omega$,

$$\lim_{l\to\infty}\int_E a^{ij}(u_{\varepsilon_l})\nabla_i u_{\varepsilon_l}\nabla_j u_{\varepsilon_l} d\mathfrak{L}^n = \int_E a^{ij}(u)\nabla_i u\nabla_j u d\mathfrak{L}^n,$$

(g) $\nabla u_{\varepsilon_l} \to \nabla u$ *weakly* $\star$ *in* $L^\infty(U)$ *as* $l \to \infty$ *for* $U \subset\subset \Omega$,

(h) $||\nabla u||_{\infty,U} \leqq K_U$ *for each* $U \subset\subset \Omega$, *where* K_U *is a positive constant as in Lemma* 7.

Proof. By (a) of Lemma 8, (u_ε) are uniform Hölder continuous locally in Ω. Moreover, since u_ε is an $\mathbb{F}_\varepsilon$-minimizer, u_ε are uniformly bounded in Ω. Thus we can apply Ascoli-Arzelà Theorem we obtain (a). From (d) of Theorem 8, it holds that

$$||u_\varepsilon||_{1,2,\Omega} \leqq |\Omega|^{\frac{1}{2}}\big(\sup_{\partial\Omega}\ \phi_0\big)+\Big\{\frac{1}{\lambda}\Big(\Lambda\int_\Omega |\nabla\phi_0|^2 d\mathfrak{L}^n + |\Omega|\Big)\Big\}^{\frac{1}{2}}. \tag{13}$$

Therefore we can extract a subsequence such that (b) and (c) hold. Especially by (a), we can show (d) and (e) in the same way as the proof of (e) and (f) of Theorem 8.

Let us show (f). By recalling the proof of (g) of Theorem 8, it turns out to be enough to show

$$\int_\Omega \chi(u) d\mathfrak{L}^n = \lim_{l\to\infty}\int_\Omega \chi_{\varepsilon_l}(u_{\varepsilon_l}) d\mathfrak{L}^n. \tag{14}$$

From (b) and (c), we can apply Lemma 3 so that

$$\int_\Omega a^{ij}(u)\nabla_i u\nabla_j u d\mathfrak{L}^n \leqq \varliminf_{l\to\infty}\int_\Omega a^{ij}(u_{\varepsilon_l})\nabla_i u_{\varepsilon_l}\nabla_j u_{\varepsilon_l} d\mathfrak{L}^n.$$

From this and (e),

$$\begin{aligned}\int_\Omega \big(a^{ij}(u)\nabla_i u\nabla_j u + \chi(u)\big) d\mathfrak{L}^n &= \mathbb{F}(u) = \lim_{l\to\infty}\mathbb{F}_{\varepsilon_l}(u_{\varepsilon_l}) \\ &\geqq \varliminf_{l\to\infty}\int_\Omega a^{ij}(u_{\varepsilon_l})\nabla_i u_{\varepsilon_l}\nabla_j u_{\varepsilon_l} d\mathfrak{L}^n + \varlimsup_{l\to\infty}\int_\Omega \chi_{\varepsilon_l}(u_{\varepsilon_l}) d\mathfrak{L}^n \\ &\geqq \int_\Omega a^{ij}(u)\nabla_i u\nabla_j u d\mathfrak{L}^n + \varlimsup_{l\to\infty}\int_\Omega \chi_{\varepsilon_l}(u_{\varepsilon_l}) d\mathfrak{L}^n.\end{aligned}$$

We thus have

$$\int_\Omega \chi(u) d\mathfrak{L}^n \geqq \varlimsup_{l\to\infty}\int_\Omega \chi_{\varepsilon_l}(u_{\varepsilon_l}) d\mathfrak{L}^n.$$

On the other hand, adopting Lemma 9 as $\eta_\varepsilon = \chi_\varepsilon$ to have

$$\int_\Omega \chi(u) d\mathfrak{L}^n \leqq \varliminf_{l\to\infty}\int_\Omega \chi_{\varepsilon_l}(u_{\varepsilon_l}) d\mathfrak{L}^n,$$

we obtain (14), and so we reach (f) in the same way as the proof of (g) of Theorem 8.

Finally, we show (g) and (h). Let $U \subset\subset \Omega$ be a subdomain, and let m be an integer large enough such that $U \subset \Omega_m$ holds. Then for sufficiently large integer l, by (i) of Theorem 8, we have $||\nabla u_{\varepsilon_l}||_{\infty,U} \leqq K_U$. Hence by the lower semicontinuity of the L^∞-norm with respect to the convergence (g), we obtain (h). □

3. Convergence Result and a Weak Non-Degeneracy Property

The outline of this section is as follows: Firstly, we define Radon measures for each $\mathbb{F}$-, $\mathbb{F}_\varepsilon$- and $\mathbb{F}_\varepsilon^{\delta_k}$-minimizers, and investigate the convergence between them. Secondly, we obtain an estimate for the Radon measure Γ_ε. Combining these results, we arrive at the criterion theorem which is the main result of this paper. Thirdly, we lead a weak non-degeneracy property as a corollary of the criterion. Let us start from the definition of Radon measures:

Definition 12. (Radon Measures $\Gamma, \Gamma_\varepsilon$ and $\Gamma_\varepsilon^{\delta_k}$) Let u be an $\mathbb{F}$-minimizer. Then by the non-decreasing property of $\chi(\cdot)$, the following functional is turned out to be non-negative:

$$\zeta \in C_c^1(\Omega) \quad \longmapsto \quad -\int_\Omega \big(a^{ij}(u)\nabla_i u \nabla_j \zeta + \frac{1}{2}\dot{a}^{ij}(u)\nabla_i u \nabla_j u \zeta\big) d\mathcal{L}^n.$$

So, by the Riesz representation theorem there exists a unique positive Radon measure Γ such that the following representation holds:

$$\int_\Omega \zeta d\Gamma = -\int_\Omega \big(a^{ij}(u)\nabla_i u \nabla_j \zeta + \frac{1}{2}\dot{a}^{ij}(u)\nabla_i u \nabla_j u \zeta\big) d\mathcal{L}^n \quad \text{for any } \zeta \in C_c^1(\Omega).$$

By the same argument, we can define for $\mathbb{F}_\varepsilon$-minimizer u_ε and $\mathbb{F}_\varepsilon^{\delta_k}$-minimizer $u_\varepsilon^{\delta_k}$ the positive Radon measures $\Gamma_\varepsilon, \Gamma_\varepsilon^{\delta_k}$ respectively.

Remark 13. (i) By the density argument, the representation formula

$$\int_\Omega \zeta d\Gamma = -\int_\Omega \big(a^{ij}(u)\nabla_i u \nabla_j \zeta + \frac{1}{2}\dot{a}^{ij}(u)\nabla_i u \nabla_j u \zeta\big) d\mathcal{L}^n \tag{15}$$

holds for any $\zeta \in W^{1,\infty}(\Omega)$ with $\operatorname{spt}\zeta \subset \Omega$. Such a fact is also true for Γ and Γ_ε.

(ii) We sum up the possibility of representation of each Radon measures.

— $\Gamma_\varepsilon^{\delta_k}$: The approximated characteristic function $\chi_\varepsilon^{\delta_k}$ is differentiable, and so the right hand of (15) with Γ replaced by $\Gamma_\varepsilon^{\delta_k}$ is equal to $\int_\Omega \frac{1}{2}(\chi_\varepsilon^{\delta_k})'(u_\varepsilon^{\delta_k}) \zeta d\mathcal{L}^n$. Hence, in particular, we have the formula

$$\int_\Omega \zeta d\Gamma_\varepsilon^{\delta_k} = \frac{1}{2}\int_\Omega (\chi_\varepsilon^{\delta_k})'(u_\varepsilon^{\delta_k})\zeta d\mathcal{L}^n \quad \text{for } \zeta \in W^{1,\infty}(\Omega) \text{ with } \operatorname{spt}\zeta \subset \Omega. \tag{16}$$

However, a kind of this formula does not hold for Radon measure Γ and Γ_ε, because χ_ε and χ are not differentiable.

— Γ: On the other hand, we have known for Γ the following so-called "*Identification formula*":

$$\Gamma = \mathcal{H}^{n-1} \llcorner \partial\Omega(u > 0), \tag{17}$$

which is shown by essentially using Lipschitz continuity of the minimizer u (see Omata et al [22, p. 28, Theorem 3.1] for the case $n = 2$, and we can show also for general n in the same way by using Lipschitz continuity of the minimizer which are known in this paper). We here notice that the coefficient of the right-hand side of (17) is one, because of the assumption $a^{ij}(0) = \delta_{ij}$.

— Γ_ε: Unfortunately, we can not lead a representation formula similar to (16) for the Radon measure Γ_ε, because $\chi_\varepsilon(t)$ is not differentiable at $t = 0$ and $t = \varepsilon$. However, we shall instead give inequalities in Lemma 20 below, which is a key in the proof of the conclusion of this paper.

The following fact shall be used in the proof of Lemma 20 below.

Proposition 14. (The Support of the Measure Γ_ε)

Let u_ε be an $\mathbb{F}_\varepsilon$-minimizer. Then the following inclusion holds:

$$\operatorname{spt}\Gamma_\varepsilon \subset \ \Omega \cap \ \overline{\Omega(0 < u_\varepsilon < \varepsilon)}\,.$$

Proof. Assume that $x_0 \notin \Omega \cap \overline{\Omega(0 < u_\varepsilon < \varepsilon)}$. If $x_0 \notin \Omega$, then $x_0 \notin \operatorname{spt}\Gamma_\varepsilon$. So, we have only to proceed our argument in the case where $x_0 \notin \Omega \setminus \overline{\Omega(0 < u_\varepsilon < \varepsilon)}$. Recalling that u_ε is continuous in Ω, and $0 \leqq u_\varepsilon \leqq \sup_{\partial\Omega}\phi_0$ holds, for sufficiently small $\varrho > 0$, it holds that $B_\varrho(x_0) \subset\subset \Omega$, and $u_\varepsilon \equiv 0$ in $B_\varrho(x_0)$ or $u_\varepsilon > \varepsilon$ in $B_\varrho(x_0)$. In the former case, we have

$$\int_{B_\varrho(x_0)} \big(a^{ij}(u_\varepsilon)\nabla_i u_\varepsilon \nabla_j \zeta + \frac{1}{2}\dot{a}^{ij}(u_\varepsilon)\nabla_i u_\varepsilon \nabla_j u_\varepsilon \zeta\big) d\mathcal{L}^n = 0$$
$$\text{for any } \zeta \in C_c^1(B_\varrho(x_0)). \tag{18}$$

We have (18) also in the latter. In fact, for $\zeta \in C_c^1(B_\varrho(x_0))$, there exists a positive number $\delta_0 > 0$ such that $|\delta| < \delta_0$ implies $u_\varepsilon + \delta\zeta > \varepsilon$ on $B_\varrho(x_0)$. Therefore, we can compute the first variation of $\mathbb{F}_\varepsilon$ restricted in the domain $B_\varrho(x_0)$ by neglecting the χ_ε-term, and as a result, we obtain (18). So from

(15),

$$\int_{B_\varrho(x_0)} \zeta d\Gamma_\varepsilon = 0 \qquad \text{for any } \zeta \in C_c^1(B_\varrho(x_0)),$$

which in particular follows $\Gamma_\varepsilon(B_{\frac{\varrho}{2}}(x_0)) = 0$. By the definition of the support of measures (see e.g. Ambrosio et al [3, p. 30, Definition 1.64]) we have $x_0 \notin \operatorname{spt}\Gamma_\varepsilon$. □

We now concentrate ourselves to investigate the convergence of Radon measures as defined above. In the following of this section, let m be an arbitrary integer, and let ε be a positive constant less than σ_m, where σ_m is a positive number introduced before Theorem 8. We denote by $u_\varepsilon^{\delta_{k_l}}$ an $\mathbb{F}_\varepsilon^{\delta_{k_l}}$-minimizers as in Theorem 8, and by u_ε the limit function. Let $\Gamma_\varepsilon^{\delta_{k_l}}$ be a Radon measure defined from $u_\varepsilon^{\delta_{k_l}}$. In particular, the limit function u_ε is an $\mathbb{F}_\varepsilon$-minimizer, and the Radon measure Γ_ε is well-defined. Moreover, let (u_{ε_l}) be a subsequence of (u_ε) above, and let u be the limit function. Also here, notice that u is an $\mathbb{F}$-minimizer, and the corresponding Radon measure Γ is well-defined.

Lemma 15.

(i) $\displaystyle \varlimsup_{l\to\infty} \int_\Omega \zeta d\Gamma_\varepsilon^{\delta_{k_l}} \leqq \int_\Omega \zeta d\Gamma_\varepsilon$ *for* $\forall\zeta \in C_c^1(\Omega)$ *with* $\zeta \geqq 0$ *in* Ω *and* $\operatorname{spt}\zeta \subset \Omega_m$,

(ii) $\displaystyle \varlimsup_{l\to\infty} \int_\Omega \zeta d\Gamma_{\varepsilon_l} \leqq \int_\Omega \zeta d\Gamma$ *for* $\forall\zeta \in C_c^1(\Omega)$ *with* $\zeta \geqq 0$ *in* Ω.

Proof. Since (ii) can be shown in the same way, we only state the proof of (i). Let $\zeta \in C_c^1(\Omega)$ with $\operatorname{spt}\zeta \subset \Omega_m$ and $\zeta \geqq 0$ in Ω. Then by (15)

$$\int_\Omega \zeta d\Gamma_\varepsilon^{\delta_{k_l}} = -\int_\Omega \big(a^{ij}(u_\varepsilon^{\delta_{k_l}})\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j\zeta + \frac{1}{2}\dot a^{ij}(u_\varepsilon^{\delta_{k_l}})\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j u_\varepsilon^{\delta_{k_l}}\zeta\big)d\mathcal{L}^n, \tag{19}$$

$$\int_\Omega \zeta d\Gamma_\varepsilon = -\int_\Omega \big(a^{ij}(u_\varepsilon)\nabla_i u_\varepsilon\nabla_j\zeta + \frac{1}{2}\dot a^{ij}(u_\varepsilon)\nabla_i u_\varepsilon\nabla_j u_\varepsilon\zeta\big)d\mathcal{L}^n. \tag{20}$$

From (b), (c) of Theorem 8 and the non-negativity assumption for $[\dot a^{ij}]$, it is possible to apply Lemma 3, so that we have

$$\int_\Omega \dot a^{ij}(u_\varepsilon)\nabla_i u_\varepsilon\nabla_j u_\varepsilon\zeta d\mathcal{L}^n \leqq \varliminf_{l\to\infty}\int_\Omega \dot a^{ij}(u_\varepsilon^{\delta_{k_l}})\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j u_\varepsilon^{\delta_{k_l}}\zeta d\mathcal{L}^n. \tag{21}$$

Henceforth, if we show

$$\lim_{l\to\infty}\int_\Omega a^{ij}(u_\varepsilon^{\delta_{k_l}})\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j\zeta d\mathcal{L}^n = \int_\Omega a^{ij}(u_\varepsilon)\nabla_i u_\varepsilon\nabla_j\zeta d\mathcal{L}^n, \tag{22}$$

then we reach our conclusion because of (19) and (20). Let us prove (22).

$$\left|\int_\Omega a^{ij}(u_\varepsilon^{\delta_{k_l}})\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j\zeta d\mathcal{L}^n - \int_\Omega a^{ij}(u_\varepsilon)\nabla_i u_\varepsilon \nabla_j \zeta d\mathcal{L}^n\right|$$
$$\leqq \left|\int_\Omega [a^{ij}(u_\varepsilon^{\delta_{k_l}}) - a^{ij}(u_\varepsilon)]\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j\zeta d\mathcal{L}^n\right|$$
$$+\left|\int_\Omega a^{ij}(u_\varepsilon)\nabla_j\zeta\big(\nabla_j u_\varepsilon^{\delta_{k_l}} - \nabla_j u_\varepsilon\big)d\mathcal{L}^n\right| =: I_{1,l} + I_{2,l}.$$

Here

$$I_{1,l} \leqq \tilde{\Lambda}\int_\Omega |u_\varepsilon^{\delta_{k_l}} - u_\varepsilon||\nabla u_\varepsilon^{\delta_{k_l}}||\nabla\zeta|d\mathcal{L}^n$$
$$\leqq \tilde{\Lambda}\sup_{\Omega_m}|u_\varepsilon^{\delta_{k_l}} - u_\varepsilon|||\nabla u_\varepsilon^{\delta_{k_l}}||_{2,\Omega}||\nabla\zeta||_{2,\Omega} \to 0 \quad \text{as } l\to\infty.$$

The last convergence follows from (b) of Theorem 8, and the uniform boundedness of $||\nabla u_\varepsilon^{\delta_{k_l}}||_{2,\Omega}$ which is followed from (9). From (c) of Theorem 8, we can immediately lead that $I_{2,l}\to 0$ as $l\to\infty$, and so we obtain (22). □

Lemma 16.

$$\text{(i)}\quad \nabla u_\varepsilon^{\delta_{k_l}} \to \nabla u_\varepsilon \quad \textit{in } L^2_{loc}(\Omega_m) \quad \textit{as } l\to\infty,$$
$$\text{(ii)}\quad \nabla u_{\varepsilon_l}\to\nabla u \quad \textit{in } L^2_{loc}(\Omega) \quad \textit{as } l\to\infty.$$

Proof. Since (i) can be shown in the same way, we only prove (ii). Let $K\subset\Omega$ be a compact subset and let ζ be a C^1-function such that $\zeta\equiv 1$ in K, $0\leqq\zeta\leqq 1$ in Ω. Then

$$\int_K |\nabla u_{\varepsilon_l} - \nabla u|^2 d\mathcal{L}^n \leqq \frac{1}{\lambda}\int_\Omega a^{ij}(u_{\varepsilon_l})\nabla_i(u_{\varepsilon_l}-u)\nabla_j(u_{\varepsilon_l}-u)\zeta d\mathcal{L}^n$$
$$= \frac{1}{\lambda}\int_\Omega a^{ij}(u_{\varepsilon_l})\nabla_i u_{\varepsilon_l}\nabla_j(u_{\varepsilon_l}-u)\zeta d\mathcal{L}^n$$
$$-\frac{1}{\lambda}\int_\Omega a^{ij}(u)\nabla_i u\nabla_j(u_{\varepsilon_l}-u)\zeta d\mathcal{L}^n + \frac{1}{\lambda}\int_\Omega [a^{ij}(u) - a^{ij}(u_{\varepsilon_l})]\nabla_i u\nabla_j(u_{\varepsilon_l}-u)\zeta d\mathcal{L}^n$$
$$= \underbrace{\frac{1}{\lambda}\int_\Omega a^{ij}(u_{\varepsilon_l})\nabla_i u_{\varepsilon_l}\nabla_j\Big[(u_{\varepsilon_l}-u)\zeta\Big]d\mathcal{L}^n}_{(I)} - \frac{1}{\lambda}\int_\Omega a^{ij}(u_{\varepsilon_l})\nabla_i u_{\varepsilon_l}\nabla_j\zeta(u_{\varepsilon_l}-u)d\mathcal{L}^n$$
$$-\frac{1}{\lambda}\int_\Omega a^{ij}(u)\nabla_i u\nabla_j(u_{\varepsilon_l}-u)\zeta d\mathcal{L}^n + \frac{1}{\lambda}\int_\Omega [a^{ij}(u) - a^{ij}(u_{\varepsilon_l})]\nabla_i u\nabla_j(u_{\varepsilon_l}-u)\zeta d\mathcal{L}^n.$$

Here we can rewrite (I)-term by Radon measure Γ_{ε_l} (see (ii) of Remark 13).

$$\int_K |\nabla u_{\varepsilon_l} - \nabla u|^2 d\mathcal{L}^n \leqq -\frac{1}{\lambda}\int_\Omega (u_{\varepsilon_l}-u)\zeta d\Gamma_{\varepsilon_l}$$

$$-\frac{1}{2\lambda}\int_\Omega \dot{a}^{ij}(u_{\varepsilon_l})\nabla_i u_{\varepsilon_l}\nabla_j u_{\varepsilon_l}(u_{\varepsilon_l}-u)\zeta d\mathcal{L}^n-\frac{1}{\lambda}\int_\Omega a^{ij}(u_{\varepsilon_l})\nabla_i u_{\varepsilon_l}\nabla_j\zeta(u_{\varepsilon_l}-u)d\mathcal{L}^n$$
$$+\frac{1}{\lambda}\int_\Omega[a^{ij}(u)-a^{ij}(u_{\varepsilon_l})]\nabla_i u\nabla_j(u_{\varepsilon_l}-u)\zeta d\mathcal{L}^n$$
$$-\frac{1}{\lambda}\int_\Omega a^{ij}(u)\nabla_i u\zeta\Big[\nabla_j u_{\varepsilon_l}-\nabla_j u\Big]d\mathcal{L}^n.$$

Estimating by the maximum norm of $(u_{\varepsilon_l}-u)$ the first four terms of the right-hand side, we have

$$\int_K|\nabla u_{\varepsilon_l}-\nabla u|^2 d\mathcal{L}^n \leqq \frac{1}{\lambda}||u_{\varepsilon_l}-u||_{\infty,\mathrm{spt}\zeta}\Big\{\Gamma_{\varepsilon_l}(\mathrm{spt}\,\zeta)+\frac{1}{2}\int_{\mathrm{spt}\zeta}\tilde{\Lambda}|\nabla u_{\varepsilon_l}|^2 d\mathcal{L}^n$$
$$+\int_{\mathrm{spt}\zeta}\Lambda|\nabla u_{\varepsilon_l}||\nabla\zeta|d\mathcal{L}^n+\int_{\mathrm{spt}\zeta}\tilde{\Lambda}|\nabla u||\nabla u_{\varepsilon_l}-\nabla u|d\mathcal{L}^n\Big\}$$
$$-\frac{1}{\lambda}\int_\Omega a^{ij}(u)\nabla_i u\zeta\Big[\nabla_j u_{\varepsilon_l}-\nabla_j u\Big]d\mathcal{L}^n=:\frac{1}{\lambda}||u_{\varepsilon_l}-u||_{\infty,\mathrm{spt}\zeta}\times I_l-J_l.$$

Here the term J_l goes to zero as $l\to\infty$ because of (c) of Theorem 11. Since from (a) of Theorem 11, we have known that $||u_{\varepsilon_l}-u||_{\infty,\mathrm{spt}\zeta}\to 0$, if we show I_l is bounded with respect to l, then we have that the right-hand side of (3) converges to zero as $l\to\infty$. For showing the boundedness of I_l, we have only to show that the boundedness of $\Gamma_{\varepsilon_l}(\mathrm{spt}\,\zeta)$ with respect to l because we have known he uniform boundedness of $||\nabla u_{\varepsilon_l}||_{2,\Omega}$.

Let L be a compact subset of Ω, and let $\varphi_0\in C_c^1(\Omega)$ be a function such that

$$\begin{cases} 0\leqq\varphi_0\leqq 1\,, & \text{in }\Omega\,,\\ \varphi_0=1\,, & \text{in }L.\end{cases}$$

Then by (ii) of Lemma 15

$$\overline{\lim_{l\to\infty}}\,\Gamma_{\varepsilon_l}(L)\leqq\overline{\lim_{l\to\infty}}\int_\Omega\varphi_0 d\Gamma_{\varepsilon_l}\leqq\int_\Omega\varphi_0 d\Gamma<+\infty.$$

Therefore, for any fixed compact subset L of Ω, $\Gamma_{\varepsilon_l}(L)$ is uniformly bounded with respect to l. In particular, by setting $L=\mathrm{spt}\,\zeta$, we attain our goal. □

By making use of Lemma 16, it is possible to obtain stronger convergence than that shown in Lemma 15.

Lemma 17.

(i) $$\lim_{l\to\infty}\int_\Omega \zeta d\Gamma_\varepsilon^{\delta_{k_l}} = \int_\Omega \zeta d\Gamma_\varepsilon \quad \textit{for } \forall\zeta \in C_c^1(\Omega) \textit{ with } \zeta \geqq 0 \textit{ in } \Omega$$
and $\operatorname{spt}\zeta \subset \Omega_m$,

(ii) $$\lim_{l\to\infty}\int_\Omega \zeta d\Gamma_{\varepsilon_l} = \int_\Omega \zeta d\Gamma \quad \textit{for } \forall\zeta \in C_c^1(\Omega) \textit{ with } \zeta \geqq 0 \textit{ in } \Omega.$$

Proof. As usual we only show (i). Let ζ be a $C_c^1(\Omega)$-function such that $\operatorname{spt}\zeta \subset \Omega_m$, $\zeta \geqq 0$ in Ω. By recalling the proof of Lemma 15, it is sufficient to show

$$\lim_{l\to\infty}\int_\Omega \dot{a}^{ij}(u_\varepsilon^{\delta_{k_l}})\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j u_\varepsilon^{\delta_{k_l}}\zeta d\mathcal{L}^n = \int_\Omega \dot{a}^{ij}(u_\varepsilon)\nabla_i u_\varepsilon\nabla_j u_\varepsilon \zeta d\mathcal{L}^n. \tag{23}$$

For showing this, we proceed our computation as follows:

$$\begin{aligned}
&\left|\int_\Omega \dot{a}^{ij}(u_\varepsilon^{\delta_{k_l}})\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j u_\varepsilon^{\delta_{k_l}}\zeta d\mathcal{L}^n - \int_\Omega \dot{a}^{ij}(u_\varepsilon)\nabla_i u_\varepsilon\nabla_j u_\varepsilon\zeta d\mathcal{L}^n\right| \\
&\quad \leqq \left|\int_\Omega \dot{a}^{ij}(u_\varepsilon^{\delta_{k_l}})\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j u_\varepsilon^{\delta_{k_l}}\zeta d\mathcal{L}^n - \int_\Omega \dot{a}^{ij}(u_\varepsilon)\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j u_\varepsilon^{\delta_{k_l}}\zeta d\mathcal{L}^n\right| \\
&\quad\quad + \left|\int_\Omega \dot{a}^{ij}(u_\varepsilon)\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j u_\varepsilon^{\delta_{k_l}}\zeta d\mathcal{L}^n - \int_\Omega \dot{a}^{ij}(u_\varepsilon)\nabla_i u_\varepsilon\nabla_j u_\varepsilon\zeta d\mathcal{L}^n\right| \\
&\quad\quad\quad \leqq \underbrace{||u_\varepsilon^{\delta_{k_l}} - u_\varepsilon||_{\infty,\operatorname{spt}\zeta}\tilde{\tilde{\Lambda}}\int_\Omega |\nabla u_\varepsilon^{\delta_{k_l}}|^2 d\mathcal{L}^n}_{(I_1)} \\
&\quad\quad\quad\quad + \underbrace{\tilde{\Lambda}\sum_{i,j=1}^n \int_\Omega |\nabla_i u_\varepsilon^{\delta_{k_l}}\nabla_j u_\varepsilon^{\delta_{k_l}} - \nabla_i u_\varepsilon\nabla_j u_\varepsilon|\zeta d\mathcal{L}^n}_{(I_2)}\,.
\end{aligned}$$

It turns out that (I_1) and (I_2) go to zero as $l \to \infty$ because of (a) of Theorem 8 and (i) of Lemma 16 respectively. □

Finally, from Lemma 17, we arrive at the final convergence result of Radon measures. Namely, we show the convergence of locally weakly $\star$ topology (see Ambrosio et al [3, p. 26, Definition 1.58] for the definition).

Corollary 18. (Locally Weakly $\star$ Convergence)

(i) $\Gamma_\varepsilon^{\delta_{k_l}} \to \Gamma_\varepsilon$ *locally weakly* $\star$ *in* Ω_m *as* $l \to \infty$,

(ii) $\Gamma_{\varepsilon_l} \to \Gamma$ *locally weakly* $\star$ *in* Ω *as* $l \to \infty$.

Proof. We state the proof of (ii) for instance. We have only to show

$$\lim_{l\to\infty}\int_\Omega \zeta d\Gamma_\varepsilon^{\delta_{k_l}} \to \int_\Omega \zeta d\Gamma_\varepsilon \quad \text{for } \zeta \in C_c(\Omega) \text{ with } \zeta \geqq 0 \text{ in } \Omega.$$

Let $(\zeta_j) \subset C_c^1(\Omega) \geqq 0$ be a sequence such that $\zeta_j \rightrightarrows \zeta$ in Ω as $j \to \infty$, $\operatorname{spt}\zeta_j \subset K$ for some compact set $K \subset \Omega$. Then

$$\begin{aligned}
&\left|\int_\Omega \zeta d\Gamma_{\varepsilon_l} - \int_\Omega \zeta d\Gamma\right| \\
&\leqq \left|\int_\Omega \zeta d\Gamma_{\varepsilon_l} - \int_\Omega \zeta_j d\Gamma_{\varepsilon_l}\right| + \left|\int_\Omega \zeta_j d\Gamma_{\varepsilon_l} - \int_\Omega \zeta_j d\Gamma\right| + \left|\int_\Omega \zeta_j d\Gamma - \int_\Omega \zeta d\Gamma\right| \\
&\leqq ||\zeta_j - \zeta||_{\infty,\Omega}\Big(\sup_{l\in\mathbb{N}}\Gamma_{\varepsilon_l}(K) + \Gamma(K)\Big) \quad + \quad \left|\int_\Omega \zeta_j d\Gamma_{\varepsilon_l} - \int_\Omega \zeta_j d\Gamma\right|.
\end{aligned}$$

Remark that Γ_{ε_l} is locally uniformly bounded with respect to l as stated in the proof of Lemma 16. Hence for any positive number ε, there exists an integer j_ε such that the first term is less than ε. For this fixed j_ε, the second term goes to zero as $l \to \infty$. Thus, for any $\varepsilon > 0$,

$$\overline{\lim_{l\to\infty}}\left|\int_\Omega \zeta d\Gamma_{\varepsilon_l} - \int_\Omega \zeta d\Gamma\right| < \varepsilon,$$

and so we obtain the conclusion. □

Having obtained the convergence result for Radon measures, we are in a position to show our conclusion of this paper. For this purpose, we need to investigate the relation between the Radon measure Γ_ε and the Lebesgue measure $\mathscr{L}^n$, as stated in Remark 13. Hereafter, u_ε is an $\mathbb{F}_\varepsilon$-minimizer which is a limit function of the $\mathbb{F}_\varepsilon^{\delta_{k_l}}$-minimizers as in Theorem 8.

Definition 19. (Approximated Free Boundary) Let u_ε be as above. We call the set

$$F_{u_\varepsilon^1} \equiv \Big\{x \in \Omega \,\Big|\, 0 < u_\varepsilon(x) < \varepsilon\Big\}.$$

the approximated free boundary for u_ε.

Our result is stated in the following way.

Lemma 20. (The Relation Between Γ_ε and $\mathscr{L}^n$)

$$\frac{1}{2\varepsilon}\mathscr{L}^n \llcorner F_{u_\varepsilon^1} \;\leqq\; \Gamma_\varepsilon \;\leqq\; \frac{1}{2\varepsilon}\mathscr{L}^n \llcorner \overline{F}_{u_\varepsilon^1} \qquad \textit{on } 2^{\Omega_m}.$$

Proof. Due to the regularity of Radon measure, we have only to show

$$\frac{1}{2\varepsilon}\mathscr{L}^n\big(U \cap F_{u_\varepsilon^1}\big) \leqq \Gamma_\varepsilon(U) \leqq \frac{1}{2\varepsilon}\mathscr{L}^n\big(U \cap \overline{F}_{u_\varepsilon^1}\big) \quad \text{for any open set } U \subset \Omega_m,$$

and moreover, it is enough to show

$$\int_{F_{u_\varepsilon^1}} \zeta \frac{1}{2\varepsilon} d\mathcal{L}^n \leqq \int_\Omega \zeta d\Gamma_\varepsilon \leqq \int_{\overline{F}_{u_\varepsilon^1}} \zeta \frac{1}{2\varepsilon} d\mathcal{L}^n$$

$$\text{for any } \zeta \in C_c^1(\Omega) \text{ with } \operatorname{spt}\zeta \subset \Omega_m \text{ and } \zeta \geqq 0 \text{ in } \Omega. \quad (24)$$

Before showing (24), we remark the following fact:

$$\begin{cases} \lim_{l\to\infty} (\chi_\varepsilon^{\delta_{k_l}})'(u_\varepsilon^{\delta_{k_l}}(x)) = \begin{cases} \dfrac{1}{\varepsilon} & \text{if } u_\varepsilon(x) \in (0,\varepsilon), \\ 0 & \text{if } u_\varepsilon(x) \in (-\infty,0)\cup(\varepsilon,+\infty), \end{cases} \\ 0 \leqq (\chi_\varepsilon^{\delta_{k_l}})'(u_\varepsilon^{\delta_{k_l}}(x)) \leqq \dfrac{1}{\varepsilon} \quad \text{if } u_\varepsilon(x) = 0 \text{ or } \varepsilon. \end{cases} \quad (25)$$

In fact, if $0 < u_\varepsilon(x_0) < \varepsilon$, then for sufficiently large l, in the neighborhood of $t = u_\varepsilon(x_0)$ it holds that $\chi_\varepsilon^{\delta_k}(t) \equiv \frac{t}{\varepsilon} - \delta_k$. Since $u_\varepsilon^{\delta_k}(x_0) \to u_\varepsilon(x_0)$, if necessary, further taking l large enough, we have $(\chi_\varepsilon^{\delta_k})'(u_\varepsilon^{\delta_{k_l}}(x_0)) = \frac{1}{\varepsilon}$, and so we have the first equality of (25). The second equality is shown in the same way. We here notice that in case $u_\varepsilon(x_0) = 0$ or ε, it is impossible to get the same type of equality as above, because the corresponding limit value depends on the way of the convergence of $u_\varepsilon^{\delta_k}(x_0) \to u_\varepsilon(x_0)$. Nevertheless, the last two inequalities are from (iii) of Lemma 2.

Thus from (25), for any $x \in \Omega$ and any positive function ϕ on Ω, it holds that

$$\frac{1}{\varepsilon}\chi_{(0,\varepsilon)}(u_\varepsilon(x))\phi(x) \leqq \varliminf_{l\to\infty} (\chi_\varepsilon^{\delta_{k_l}})'(u_\varepsilon^{\delta_{k_l}}(x))\phi(x)$$
$$\leqq \varlimsup_{l\to\infty} (\chi_\varepsilon^{\delta_{k_l}})'(u_\varepsilon^{\delta_{k_l}}(x))\phi(x) \leqq \frac{1}{\varepsilon}\chi_{[0,\varepsilon]}(u_\varepsilon(x))\phi(x). \quad (26)$$

Now, in the following, let ζ be a $C_c^1(\Omega)$-function satisfying $\operatorname{spt}\zeta \subset \Omega_m$ and $\zeta \geqq 0$ in Ω.

Proof of the First Inequality of (24). Let us use the first inequality of (26). By virtue of the Fatou Lemma, we deduce

$$\frac{1}{\varepsilon}\int_{\Omega(0<u_\varepsilon<\varepsilon)} \zeta d\mathcal{L}^n \leqq \varliminf_{l\to\infty} \int_\Omega (\chi_\varepsilon^{\delta_{k_l}})'(u_\varepsilon^{\delta_{k_l}}(x))\zeta(x) d\mathcal{L}^n(x) =$$
$$= \lim_{l\to\infty} 2\int_\Omega \zeta d\Gamma_\varepsilon^{\delta_{k_l}} = 2\int_\Omega \zeta d\Gamma_\varepsilon,$$

where the second and third equality follow from (16) and (ii) of Lemma 17 respectively. □

Proof of the Second Inequality of (24). Since from the regularity of the Lebesgue measure $\mathcal{L}^n$, for any positive constant σ we can choose an open subset $U_\sigma \subset\subset \Omega$ such that

$$\begin{cases} U_\sigma \supset [\overline{F}_{u^1_\varepsilon} \cap \operatorname{spt}\zeta], \\ \mathcal{L}^n(U_\sigma \setminus [\overline{F}_{u^1_\varepsilon} \cap \operatorname{spt}\zeta]) < \sigma. \end{cases}$$

Moreover, let $\tilde{U}_\sigma$ be an open set such that $[\overline{F}_{u^1_\varepsilon} \cap \operatorname{spt}\zeta] \subset \tilde{U}_\sigma \subset\subset U_\sigma$ holds. Then we can choose a function $\varphi_\sigma \in C^1_c(\Omega)$ such that

$$\begin{cases} \varphi_\sigma \equiv 1 \text{ in } \tilde{U}_\sigma, \\ 0 \leqq \varphi_\sigma \leqq 1 \text{ in } \Omega, \\ \operatorname{spt}\varphi_\sigma \subset U_\sigma. \end{cases}$$

From Proposition 14, $\operatorname{spt}\Gamma_\varepsilon \subset \overline{F}_{u^1_\varepsilon}$ holds, and so $d\Gamma_\varepsilon = d\Gamma_\varepsilon \llcorner \overline{F}_{u^1_\varepsilon} = \chi_{\overline{F}_{u^1_\varepsilon}} d\Gamma_\varepsilon$. Hence

$$\begin{aligned} \int_\Omega \zeta d\Gamma_\varepsilon &= \int_\Omega \zeta\varphi_\sigma d\Gamma_\varepsilon + \int_\Omega \zeta(1-\varphi_\sigma) d\Gamma_\varepsilon \\ &= \int_\Omega \zeta\varphi_\sigma d\Gamma_\varepsilon + \int_{\overline{F}_{u^1_\varepsilon}} \zeta(1-\varphi_\sigma) d\Gamma_\varepsilon. \end{aligned} \tag{27}$$

Notice here the inclusion $\operatorname{spt}[\zeta(1-\varphi_\sigma)] \subset \Omega \setminus \overline{F}_{u^1_\varepsilon}$. Then we have

$$\int_\Omega \zeta d\Gamma_\varepsilon = \int_\Omega \zeta\varphi_\sigma d\Gamma_\varepsilon. \tag{28}$$

Now let us show the second inequality of (24). From (iii) of Lemma 2, the estimate $|(\chi_\varepsilon^{\delta_{k_l}})'(u_\varepsilon^{\delta_{k_l}}(x))| \leqq \frac{1}{\varepsilon}$ holds for $x \in \Omega$, and therefore, by the last inequality of (26) and the Lebesgue convergence theorem, we obtain

$$\overline{\lim_{l\to\infty}} \int_\Omega (\chi_\varepsilon^{\delta_{k_l}})'(u_\varepsilon^{\delta_{k_l}}(x))\zeta(x)\varphi_\sigma(x) d\mathcal{L}^n(x) \leqq \int_{\Omega(0\leqq u_\varepsilon \leqq \varepsilon)} \frac{1}{\varepsilon}\zeta\varphi_\sigma d\mathcal{L}^n. \tag{29}$$

On the other hand, with the help of (16) we rewrite the left-hand side of (29) as follows:

$$\begin{aligned} &\overline{\lim_{l\to\infty}} \int_\Omega (\chi_\varepsilon^{\delta_{k_l}})'(u_\varepsilon^{\delta_{k_l}}(x))\zeta(x)\varphi_\sigma(x) d\mathcal{L}^n(x) \\ &= \overline{\lim_{l\to\infty}}\, 2\int_\Omega \zeta\varphi_\sigma d\Gamma_\varepsilon^{\delta_{k_l}} = 2\int_\Omega \zeta\varphi_\sigma d\Gamma_\varepsilon = 2\int_\Omega \zeta d\Gamma_\varepsilon, \end{aligned} \tag{30}$$

where the second and third equality follow from (i) of Lemma 17 and (28) respectively. We thus from (29)

$$\int_\Omega \zeta d\Gamma_\varepsilon \leqq \int_{\Omega(0\leqq u_\varepsilon \leqq \varepsilon)} \frac{1}{2\varepsilon}\zeta\varphi_\sigma d\mathcal{L}^n.$$

Letting here $\sigma \downarrow 0$, we obtain

$$\int_\Omega \zeta d\Gamma_\varepsilon \leqq \int_{\Omega(0\leqq u_\varepsilon \leqq \varepsilon)\cap \overline{F}_{u^1_\varepsilon}\cap \text{ spt } \zeta} \frac{1}{2\varepsilon}\zeta d\mathcal{L}^n.$$

Since u_ε is continuous, then $\overline{F}_{u^1_\varepsilon} \cap \operatorname{spt}\zeta \subset \Omega(0 \leqq u_\varepsilon \leqq \varepsilon) \cap \operatorname{spt}\zeta$ holds. So that the last inequality is rewritten by

$$\int_\Omega \zeta d\Gamma_\varepsilon \leqq \int_{\overline{F}_{u^1_\varepsilon}} \frac{1}{2\varepsilon}\zeta d\mathcal{L}^n,$$

which is the second inequality of (24). □

Due to the general theory of Radon measures (see e.g. Evans et al [14, p. 54, Theorem 1]), we have from (ii) of Corollary 18 the following result.

Lemma 21. *Let* $\{u_{\varepsilon_l}\}_{l=1}^\infty$ *and* u *be as in Theorem* 11, *and denote by* Γ_{ε_l} *and* Γ *the corresponding Radon measures. Then*

$$\overline{\lim_{l\to\infty}}\, \Gamma_{\varepsilon_l}(K) \leqq \Gamma(K) \quad \textit{for compact set } \forall K \subset \Omega,$$
$$\Gamma(U) \leqq \underline{\lim_{l\to\infty}}\, \Gamma_{\varepsilon_l}(U) \quad \textit{for open set } \forall U \subset\subset \Omega.$$

Combining Lemma 20 with Lemma 21, we finally arrive at the conclusion of this paper.

Theorem 22. (A Criterion) *Let* $\{u_{\varepsilon_l}\}_{l=1}^\infty$ *and* u *be as in Theorem* 11.

(i) *Let* $U \subset\subset \Omega$ *be an open subset. Then if there exists at least one subsequence* $\{\varepsilon_{l_k}\}_{k=1}^\infty \subset \{\varepsilon_l\}_{l=1}^\infty$ *such that*

$$\frac{1}{\varepsilon_{l_k}}\mathcal{L}^n\big(U \cap \overline{F}_{u_{\varepsilon_{l_k}}}\big) \to 0 \quad \textit{as } k \to \infty,$$

then it holds that $\mathcal{H}^{n-1}(U \cap \partial\Omega(u > 0)) = 0$.

(ii) *Let* $K \subset \Omega$ *be a compact subset. If there exists at least one subsequence* $\{\varepsilon_{l_k}\}_{k=1}^\infty \subset \{\varepsilon_l\}_{l=1}^\infty$ *such that*

$$\frac{1}{\varepsilon_{l_k}}\mathcal{L}^n\big(K \cap F_{u_{\varepsilon_{l_k}}}\big) \geqq \delta > 0$$

holds for some positive constant δ *and for any* $k \in \mathbb{N}$, *then it holds that* $\mathcal{H}^{n-1}(K \cap \partial\Omega(u > 0)) \geqq \delta$.

In the remainder of this section, we explain about "a non-degeneracy property". In the reference Alt et al [1, p. 114, Theorem 2.7], (see also Alt et al [2, p. 12, Lemma 2.5], Omata et al [22, p. 17, Theorem 2.7]) the following property is shown: *Let* u *be an* $\mathbb{F}$*-minimizer. There exists a positive number* *such that for any open ball* $B_\varrho(x_0) \subset\subset \Omega$, *the following holds:*

$$\frac{1}{\varrho}\sup_{B_\varrho(x_0)} u < \delta \quad \Longrightarrow \quad \mathcal{L}^n\Big(B_{\frac{\varrho}{2}}(u \neq 0)\Big) = 0\,. \tag{31}$$

Since we have known that $u \geqq 0$ in Ω, the contraposition of (31) is as follows:

$$u \not\equiv B_{\frac{\varrho}{2}}(x_0) \quad \Longrightarrow \quad \frac{1}{\varrho}\sup_{B_\varrho(x_0)} u \geqq \delta\,.$$

Especially, any free boundary point x_0 satisfies the sufficient condition, and so we obtain the following implication:

$$x_0 \in \partial\Omega(u>0) \quad \Longrightarrow \quad \frac{1}{\varrho}\sup_{B_\varrho(x_0)} u \geqq \delta\,. \tag{32}$$

The property (32) tells us that the graph of u is not degenerate on the free boundary point. For this reason, this property (31) is called "*non-degeneracy property*". This property is proved by using essentially the singularity of χ, and so it cannot be expected to hold for $\mathbb{F}_\varepsilon$-minimizer u_ε. In fact, the gradient on the free boundary of u_ε must equal to zero since u_ε satisfies the Euler-Lagrange equation in the whole of the domain Ω. However, we can show some weak property not for each u_ε but a subsequence of (u_ε), which we call "*weak non-degeneracy property*" corresponding to (31).

Corollary 23. *Let* (u_{ε_l}) *be as in Theorem* 11.

(i) *Let* $U \subset \Omega$ *be an open subset. Then it holds that*

$$u_{\varepsilon_l} < \varepsilon_l \ \text{ in } U \ (\text{sufficiently large } l) \implies \lim_{l\to\infty} \mathcal{L}^n\Big(U(u_{\varepsilon_l} \neq 0)\Big) = 0.$$

(ii) (Weak Non-Degeneracy Property) *Let* C_0 *be a positive number, and* $x_0 \in \overline{\Omega}$, $\varrho \leqq \frac{1}{C_0}$. *Then it holds that*

$$\frac{1}{\varrho}\sup_{\Omega_\varrho(x_0)} u_{\varepsilon_l} < C_0\varepsilon_l \ (\text{sufficiently large } l) \implies \lim_{l\to\infty} \mathcal{L}^n\Big(\Omega_\varrho(x_0)(u_{\varepsilon_l} \neq 0)\Big) = 0,$$

where $\Omega_\varrho(x_0) = \Omega \cap B_\varrho(x_0)$.

Proof. (i) Suppose on the contrary that there exists a subsequence (u_{ε_j}) and an open subset U such that

$$\begin{cases} \text{(a)} \ \ u_{\varepsilon_l} < \varepsilon_l \quad \text{ on } U \ (\text{sufficiently large } l), \\ \text{(b)} \ \ \overline{\lim_{l\to\infty}}\, \mathcal{L}^n\big(U \cap \{u_{\varepsilon_l} \neq 0\}\big) > 0 \end{cases}$$

hold at the same time. Then, we can take a positive number σ and select a subsequence $(u_{\varepsilon_{l_j}}) \subset (u_{\varepsilon_l})$ with $\varepsilon_{l_j} \leqq 1$ for $j \in \mathbb{N}$ such that

$$\begin{cases} \text{(a)}' \ u_{\varepsilon_{l_j}} < \varepsilon_{l_j} \quad \text{on } U \ (\forall j \in \mathbb{N}), \\ \text{(b)}' \ \mathcal{L}^n(U(u_{\varepsilon_{l_j}} \neq 0)) = \mathcal{L}^n(U(u_{\varepsilon_{l_j}} > 0)) \geqq 2\sigma > 0 \quad \text{for } j \in \mathbb{N}. \end{cases}$$

From (a)$'$ and (b)$'$, we have

$$\mathcal{L}^n(U \cap F_{u_{\varepsilon_{l_j}}}) \geqq 2\sigma > 0 \quad \text{for } j \in \mathbb{N},$$

where $F_{u_{\varepsilon_{l_j}}}$ is an approximated free boundary (see Definition 19). If we can choose a compact subset $K \subset U$ such that $\mathcal{L}^n(U \setminus K) < \sigma$, then it holds that

$$2\sigma \leqq \mathcal{L}^n(U \cap F_{u_{\varepsilon_{l_j}}}) \leqq \mathcal{L}^n(K \cap F_{u_{\varepsilon_{l_j}}}) + \mathcal{L}^n(U \setminus K) \leqq \mathcal{L}^n(K \cap F_{u_{\varepsilon_{l_j}}}) + \sigma.$$

Hence we obtain

$$\frac{1}{\varepsilon_{l_j}} \mathcal{L}^n(K \cap F_{u_{\varepsilon_{l_j}}}) \geqq \mathcal{L}^n(K \cap F_{u_{\varepsilon_{l_j}}}) \geqq \sigma > 0 \quad \text{for } j \in \mathbb{N}.$$

With the aid of (ii) of Theorem 22, $\mathcal{H}^{n-1}(K \cap \partial\Omega(u > 0)) > 0$. This contradicts the fact $u \equiv 0$ in U which is obtained from (a)$'$.

(ii) From the assumption $u_{\varepsilon_l} < C_0 \varepsilon_l \varrho \leqq \varepsilon_l$, and so by applying (i) as $U = \Omega_\varrho(x_0)$, the conclusion follows immediately. □

Remark 24. All the argument above was achieved by taking a piecewise linear function as an approximated characteristic function. Such a choice is considered to be a simplest and best in order to obtain the weak non-degeneracy property as (ii) of Corollary 23. In fact, it is understood by referring the consideration below that if the singularity of the approximation to χ increases, we can get more sharp criterion and strong non-degeneracy property. On the other hand, for showing the convergence of Radon measures as in Corollary 18, it is essential for minimizers to have Lipschitz continuity. Moreover, for that, the order

$$\sup_{\varepsilon > 0} \varepsilon ||\chi'_\varepsilon||_{\infty, \mathbb{R}} < +\infty$$

is essential. Thus, we are required to choose as χ_ε a Lipschitz continuous function whose Lipschitz constant is of order $\frac{1}{\varepsilon}$ as $\varepsilon \downarrow 0$.

A Consideration to the Case of Smooth Approximation. Let us show a technical reason why the assertion of Corollary 23 does not necessarily hold if we choose as χ_ε a smooth function (we shall construct a concrete counter-example in the final section). Let $\widehat{\chi}_1$ be a non-decreasing $C^\infty(\mathbb{R})$-function such that $\widehat{\chi}''_1(t) \geqq 0$ for $t \in [0, \frac{1}{2}]$, and $\widehat{\chi}_1(t) = 0$ if $t \leqq 0$, $= 1$ if $t \geqq 1$. We set

$\widehat{\chi}_\varepsilon(t) = \widehat{\chi}_1(\frac{t}{\varepsilon})$ for $t \in \mathbb{R}$. Then

$$\int_\Omega \zeta d\Gamma_\varepsilon = \frac{1}{2}\int_\Omega \widehat{\chi}'_\varepsilon(u_\varepsilon)\zeta d\mathcal{L}^n \quad \text{for } \zeta \in C^1_c(\Omega).$$

We therefore have by the regularity of Radon measures

$$\Gamma_\varepsilon = \frac{1}{2}\widehat{\chi}'_\varepsilon(u_\varepsilon)d\mathcal{L}^n = \frac{1}{2}\widehat{\chi}'_\varepsilon(u_\varepsilon)d\mathcal{L}^n \llcorner F_{u_\varepsilon} \quad \text{on } 2^\Omega.$$

Thus we obtain instead of the result of Theorem 22 the following:

$$\begin{aligned} &\frac{1}{2}\int_{U\cap F_{u_{\varepsilon_{l_k}}}} \widehat{\chi}'_{\varepsilon_{l_k}}(u_{\varepsilon_{l_k}})d\mathcal{L}^n \to 0 \quad \text{as } k \downarrow 0 \\ &\qquad \Longrightarrow \quad \mathcal{H}^{n-1}(U \cap \partial\Omega(u>0)) = 0, \end{aligned} \tag{33}$$

$$\begin{aligned} &\frac{1}{2}\int_{K\cap F_{u_{\varepsilon_{l_k}}}} \widehat{\chi}'_{\varepsilon_{l_k}}(u_{\varepsilon_{l_k}})d\mathcal{L}^n \geqq \delta > 0 \quad \text{for } k \in \mathbb{N} \\ &\qquad \Longrightarrow \quad \mathcal{H}^{n-1}(K \cap \partial\Omega(u>0)) \geqq \delta. \end{aligned} \tag{34}$$

Now, we choose a sequence of positive numbers (δ_ε) such that $\delta_\varepsilon \leqq M\varepsilon^2$ for any $\varepsilon > 0$, where M is a positive constant. Then

$$\widehat{\chi}'_\varepsilon(\delta_\varepsilon) := \frac{1}{\varepsilon}\widehat{\chi}'_1\Big(\frac{\delta_\varepsilon}{\varepsilon}\Big) \leqq \frac{1}{\varepsilon}\widehat{\chi}'_1(M\varepsilon) \to 0 \quad \text{as } \varepsilon \downarrow 0, \tag{35}$$

where the last convergence is followed from the fact that $\widehat{\chi}'_1(0) = \widehat{\chi}''_1(0) = 0$. Assume now that

$$0 < u_{\varepsilon_{l_k}} \leqq \delta_{\varepsilon_{l_k}} \quad \text{on } U \quad \text{for } \forall k \in \mathbb{N}$$

holds. In particular, (a) and (b) of Corollary 23 are satisfied for $j \in \mathbb{N}$ sufficiently large. However, from this we cannot reach a contradiction. Because by noticing the non-decreasing property of $\widehat{\chi}'_\varepsilon(\cdot)$ and (35), we have

$$\int_{U\cap F_{u_{\varepsilon_{l_k}}}} \widehat{\chi}'_{\varepsilon_{l_k}}(u_{\varepsilon_{l_k}})d\mathcal{L}^n \leqq \int_{U\cap F_{u_{\varepsilon_{l_k}}}} \widehat{\chi}'_{\varepsilon_{l_k}}(\delta_{\varepsilon_{l_k}})d\mathcal{L}^n \to 0 \quad \text{as } k \to \infty.$$

Hence from (33) we have $\mathcal{H}^{n-1}(U \cap \partial\Omega(u>0)) = 0$. This is *not* a contradiction to the fact $u \equiv 0$ on U.

The argument of this remark is substantiated in Section 5 by constructing a counter-example in the one-dimension model. □

4. The First Variation

Under some regularity hypothesis, we compute the first variation of the functional $\mathbb{F}_\varepsilon$ with the piecewise linear approximation χ_ε.

We sum up notation. In the argument of the preceding sections, we let $n \geqq 1$, and $a^{ij}(\cdot) = \delta_{ij}$. Moreover, let χ_ε be a piecewise linear function as in (i) of Definition 1. For a given non-negative function $\phi_0 \in W^{1,2}(\Omega)$, we consider the following variational problem:

$$\mathbb{F}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \big(|\nabla u|^2 + \chi_\varepsilon(u)\big) d\mathcal{L}^n & \text{for } u \in \mathcal{K}_{\phi_0}\,, \\ +\infty & \text{for } u \in L^2(\Omega) \setminus \mathcal{K}_{\phi_0}, \end{cases}$$

where $\mathcal{K}_{\phi_0} := \{w \in W^{1,2}(\Omega) \,:\, w = \phi_0 \text{ on } \partial\Omega, 0 \leqq w \leqq \sup_{\partial\Omega} \phi_0\}$.

Remark. In the preceding sections, we exclude the case $n = 1$ because the condition $n \geqq 2$ is needed when we adopt the general results as the Harnack's inequality and the regularity theory of the weak solutions of PDE. On the other hand, in this section, we consider also the case $n = 1$. In fact, the result of this section shall be used in the final section in order to construct a one-dimensional example (see the proof of Proposition 31).

Before stating the main result, we give a notation and a terminology used throughout this section:

Terminology. Let $A \subset \mathbb{R}^n$ be a bounded open subset. We say that A *is in C^1-class in a compact set $K \subset \mathbb{R}^n$* when the followings are satisfied:

— The case where $n = 1$.

The set $\partial A \cap K$ is singleton.

— The case where $n \geqq 2$.

It holds that $\partial A \cap K \neq \emptyset$ and for any $\xi_0 \in \partial A \cap K$, there exists an open neighbourhood N_{ξ_0} of ξ_0 and a C^1-diffeomorphism $\Phi_{\xi_0} : N_{\xi_0} \to B_1(0)$ such that $\Phi_{\xi_0}(N_{\xi_0} \cap A) = B_1(0) \cap \{x_n < 0\}$.

Notation. For $u : \Omega \to \mathbb{R}$, and an open or closed subset $A \subset \Omega$, we denote $u \in C^k(A)$ in case the function $u\big|_A$ defined by restricting the domain of definition of u to A is in $C^k(A)$-class.

Theorem 25. (First Variation) *Let u_ε be an $\mathbb{F}_\varepsilon$-minimizer. Assume that $u_\varepsilon \in C^0(\Omega)$.*

(i) *Let $\eta \in C_c^1(\Omega, \mathbb{R}^n)$ with $\operatorname{spt}\eta \cap \Omega(u_\varepsilon \geqq \varepsilon) = \emptyset$. Assume that the set $\Omega(u_\varepsilon > 0)$ is in C^1-class in $\operatorname{spt}\eta$. If $u_\varepsilon \in C^1\big(\overline{\Omega(u_\varepsilon > 0) \cap \operatorname{spt}\eta}\big)$, then the* *following holds:*

$$\int_{\partial\Omega(u_\varepsilon>0)} |\nabla u_\varepsilon|_+^2 \langle \eta, \nu_\varepsilon\rangle d\mathcal{H}^{n-1} = 0,$$

where ν_ε *is the unit outer normal to the boundary of the set* $\Omega(u_\varepsilon > 0)$*, and* $|\nabla u_\varepsilon|_+(\xi) = \lim_{\substack{x\to\xi \\ u_\varepsilon(x)>0}} |\nabla u_\varepsilon(x)|$ *for* $\xi \in \partial\Omega(u_\varepsilon > 0)$*.*

(ii) *Let* $\eta \in C_c^1(\Omega, \mathbb{R}^n)$ *with* $\operatorname{spt}\eta \cap \Omega(u_\varepsilon = 0) = \emptyset$*. Assume that the boundary of* $\Omega(u_\varepsilon > \varepsilon)$ *coincides with* $\Omega(u_\varepsilon = \varepsilon)$ *in* $\operatorname{spt}\eta$*, and the set* $\Omega(u_\varepsilon > \varepsilon)$ *is in* C^1*-class in* $\operatorname{spt}\eta$*. If*

$$u_\varepsilon \in C^1\big(\overline{\Omega(u_\varepsilon > \varepsilon) \cap \operatorname{spt}\eta}\big) \cap C^1\big(\overline{\Omega(u_\varepsilon < \varepsilon) \cap \operatorname{spt}\eta}\big),$$

then the following holds:

$$\int_{\Omega(u_\varepsilon=\varepsilon)} \big(|\nabla u_\varepsilon|_+^2 - |\nabla u_\varepsilon|_-^2\big)\langle \eta, \overline{\nu}_\varepsilon\rangle d\mathcal{H}^{n-1} = 0,$$

where $\overline{\nu}_\varepsilon$ *is the unit outer normal to the boundary of the set* $\Omega(u_\varepsilon > \varepsilon)$*, and* $|\nabla u_\varepsilon|_+(\xi) = \lim_{\substack{x\to\xi \\ u_\varepsilon(x)>\varepsilon}} |\nabla u_\varepsilon(x)|$*,* $|\nabla u_\varepsilon|_-(\xi) = \lim_{\substack{x\to\xi \\ u_\varepsilon(x)<\varepsilon}} |\nabla u_\varepsilon(x)|$ *for* $\xi \in \Omega(u_\varepsilon = \varepsilon)$*.*

(iii) *Let* $\eta \in C_c^1(\Omega, \mathbb{R}^n)$ *with* $\operatorname{spt}\eta \cap \Omega(u_\varepsilon < \varepsilon) = \emptyset$*. Assume that the set* $\Omega(u_\varepsilon > \varepsilon)$ *is in* C^1*-class in* $\operatorname{spt}\eta$*. If* $u_\varepsilon \in C^1\big(\overline{\Omega(u_\varepsilon > \varepsilon) \cap \operatorname{spt}\eta}\big)$*, then the following holds:*

$$\int_{\partial\Omega(u_\varepsilon>\varepsilon)} |\nabla u_\varepsilon|_+^2 \langle \eta, \tilde{\nu}_\varepsilon\rangle d\mathcal{H}^{n-1} = 0,$$

where $\tilde{\nu}_\varepsilon$ *is the unit outer normal to the boundary of the set* $\Omega(u_\varepsilon > \varepsilon)$*, and* $|\nabla u_\varepsilon|_+(\xi) = \lim_{\substack{x\to\xi \\ u_\varepsilon(x)>\varepsilon}} |\nabla u_\varepsilon(x)|$ *for* $\xi \in \partial\Omega(u_\varepsilon > \varepsilon)$*.*

Proof. We adopt the same test function as in Alt et al [1, p. 109]. Let $\eta \in C_c^1(\Omega, \mathbb{R}^n)$. For $\delta \in \mathbb{R}$, we define $\tau_\delta(x) := x + \delta\eta(x)$ for $x \in \Omega$. Then there exists a positive number δ_0 such that

$$|\delta| < \delta_0 \quad \Longrightarrow \quad \begin{cases} |\det \nabla\tau_\delta| \neq 0 & \text{in } \Omega\,, \\ \tau_\delta(\operatorname{spt}\eta) \subset \Omega\,. \end{cases}$$

In the following, let δ be a real number such that $|\delta| < \delta_0$. Now we set $(u_\varepsilon)_\delta(y) := (u_\varepsilon \circ \tau_\delta^{-1})(y)$ for $y \in \Omega$. Since $(u_\varepsilon)_\delta \in \mathcal{K}_{\phi_0}$, we have

$$\mathbb{F}_\varepsilon\big((u_\varepsilon)_\delta\big) = \int_\Omega \big(|\nabla(u_\varepsilon)_\delta|^2 + \chi_\varepsilon\big((u_\varepsilon)_\delta\big)\big) d\mathcal{L}^n. \tag{36}$$

Let us rewrite the right-side hand of (36) to the form using u_ε.

—[1] $|\nabla(u_\varepsilon)_\delta|^2$-term: We at first have for $y \in \Omega$ that

$$\nabla(u_\varepsilon)_\delta(y) = (\nabla u_\varepsilon)(\tau_\delta^{-1}(y)) \cdot \Big[(\nabla\tau_\delta)(\tau_\delta^{-1}(y))\Big]^{-1}$$
$$= (\nabla u_\varepsilon)(\tau_\delta^{-1}(y)) \cdot (I + \delta\nabla\eta(\tau_\delta^{-1}(y)))^{-1}\,. \quad (37)$$

Since it holds that

$$(I + \delta\nabla\eta)^{-1} = (I - \delta\nabla\eta) + \delta^2(\nabla\eta)^2(I + \delta\nabla\eta)^{-1}, \quad (38)$$

we have

$$(I + \delta\nabla\eta(\tau_\delta^{-1}(y)))^{-1} = I - \delta\nabla\eta(\tau_\delta^{-1}(y)) + \sigma(\delta, y) \quad \text{for } y \in \Omega,$$

where $\sigma(\delta, y)$ is an $n \times n$-matrix whose (i,j)-coefficient $(1 \leqq i, j \leqq n)$ satisfies

$$\frac{1}{\delta}\sup_{y\in\Omega}|\sigma_{i,j}(\delta, y)| \to 0 \quad \text{as } \delta \to 0.$$

We thus have from (37)

$$\nabla(u_\varepsilon)_\delta(y)$$
$$= (\nabla u_\varepsilon)(\tau_\delta^{-1}(y)) - \delta(\nabla u_\varepsilon)(\tau_\delta^{-1}(y)) \cdot \nabla\eta + (\nabla u_\varepsilon)(\tau_\delta^{-1}(y)) \cdot \sigma(\delta, \sigma),$$

and hence

$$|\nabla(u_\varepsilon)_\delta(y)|^2 = |\nabla u_\varepsilon(\tau_\delta^{-1}(y))|^2$$
$$- 2\delta\nabla u_\varepsilon(\tau_\delta^{-1}(y)) \cdot (\nabla\eta)(\tau_\delta^{-1}(y)) \cdot \nabla u_\varepsilon(\tau_\delta^{-1}(y)) + \sigma_1(\delta, y), \quad (39)$$

where σ_1 is a function satisfying

$$|\sigma_1(\delta, y)| \leqq \beta_1(\delta)|\nabla u_\varepsilon(\tau_\delta^{-1}(y))|^2 \quad \text{for } y \in \Omega,\ \delta \in (-\delta_0, \delta_0)$$

for some function $\beta_1 : (-\delta_0, \delta_0) \to \mathbb{R}^+$ such that $\frac{1}{\delta}\beta_1(\delta) \to 0$ as $\delta \to 0$.

We here sum up estimates related to the value $\det(\nabla\tau_\delta)$. We firstly have

$$\det(\nabla\tau_\delta(\tau_\delta^{-1}(y)))$$
$$= \det(I + \delta\nabla\eta(\tau_\delta^{-1}(y))) = 1 + \delta(\mathrm{div}\eta)(\tau_\delta^{-1}(y)) + \sigma_2(\delta, y), \quad (40)$$

where $\sigma_2(\delta, y)$ is a finite sum of functions of the form $\delta^k(\nabla\eta(\tau_\delta^{-1}(y)))^k$

$(k \geqq 2)$ So that if we set

$$\beta_2(\delta) := \sup_{y\in\Omega}|\sigma_2(\delta, y)| \quad \text{for each } \delta \in (-\delta_0, \delta_0),$$

then the following holds:

$$\frac{1}{\delta}\beta_2(\delta) \to 0 \quad \text{as } \delta \to 0. \quad (41)$$

Moreover, setting $\beta_3(\delta) := \sup_{y\in\Omega}\left|\delta(\mathrm{div}\eta)(\tau_\delta^{-1}(y)) + \sigma_2(\delta, y)\right|$ for $\delta \in (-\delta_0,\, \delta_0)$, we have

$$\begin{cases} \sup_{y\in\Omega} |\det(\nabla(u_\varepsilon)_\delta(\tau_\delta^{-1}(y)) - 1| = \sup_{y\in\Omega} |\det(I + \delta\nabla\eta(\tau_\delta^{-1}(y))) - 1| \\ \leqq \beta_3(\delta), \\ \beta_3(\delta) \to 0 \quad \text{as } \delta \to 0. \end{cases} \tag{42}$$

Now we have from (42)

$$\begin{aligned} &\frac{1}{\delta}\int_\Omega |\sigma_1(\delta, y)| dy \leqq \beta_1(\delta) \int_\Omega |\nabla u_\varepsilon(\tau_\delta^{-1}(y)|^2 dy \\ &\quad = \frac{1}{\delta}\beta_1(\delta)\int_\Omega |\nabla u_\varepsilon(x)|^2 |\det \nabla\tau_\delta(x)| dx \leqq \frac{1}{\delta}\beta_1(\delta)(1+\beta_3(\delta))\int_\Omega |\nabla u_\varepsilon|^2 dx \\ &\qquad \to 0 \text{ as } \delta \to 0. \end{aligned}$$

As a result we obtain

$$\int_\Omega \sigma_1(\delta, y) dy = o(\delta) \quad \text{as } \delta \to 0. \tag{43}$$

Remarking (40) and (43), we integrate on Ω the both sides of the equality (39) in order to have

$$\begin{aligned} &\int_\Omega |\nabla(u_\varepsilon)_\delta(y)|^2 dy \\ &= \int_\Omega \Big\{ |\nabla u_\varepsilon(\tau_\delta^{-1}(y))|^2 - 2\delta\nabla u_\varepsilon(\tau_\delta^{-1}(y)) \cdot (\nabla\eta)(\tau_\delta^{-1}(y)) \cdot \nabla u_\varepsilon(\tau_\delta^{-1}(y)) \Big\} dy \\ &\qquad + \int_\Omega \sigma_1(\delta, y) dy \\ &= \int_\Omega \Big\{ |\nabla u_\varepsilon(\tau_\delta^{-1}(y))|^2 - 2\delta\nabla u_\varepsilon(\tau_\delta^{-1}(y)) \cdot (\nabla\eta)(\tau_\delta^{-1}(y)) \cdot \nabla u_\varepsilon(\tau_\delta^{-1}(y)) \Big\} dy \\ &\qquad + o(\delta) = \int_\Omega \Big(|\nabla u_\varepsilon|^2 - 2\delta\nabla u_\varepsilon \nabla\eta \nabla u_\varepsilon \Big) |\det \nabla\tau_\delta| dx + o(\delta) \\ &\quad = \int_\Omega \Big(|\nabla u_\varepsilon|^2 - 2\delta\nabla u_\varepsilon \nabla\eta \nabla u_\varepsilon \Big) \big(1 + \delta \mathrm{div}\eta + \sigma_2(\delta, \tau_\delta(x))\big) dx + o(\delta) \\ &\qquad = \int_\Omega \Big(|\nabla u_\varepsilon|^2 + \delta \mathrm{div}\eta |\nabla u_\varepsilon|^2 - 2\delta\nabla u_\varepsilon \nabla\eta \nabla u_\varepsilon \Big) d\mathcal{L}^n + \\ &\underbrace{\int_\Omega \Big(\sigma_2(\delta, \tau_\delta(x)) |\nabla u_\varepsilon|^2 - 2\delta^2 \nabla u_\varepsilon \nabla\eta \nabla u_\varepsilon \mathrm{div}\eta - 2\delta\nabla u_\varepsilon \nabla\eta \nabla u_\varepsilon \sigma_2(\delta, \tau_\delta(x)) \Big) dx}_{I} \\ &\qquad + o(\delta) \quad \text{as } \delta \to 0. \end{aligned} \tag{44}$$

By remarking (41), it turns out that I-term of (44) is small order of δ as $\delta \to 0$, and finally we reach

$$\int_\Omega |\nabla(u_\varepsilon)_\delta|^2 d\mathcal{L}^n = \int_\Omega \Big(|\nabla u_\varepsilon|^2 + \delta \mathrm{div}\eta |\nabla u_\varepsilon|^2 - 2\delta \nabla u_\varepsilon \nabla\eta \nabla u_\varepsilon\Big) d\mathcal{L}^n + o(\delta) \quad \text{as } \delta \to 0. \quad (45)$$

—[2] $\chi_\varepsilon(u_\varepsilon)$-term. Remarking that $0 \leqq \chi_\varepsilon \leqq 1$, we have from (41)

$$\begin{aligned}\int_\Omega \chi_\varepsilon((u_\varepsilon)_\delta) d\mathcal{L}^n &= \int_\Omega (\chi_\varepsilon \circ u_\varepsilon)(\tau_\delta^{-1}(y)) dy = \int_\Omega (\chi_\varepsilon \circ u_\varepsilon)(x) |\det \nabla \tau_\delta(x)| dx \\ &= \int_\Omega \chi_\varepsilon(u_\varepsilon(x))(1 + \delta \mathrm{div}\eta(x) + \sigma_2(\delta, \tau_\delta(x))) dx + o(\delta) \\ &= \int_\Omega (\chi_\varepsilon(u_\varepsilon) + \delta \mathrm{div}\eta \chi_\varepsilon(u_\varepsilon)) d\mathcal{L}^n + o(\delta) \quad \text{as } \delta \to 0. \quad (46)\end{aligned}$$

With the help of (45) and (46) (36) is finally rewritten as follows:

$$\begin{aligned}\mathbb{F}_\varepsilon((u_\varepsilon)_\delta) = \int_\Omega \Big(&|\nabla u_\varepsilon|^2 + \chi_\varepsilon(u_\varepsilon) + \delta \mathrm{div}\eta |\nabla u_\varepsilon|^2 \\ &- 2\delta \nabla u_\varepsilon \nabla\eta \nabla u_\varepsilon + \delta \mathrm{div}\eta \chi_\varepsilon(u_\varepsilon)\Big) d\mathcal{L}^n + o(\delta) \quad \text{as } \delta \to 0. \quad (47)\end{aligned}$$

By making use of (47), we can compute the first variation as follows:

$$\begin{aligned}0 &= \lim_{\delta\to 0} \frac{1}{\delta}\big[\mathbb{F}_\varepsilon((u_\varepsilon)_\delta) - \mathbb{F}_\varepsilon(u_\varepsilon)\big] \\ &= \int_\Omega \Big(\mathrm{div}\eta |\nabla u_\varepsilon|^2 - 2\nabla u_\varepsilon \nabla\eta \nabla u_\varepsilon + \mathrm{div}\eta \chi_\varepsilon(u_\varepsilon)\Big) d\mathcal{L}^n. \quad (48)\end{aligned}$$

Let us show the assertions of theorem by using (48):

(i) Let $\eta \in C_c^1(\Omega, \mathbb{R}^n)$ with $\mathrm{spt}\,\eta \cap \Omega(u_\varepsilon \geqq \varepsilon) = \emptyset$. Then from (48) we have

$$\int_{\Omega(0<u_\varepsilon<\varepsilon)} \Big(\mathrm{div}\eta |\nabla u_\varepsilon|^2 - 2\nabla u_\varepsilon \nabla\eta \nabla u_\varepsilon + \mathrm{div}\eta \chi_\varepsilon(u_\varepsilon)\Big) d\mathcal{L}^n = 0. \quad (49)$$

Since from the assumption that $u_\varepsilon \in C^0(\Omega)$ the set $\Omega(0 < u_\varepsilon < \varepsilon)$ is open, we can compute the first variation restricted on this open set, and so u_ε turns out to be a weak solution of

$$\triangle u_\varepsilon = \frac{1}{2}\chi_\varepsilon'(u_\varepsilon) \equiv \frac{1}{2\varepsilon} \quad \text{in } \Omega(0 < u_\varepsilon < \varepsilon).$$

With the aid of the regularity result Ladyzhenskaya et al [19, p. 284, Theorem 6.4], we have $u_\varepsilon \in C^\infty\big(\Omega(0 < u_\varepsilon < \varepsilon)\big)$ (in case $n = 1$, the equation above tells us that the second weak derivative of u_ε is C^∞-function, and so the regularity is evident). Hence

$$u_\varepsilon \in C^\infty\big(\Omega(0 < u_\varepsilon < \varepsilon)\big) \quad \text{and} \quad \triangle u_\varepsilon = \frac{1}{2}\chi_\varepsilon'(u_\varepsilon) \quad \text{in } \Omega(0 < u_\varepsilon < \varepsilon). \quad (50)$$

Thus the following computation is possible:

$$\begin{aligned}\operatorname{div}\eta|\nabla u_\varepsilon|^2 - 2\nabla u_\varepsilon\nabla\eta\nabla u_\varepsilon &= \operatorname{div}\Big(|\nabla u_\varepsilon|^2\eta - 2\langle\eta,\nabla u_\varepsilon\rangle\nabla u_\varepsilon\Big)\\ &\quad + 2\langle\eta,\nabla u_\varepsilon\rangle\triangle u_\varepsilon = \operatorname{div}\Big(|\nabla u_\varepsilon|^2\eta - 2\langle\eta,\nabla u_\varepsilon\rangle\nabla u_\varepsilon\Big)\\ &\quad + \langle\eta,\nabla u_\varepsilon\rangle\chi_\varepsilon'(u_\varepsilon) \quad \text{in } \Omega(0<u_\varepsilon<\varepsilon). \qquad (51)\end{aligned}$$

Substituting (51) to (49), we have

$$\begin{aligned}0 &= \int_{\Omega(0<u_\varepsilon<\varepsilon)} \Big\{\operatorname{div}\Big(|\nabla u_\varepsilon|^2\eta - 2\langle\eta,\nabla u_\varepsilon\rangle\nabla u_\varepsilon\Big)\\ &\qquad + \langle\eta,\nabla u_\varepsilon\rangle\chi_\varepsilon'(u_\varepsilon) + \operatorname{div}\eta\chi_\varepsilon(u_\varepsilon)\Big\}d\mathcal{L}^n\\ &= \int_{\Omega(0<u_\varepsilon<\varepsilon)} \operatorname{div}\Big(|\nabla u_\varepsilon|^2\eta - 2\langle\eta,\nabla u_\varepsilon\rangle\nabla u_\varepsilon + \chi_\varepsilon(u_\varepsilon)\eta\Big)d\mathcal{L}^n\\ &= \int_{\partial\Omega(u_\varepsilon>0)} \Big\langle|\nabla u_\varepsilon|_+^2\eta - 2\langle\eta,(\nabla u_\varepsilon)_+\rangle(\nabla u_\varepsilon)_+ + \chi_\varepsilon(0)\eta,\ \nu_\varepsilon\Big\rangle d\mathcal{H}^{n-1},\end{aligned}$$

where in the last equality we used the Gauss-Green formula which is guaranteed by the smoothness assumption on $\partial\Omega(u_\varepsilon > 0)$ and u_ε. Since the equality $\nu_\varepsilon|\nabla u_\varepsilon|_+ = -(\nabla u_\varepsilon)_+$ holds on $\partial\Omega(u_\varepsilon > 0)\cap\operatorname{spt}\eta$, it holds that

$$|\nabla u_\varepsilon|_+^2\langle\eta,\nu_\varepsilon\rangle - 2\langle\eta,(\nabla u_\varepsilon)_+\rangle\langle(\nabla u_\varepsilon)_+,\nu_\varepsilon\rangle = -|\nabla u_\varepsilon|_+^2\langle\eta,\nu_\varepsilon\rangle. \qquad (52)$$

Therefore,

$$\int_{\partial\Omega(u_\varepsilon>0)} \big(\chi_\varepsilon(0) - |\nabla u_\varepsilon|_+^2\big)\langle\eta,\nu_\varepsilon\rangle d\mathcal{H}^{n-1} = 0.$$

By noticing the fact $\chi_\varepsilon(0) = 0$, we reach the assertion (i).

(ii) Let $\eta \in C_c^1(\Omega,\mathbb{R}^n)$ with $\operatorname{spt}\eta\cap\Omega(u_\varepsilon = 0) = \emptyset$. Then by (48)

$$\int_{\Omega(u_\varepsilon>0)} \Big(\operatorname{div}\eta|\nabla u_\varepsilon|^2 - 2\nabla u_\varepsilon\nabla\eta\nabla u_\varepsilon + \operatorname{div}\eta\chi_\varepsilon(u_\varepsilon)\Big)d\mathcal{L}^n = 0. \qquad (53)$$

Since from the assumption $\Omega(u_\varepsilon = \varepsilon)\cap\operatorname{spt}\eta$ is an $(n-1)$-dimensional C^1-submanifold, $\mathcal{L}^n\big(\Omega(u_\varepsilon = \varepsilon)\cap\operatorname{spt}\eta\big) = 0$ holds. So that

$$\begin{aligned}&\int_{\Omega(u_\varepsilon<\varepsilon)} \Big(\operatorname{div}\eta|\nabla u_\varepsilon|^2 - 2\nabla u_\varepsilon\nabla\eta\nabla u_\varepsilon + \operatorname{div}\eta\chi_\varepsilon(u_\varepsilon)\Big)d\mathcal{L}^n\\ &\quad + \int_{\Omega(u_\varepsilon>\varepsilon)} \Big(\operatorname{div}\eta|\nabla u_\varepsilon|^2 - 2\nabla u_\varepsilon\nabla\eta\nabla u_\varepsilon + \operatorname{div}\eta\chi_\varepsilon(u_\varepsilon)\Big)d\mathcal{L}^n = 0.\end{aligned}$$

By the assumption $u_\varepsilon \in C^0(\Omega)$, the set $\Omega(u_\varepsilon > \varepsilon)$ is open, and so

$$u_\varepsilon \in C^\infty\big(\Omega(u_\varepsilon > \varepsilon)\big) \quad \text{and} \quad \triangle u_\varepsilon = 0 \quad \text{on } \Omega(u_\varepsilon > \varepsilon). \qquad (54)$$

Therefore, the equality (51) holds in $\Omega(0 < u_\varepsilon < \varepsilon)$ and $\Omega(u_\varepsilon > \varepsilon)$. Remarking

(50), (54) and that $\chi_\varepsilon(u_\varepsilon) \equiv 1$ on $\Omega(u_\varepsilon > \varepsilon)$, we have

$$\begin{aligned} 0 &= \int_{\Omega(u_\varepsilon<\varepsilon)} \operatorname{div}\Big(|\nabla u_\varepsilon|^2\eta - 2\langle \eta, \nabla u_\varepsilon\rangle \nabla u_\varepsilon + \chi_\varepsilon(u_\varepsilon)\eta\Big)\, d\mathcal{L}^n \\ &\quad + \int_{\Omega(u_\varepsilon>\varepsilon)} \operatorname{div}\Big(|\nabla u_\varepsilon|^2\eta - 2\langle \eta, \nabla u_\varepsilon\rangle \nabla u_\varepsilon + \eta\Big)\, d\mathcal{L}^n \\ &= \int_{\partial\Omega(u_\varepsilon<\varepsilon)} \Big\langle |\nabla u_\varepsilon|_-^2\eta - 2\langle \eta, (\nabla u_\varepsilon)_-\rangle (\nabla u_\varepsilon)_- + \chi_\varepsilon(\varepsilon)\eta,\ (-\overline{\nu}_\varepsilon)\Big\rangle\, d\mathcal{H}^{n-1} \\ &\quad + \int_{\partial\Omega(u_\varepsilon>\varepsilon)} \Big\langle |\nabla u_\varepsilon|_+^2\eta - 2\langle \eta, (\nabla u_\varepsilon)_+\rangle (\nabla u_\varepsilon)_+ + \eta,\ \overline{\nu}_\varepsilon\Big\rangle\, d\mathcal{H}^{n-1}\,. \end{aligned}$$

In the last equality, we use the Gauss-Green formula. Moreover, we used the fact $u_\varepsilon = \varepsilon$ on $\partial\Omega(u_\varepsilon < \varepsilon)$ which is from the continuity assumption of u_ε. Remarking that $\overline{\nu}_\varepsilon|\nabla u_\varepsilon|_\pm = -(\nabla u_\varepsilon)_\pm$ holds for each point of $\partial\Omega(u_\varepsilon > \varepsilon)$, $\chi_\varepsilon(\varepsilon) = 1$, the equality for $(\nabla u_\varepsilon)_\pm$ corresponding to (52), and $\partial\Omega(u_\varepsilon < \varepsilon) = \partial\Omega(u_\varepsilon > \varepsilon) = \Omega(u_\varepsilon = \varepsilon)$ which is from the continuity assumption of u_ε, we can attain the conclusion.

(iii) Replacing, in the argument of the proof of (ii), the set $\Omega(u_\varepsilon < \varepsilon)$ with the interior of the set $\Omega(u_\varepsilon = \varepsilon)$, we obtain

$$\int_{\partial\Omega(u_\varepsilon>\varepsilon)} \big(|\nabla u_\varepsilon|_+^2 - |\nabla u_\varepsilon|_-^2\big)\langle \eta, \tilde{\nu}_\varepsilon\rangle\, d\mathcal{H}^{n-1} = 0.$$

Noticing that $|\nabla u_\varepsilon|_- \equiv 0$ on $\partial\Omega(u_\varepsilon > \varepsilon)$, we arrive at the conclusion (iii). □

5. A One-Dimensional Model

The final aim of this section is the followings.

(a) We construct an example which satisfies the assertion of Corollary 23 in one dimension (see Remark 33).

(b) If we replace the piecewise linear function χ_ε in Corollary 23 with C^1-approximation, the assertion no longer hold. We show the example (see Remark 36).

We at first sum up the energy functional and the variational problem treated in this section. Let H_0 be a constant with $0 < H_0 < 1$, and set $\phi_0(t) := H_0 t$ for $t \in (0,1)$. Let

$$\mathcal{K}_{0,H_0} := \Big\{ u \in W^{1,2}(0,1) \ \Big|\ u(0) = 0,\ u(1) = H_0,\ 0 \leqq u \leqq H_0 \text{ in } (0,1) \Big\}.$$

In the following, we let $0 < \varepsilon < \min(H_0, 1 - H_0)$. Let χ_ε be a piecewise linear function as stated in (i) of Definition 1, and $\widetilde{\chi}_\varepsilon(t)$ be as follows:

$$\widetilde{\chi}_\varepsilon(t) := \begin{cases} 0 & \text{for } t \in (-\infty, 0], \\ 2\Big(\dfrac{t}{\varepsilon}\Big)^2 & \text{for } t \in \Big[0, \dfrac{\varepsilon}{2}\Big], \\ 1 - 2\Big(1 - \dfrac{t}{\varepsilon}\Big)^2 & \text{for } t \in \Big[\dfrac{\varepsilon}{2}, 1\Big], \\ 1 & \text{for } t \in [\varepsilon, +\infty). \end{cases} \tag{55}$$

Now, we set

$$\mathbb{F}^1(u) := \begin{cases} \displaystyle\int_0^1 \big(|u'|^2 + \chi(u)\big) d\mathcal{L}^1 & \text{for } u \in \mathcal{K}_{0,H_0}, \\ +\infty & \text{for } u \in L^2(0,1) \setminus \mathcal{K}_{0,H_0}, \end{cases} \tag{56}$$

and let $\mathbb{F}^1_\varepsilon$ and $\widetilde{\mathbb{F}}^1_\varepsilon$ be functionals defining by replacing χ of (56) with χ_ε and $\widetilde{\chi}_\varepsilon$ respectively. We then define the variational problem

$$(P^1) : \text{Minimize } \mathbb{F} \text{ in } L^2(\Omega),$$

and (P^1_ε) and $(\widetilde{P}^1_\varepsilon)$ are the variational problem defined by replacing the functional $\mathbb{F}$ with $\mathbb{F}^1_\varepsilon$ and $\widetilde{\mathbb{F}}^1_\varepsilon$ respectively. We denote a minimizer of the variational problem (P^1), (P^1_ε) and $(\widetilde{P}^1_\varepsilon)$ by $\mathbb{F}^1$-, $\mathbb{F}^1_\varepsilon$- and $\widetilde{\mathbb{F}}^1_\varepsilon$-minimizer, respectively. We denote the free boundary of an $\mathbb{F}$-minimizer u^1 by

$$\partial(0,1)(u_0 > 0) := \partial\{t \in (0,1) | u_0(t) > 0\} \cap (0,1),$$

the approximated free boundary of an $\mathbb{F}_\varepsilon$-minimizer u^1_ε by

$$F_{u^1_\varepsilon} := (0,1)(0 < u_\varepsilon < \varepsilon) = \big\{t \in (0,1) \,|\, 0 < u_\varepsilon(t) < \varepsilon\big\},$$

and the approximated free boundary of an $\widetilde{\mathbb{F}}_\varepsilon$-minimizer $\tilde{u}^1_\varepsilon$ by

$$F_{\tilde{u}^1_\varepsilon} := (0,1)(0 < \tilde{u}_\varepsilon < \varepsilon) = \big\{t \in (0,1) \,|\, 0 < \tilde{u}_\varepsilon(t) < \varepsilon\big\}.$$

Remark 26. It is well-known (for e.g. Brezis [8, p. 169, Theorem VIII.2]) that for any $u \in W^{1,2}(0,1)$, it holds that $u \in C^0[0,1]$.

5.1. On the $\mathbb{F}^1$-Minimizer

Lemma 27. *Let u^1 be an $\mathbb{F}^1$-minimizer. Then u^1 is non-decreasing in* $(0,1)$.

Proof. Suppose on the contrary that

$$0 < \exists t_1 < \exists t_2 < 1 \quad \text{s.t. } u^1(t_1) > u^1(t_2) \geqq 0.$$

Then, noticing that $u^1(0) = 0$, it follows from Remark 26 that there exists a

$t_0 \in [0, t_1)$ such that $u^1(t_0) = u^1(t_2)$. Therefore we have

$$0 < \exists t_0 < \exists t_1 < \exists t_2 < 1 \quad \text{s.t. } u^1(t_0) = u^1(t_2) < u^1(t_1). \tag{57}$$

Set

$$\overline{u}^1 = \begin{cases} u^1(t_0) & \text{in } [t_0, t_2], \\ u^1 & \text{else in } (0,1). \end{cases}$$

Since $\overline{u}^1 \leqq u^1$ in $(0,1)$ and $\chi(\cdot)$ is non-decreasing in $\mathbb{R}$, we have $\chi(\overline{u}^1) \leqq \chi(u^1)$ in $(0,1)$. Noticing that $\overline{u}^1 \in \mathcal{K}_{0,H_0}$, we have from the $\mathbb{F}^1$-minimality

$$\int_{t_0}^{t_2} |(u^1)'|^2 dt \leqq \int_{t_0}^{t_2} |(\overline{u}^1)'|^2 dt = 0.$$

Thus we obtain $(u^1)' = 0$ on (t_0, t_2), and so $u^1 \equiv u^1(t_0)$ on (t_0, t_2), which contradicts (57). ☐

Proposition 28. ($\mathbb{F}$-Minimizer) *$\mathbb{F}^1$-minimizer u^1 satisfies*

$$u^1(t) = \max(0, t - 1 + H_0) \quad \text{for } t \in (0,1).$$

In particular, it holds that $\partial(0,1)(u^1 > 0) = \{1 - H_0\}$.

Proof. Let u^1 be an $\mathbb{F}^1$-minimizer. We proceed our argument dividing into the following two cases:

Case 1. The case where $\partial(0,1)(u^1 > 0) = \emptyset$. In this case, it is possible either that $u^1 \equiv 0$ in $(0,1)$ or that $u^1 > 0$ in $(0,1)$. Taking into account the Dirichlet condition, the former case is excluded. Therefore, u^1 is positive everywhere in $(0,1)$. Since $u^1 \in C^0[0,1]$, by neglecting the χ-term, we can compute the first variation to have $(u^1)'' = 0$ in $(0,1)$. Considering this with the Dirichlet condition, we obtain a candidate of minimizer $v_1(t) = H_0 t$ for $t \in (0,1)$, and

$$\mathbb{F}^1(v_1) = \int_0^1 \big((v_1')^2 + \chi(v_1)\big) dt = \int_0^1 (H_0^2 + 1) dt = H_0^2 + 1. \tag{58}$$

Case 2. The case where $\partial(0,1)(u^1 > 0) \neq \emptyset$. By recalling the non-decreasing property shown in Lemma 27, it turns out that there exists $\alpha \in (0,1)$ such that

$$u^1 \begin{cases} = 0 & \text{in } [0, \alpha], \\ > 0 & \text{in } (\alpha, 1]. \end{cases}$$

By the same reason as in Case 1, minimizer u^1 is an affine function in $(\alpha, 1)$. Moreover, due to Alt et al [1, Theorem 2.5], it holds that $(u^1)'_+(\alpha) = 1$. So the gradient of u^1 in $(\alpha, 1)$ is one. Thus, we also obtain a candidate of minimizer

$v_2(t) = \max\big(0, t-1+H_0\big)$ for $t \in (0,1)$, and

$$\mathbb{F}^1(v_2) = \int_0^1 \big((v_2')^2 + \chi(v_2)\big)dt = \int_{1-H_0}^1 \big((v_2')^2 + \chi(v_2)\big)dt = 2H_0. \tag{59}$$

Let v_1 and v_2 be as above. Then the variational problem (P^1) is equivalent to the problem

$$\text{Minimize } \mathbb{F}^1 \text{ in } \{v_1, v_2\}.$$

Hence, we have only to compare the values of $\mathbb{F}^1(v_1)$ and $\mathbb{F}^1(v_2)$. By (58), (59) and the condition $H_0 < 1$,

$$\mathbb{F}^1(v_1) = H_0^2 + 1 > 2H_0 = \mathbb{F}^1(v_2),$$

and so we conclude that v_2 is the desired $\mathbb{F}^1$-minimizer. □

5.2. On the $\mathbb{F}^1_\varepsilon$-Minimizer

Let u^1_ε be an $\mathbb{F}^1_\varepsilon$-minimizer. Since also the function $\chi_\varepsilon(\cdot)$ is non-decreasing, the assertion of Lemma 27 remains valid by replacing u^1 with u^1_ε, and therefore there exist $\alpha^\varepsilon_0, \alpha^\varepsilon_1$ and α^ε_2 with $0 \leqq \alpha^\varepsilon_0 < \alpha^\varepsilon_1 \leqq \alpha^\varepsilon_2 < 1$ such that

$$\begin{cases} u^1_\varepsilon \equiv 0 & \text{in } [0, \alpha^\varepsilon_0], \\ 0 < u^1_\varepsilon < \varepsilon & \text{in } (\alpha^\varepsilon_0, \alpha^\varepsilon_1), \\ u^1_\varepsilon \equiv \varepsilon & \text{in } [\alpha^\varepsilon_1, \alpha^\varepsilon_2], \\ u^1_\varepsilon > \varepsilon & \text{in } (\alpha^\varepsilon_2, 1]. \end{cases} \tag{60}$$

With the help of Theorem 25, we can furthermore obtain the following.

Lemma 29. *Let α^ε_1 and α^ε_2 be as in* (60). *Then the equality $\alpha^\varepsilon_1 = \alpha^\varepsilon_2$ holds.*

Proof. Suppose on the contrary that $\alpha^\varepsilon_1 < \alpha^\varepsilon_2$. u^1_ε is a weak solution of $u'' = 0$ in the interval $(\alpha^\varepsilon_2, 1) = (0,1)(u^1_\varepsilon > \varepsilon)$, and hence u^1_ε is an affine function there which in particular implies that $u^1_\varepsilon \in C^1[\alpha^\varepsilon_2, 1]$. Remarking this fact, we can apply (iii) of Theorem 25 as $n = 1$, $\Omega = (0,1)$ and $\eta \in C^1_c((0,1), \mathbb{R})$ with $\operatorname{spt}\eta \subset (\frac{\alpha^\varepsilon_1+\alpha^\varepsilon_2}{2}, \frac{\alpha^\varepsilon_2+1}{2})$. As a result, we obtain $\big((u^1_\varepsilon)'_+(\alpha^\varepsilon_2)\big)^2 \times \eta(\alpha^\varepsilon_2) = 0$. Since η is arbitrary, this implies that

$$(u^1_\varepsilon)'_+(\alpha^\varepsilon_2) = 0. \tag{61}$$

As stated above, it holds that $u^1_\varepsilon(t) = At + B$ for $t \in [\alpha^\varepsilon_2, 1]$ for some real numbers A and B. From (61) and the Dirichlet condition, it turns out that $A = 0, B = \varepsilon$, and so we arrive at the fact $u^1_\varepsilon \equiv \varepsilon$ on $[\alpha^\varepsilon_2, 1)$ which contradicts the way of choosing α^ε_2. □

As a result of Lemma 29, we obtain the following theorem.

Theorem 30. *Let u_ε^1 be an $\mathbb{F}_\varepsilon^1$-minimizer. Then there exist a real numbers α_0^ε and α_1^ε with $0 \leqq \alpha_0^\varepsilon < \alpha_1^\varepsilon < 1$ such that*

$$\begin{cases} u_\varepsilon^1 = 0 \quad \text{in } [0, \alpha_0^\varepsilon], \\ 0 < u_\varepsilon^1 < \varepsilon \quad \text{in } (\alpha_0^\varepsilon, \alpha_1^\varepsilon), \\ u_\varepsilon^1(\alpha_1^\varepsilon) = \varepsilon, \\ u_\varepsilon^1 > \varepsilon \quad \text{in } (\alpha_1^\varepsilon, 1]. \end{cases}$$

u_ε^1 satisfies the Euler-Lagrange equation

$$(u_\varepsilon^1)'' = \frac{1}{2}\chi_\varepsilon'(u_\varepsilon^1) \quad \text{in } (0,1). \tag{62}$$

Since χ_ε' is not continuous at $t = 0$ and ε. Therefore, we cannot lead by using the general regularity theory the regularity $u_\varepsilon^1 \in C^2[0,1]$. However, by making use of Theorem 25, we obtain the following.

Proposition 31. (C^1-Regularity of u_ε^1) *Let u_ε^1 be an $\mathbb{F}_\varepsilon^1$-minimizer. Then $u_\varepsilon^1 \in C^1[0,1]$.*

Proof. From (62), we have

$$\begin{cases} (u_\varepsilon^1)'' = \dfrac{1}{2\varepsilon} \quad \text{in } (0,1)(0 < u_\varepsilon^1 < \varepsilon) = (\alpha_0^\varepsilon, \alpha_1^\varepsilon), \\ (u_\varepsilon^1)'' = 0 \quad \text{in } (0,1)(u_\varepsilon^1 > \varepsilon) = (\alpha_1^\varepsilon, 1). \end{cases}$$

Hence, we in particular have

$$u_\varepsilon^1 \in C^1[\alpha_0^\varepsilon, \alpha_1^\varepsilon] \cap C^1[\alpha_1^\varepsilon, 1]. \tag{63}$$

In order to proceed our argument, we divide our argument into the following two cases:

Case 1. The case where $\alpha_0^\varepsilon > 0$. By Remark 26, we have known that $u_\varepsilon^1 \in C^0[0,1]$. From this with (63), for accomplish the assertion of the proposition, it suffices to show that

$$(u_\varepsilon^1)'_+(\alpha_0^\varepsilon) = (u_\varepsilon^1)'_-(\alpha_0^\varepsilon), \tag{64}$$

$$(u_\varepsilon^1)'_+(\alpha_1^\varepsilon) = (u_\varepsilon^1)'_-(\alpha_1^\varepsilon). \tag{65}$$

By remarking (63), we can apply (i) of Theorem 25 as $n = 1$, $\Omega = (0,1)$, and $\eta \in C_c^1(0,1)$ with $\operatorname{spt}\eta \subset (\frac{\alpha_0^\varepsilon}{2}, \frac{\alpha_0^\varepsilon + \alpha_1^\varepsilon}{2})$. So that we obtain $(u_\varepsilon^1)'_+(\alpha_0^\varepsilon) = 0$. Since $u_\varepsilon^1 \equiv 0$ in the interval $(0, \alpha_0^\varepsilon)$, it holds that $(u_\varepsilon^1)'_-(\alpha_0^\varepsilon) = 0$, and so we have (64). In the same way, by applying (ii) of Theorem 25 as $n = 1$, $\Omega = (0,1)$, and

$\eta \in C_c^1(0,1)$ with $\operatorname{spt}\eta \subset (\frac{\alpha_0^\varepsilon+\alpha_1^\varepsilon}{2}, \frac{\alpha_1^\varepsilon+1}{2})$,

$$|(u_\varepsilon^1)'|_+^2(\alpha_1^\varepsilon) = |(u_\varepsilon^1)'|_-^2(\alpha_1^\varepsilon). \tag{66}$$

Since u_ε^1 is non-decreasing in $(0,1)$, $(u_\varepsilon^1)'_\pm(\alpha_1^\varepsilon) \geqq 0$. Hence from (66) we have (65).

Case 2. The case where $\alpha_0^\varepsilon = 0$.

In this case, we have only to prove (65), which is done in the same way as in Case 1. □

Theorem 32. ($\mathbb{F}_\varepsilon$-Minimizer) *Let u_ε^1 be a minimizer of the variational problem (P_ε^1), and u^1 a minimizer of (P^1). Then:*

(i) $u_\varepsilon^1 \to u^1$ *uniformly in* $[0,1]$ *as* $\varepsilon \to 0$,

(ii) $\lim\limits_{\varepsilon\to 0} \mathbb{F}_\varepsilon^1(u_\varepsilon^1) = \mathbb{F}(u^1)$,

(iii) *For sufficiently small $\varepsilon > 0$, it hold that $F_{u_\varepsilon^1} = (1-H_0-\varepsilon, 1-H_0+\varepsilon)$, and in the sense of convergence of sets*

$$\lim_{\varepsilon\to 0} F_{u_\varepsilon^1} = \lim_{\varepsilon\to 0} \overline{F}_{u_\varepsilon^1} = \partial(0,1)(u^1 > 0).$$

Proof. Let $\alpha_0^\varepsilon \geqq 0$ be as in Corollary 30. Then let us show the following fact:

Fact. $\alpha_0^\varepsilon > 0$ *for $\varepsilon > 0$ small enough.* (67)

The outline of the proof of (67) is as follows: By assuming [a] $\alpha_0^\varepsilon = 0$, [b] $\alpha_0^\varepsilon > 0$, and on each cases we study about the form of minimizer u_ε^1, and moreover compute the value of $\mathbb{F}_\varepsilon^1(u_\varepsilon^1)$. After that, we compare those values and make sure that the value in the case of [b] is strictly smaller than that in the case of [a] for ε small enough.

— [a] Assume that $\alpha_0^\varepsilon = 0$. Recall the Euler-Lagrange equation:

$$\begin{cases} (u_\varepsilon^1)'' \equiv \dfrac{1}{2\varepsilon} & \text{on } (0,\alpha_1^\varepsilon), \\ (u_\varepsilon^1)'' \equiv 0 & \text{on } (\alpha_1^\varepsilon, 1). \end{cases}$$

In the interval $(0,\alpha_1^\varepsilon)$, u_ε^1 is solved as follows:

$$u_\varepsilon^1(t) = \frac{t^2}{4\varepsilon} + A_\varepsilon t \quad \text{in } (0,\alpha_1^\varepsilon) \qquad \text{for some constant } A_\varepsilon. \tag{68}$$

On the other hand, in the interval $(\alpha_1^\varepsilon, 1)$, u_ε^1 is an affine function with $u_\varepsilon^1(\alpha_1^\varepsilon) = \varepsilon$ and the Dirichlet condition $u_\varepsilon^1(1) = H_0$:

$$u_\varepsilon^1(t) = \frac{H_0-\varepsilon}{1-\alpha_1^\varepsilon}t + \left(H_0 - \frac{H_0-\varepsilon}{1-\alpha_1^\varepsilon}\right) \quad \text{in } (\alpha_1^\varepsilon, 1). \tag{69}$$

Since from Proposition 31, $u_\varepsilon^1 \in C^1[0,1]$, the both-sides derivative of u_ε^1 at α_1^ε must be coincide. So from (68) and (69),

$$A_\varepsilon = \frac{H_0 - \varepsilon}{1-\alpha_1^\varepsilon} - \frac{\alpha_1^\varepsilon}{2\varepsilon}. \tag{70}$$

Again by Proposition 31, $(u_\varepsilon^1)'(t)$ is continuous at $t=0$, and therefore, $(u_\varepsilon^1)'_+(0) = \left(\frac{t^2}{4\varepsilon} + A_\varepsilon t\right)'\Big|_{t=0} = A_\varepsilon$. Because $u_\varepsilon^1 \geqq 0$ in $(0,1)$, we in particular have $(u_\varepsilon^1)'_+(0) = A_\varepsilon \geqq 0$. Hence from (70) we have the quadratic inequality $(\alpha_1^\varepsilon)^2 - \alpha_1^\varepsilon + 2\varepsilon(H_0 - \varepsilon) \geqq 0$. Here, there is no loss of generality if we restrict $\varepsilon < \frac{1}{8H_0}$. Then the inequality above is solved as follows:

$$\alpha_1^\varepsilon \in \left(0, \frac{1-\sqrt{1-8\varepsilon(H_0-\varepsilon)}}{2}\right] \cup \left[\frac{1+\sqrt{1-8\varepsilon(H_0-\varepsilon)}}{2}, 1\right). \tag{71}$$

Now let us compute the value of $\mathbb{F}_\varepsilon^1(u_\varepsilon^1)$. Since $(0,\alpha_1^\varepsilon) = (0,1)(0 < u_\varepsilon^1 < \varepsilon)$, by the definition of piecewise linear function χ_ε, $\chi_\varepsilon(u_\varepsilon^1) = \frac{u_\varepsilon^1}{\varepsilon}$ in $(0,\alpha_1^\varepsilon)$ holds. Also recalling (68) and (69), we have

$$\begin{aligned}
\mathbb{F}_\varepsilon^1(u_\varepsilon) &= \int_0^{\alpha_1^\varepsilon} \big(((u_\varepsilon^1)')^2 + \chi_\varepsilon(u_\varepsilon^1)\big)dt + \int_{\alpha_1^\varepsilon}^1 \big(((u_\varepsilon^1)')^2 + \chi_\varepsilon(u_\varepsilon^1)\big)dt \\
&= \int_0^{\alpha_1^\varepsilon} \Big\{\Big(\frac{t}{2\varepsilon} + A_\varepsilon\Big)^2 + \frac{1}{\varepsilon}\Big(\frac{t^2}{4\varepsilon} + A_\varepsilon t\Big)\Big\}dt + (1-\alpha_1^\varepsilon)\Big\{\Big(\frac{H_0-\varepsilon}{1-\alpha_1^\varepsilon}\Big)^2 + 1\Big\} \\
&= \frac{(\alpha_1^\varepsilon)^3}{6\varepsilon^2} + \frac{(\alpha_1^\varepsilon)^2}{\varepsilon}A_\varepsilon + A_\varepsilon^2\alpha_1^\varepsilon + (1-\alpha_1^\varepsilon)\Big\{\Big(\frac{H_0-\varepsilon}{1-\alpha_1^\varepsilon}\Big)^2 + 1\Big\}.
\end{aligned} \tag{72}$$

From (71), we divide our argument into following two cases:

$$\begin{cases} \textit{Case 1.} \quad \alpha_1^\varepsilon \in \left(0, \dfrac{1-\sqrt{1-8\varepsilon(H_0-\varepsilon)}}{2}\right]. \\ \textit{Case 2.} \quad \alpha_1^\varepsilon \in \left[\dfrac{1+\sqrt{1-8\varepsilon(H_0-\varepsilon)}}{2}, 1\right). \end{cases}$$

In Case 1, we can demonstrate

$$\begin{cases} \circ\ \alpha_1^\varepsilon \to 0 \quad \text{as } \varepsilon \downarrow 0, \\ \circ\ \dfrac{\alpha_1^\varepsilon}{\varepsilon} \leqq \dfrac{1-\sqrt{1-8\varepsilon(H_0-\varepsilon)}}{2\varepsilon} = \dfrac{8\varepsilon(H_0-\varepsilon)}{2\varepsilon(1+\sqrt{1-8\varepsilon(H_0-\varepsilon)})} \\ \longrightarrow\ 2H_0 \quad \text{as } \varepsilon \downarrow 0, \\ \circ\ (A_\varepsilon) \text{ is bounded for } \varepsilon \text{ small enough,} \end{cases} \tag{73}$$

where the boundedness of (A_ε) is obtained from

$$0 \leqq A_\varepsilon = \frac{H_0-\varepsilon}{1-\alpha_1^\varepsilon} - \frac{\alpha_1^\varepsilon}{2\varepsilon} \leqq \frac{H_0-\varepsilon}{1-\alpha_1^\varepsilon} \longrightarrow H_0 \quad \text{as } \varepsilon \downarrow 0.$$

Now noticing (73), we let $\varepsilon \downarrow 0$ in (72),

$$\mathbb{F}_\varepsilon^1(u_\varepsilon^1) \to 1 + H_0^2 \quad \text{as } \varepsilon \downarrow 0. \tag{74}$$

On the other hand, for Case 2, $\alpha_1^\varepsilon \to 1$ as $\varepsilon \downarrow 0$, and so $A_\varepsilon \geqq 0$. Therefore,

$$\mathbb{F}_\varepsilon^1(u_\varepsilon^1) \geqq \frac{(\alpha_1^\varepsilon)^3}{6\varepsilon^2} \to +\infty \quad \text{as } \varepsilon \downarrow 0. \tag{75}$$

Denote by w^0 the minimizer u_ε^1 under the hypothesis $\alpha_0^\varepsilon = 0$. Then by (74) and (75),

$$\varliminf_{\varepsilon\to 0} \mathbb{F}_\varepsilon^1(w^0) \geqq 1 + H_0^2. \tag{76}$$

—[b] Assume that $\alpha_0^\varepsilon > 0$. Recall the Euler-Lagrange equation:

$$\begin{cases} (u_\varepsilon^1)'' \equiv \dfrac{1}{2\varepsilon} & \text{on } (\alpha_0^\varepsilon, \alpha_1^\varepsilon), \\ (u_\varepsilon^1)'' \equiv 0 & \text{on } (\alpha_1^\varepsilon, 1). \end{cases} \tag{77}$$

Hence, there exist constants C and D such that $u_\varepsilon^1(t) = \frac{1}{4\varepsilon}t^2 + Ct + D$ for $t \in (\alpha_0^\varepsilon, \alpha_1^\varepsilon)$. Since from Proposition 31 $u_\varepsilon^1 \in C^1[0,1]$ holds, $u_\varepsilon^1(\alpha_0^\varepsilon) = 0$ and $(u_\varepsilon^1)'(\alpha_0^\varepsilon) = 0$. So we obtain $C = -\frac{\alpha_0^\varepsilon}{2\varepsilon}$ and $D = \frac{(\alpha_0^\varepsilon)^2}{4\varepsilon}$. Thus, $u_\varepsilon^1(t) = \frac{1}{4\varepsilon}(t-\alpha_0^\varepsilon)^2$ for $t \in (\alpha_0^\varepsilon, \alpha_1^\varepsilon)$. Taking here the fact $u_\varepsilon^1(\alpha_1^\varepsilon) = \varepsilon$ into account, we obtain $\alpha_1^\varepsilon = \alpha_0^\varepsilon + 2\varepsilon$, and so $(u_\varepsilon^1)'(\alpha_1^\varepsilon) = \frac{1}{2\varepsilon}(\alpha_1^\varepsilon - \alpha_0^\varepsilon) = 1$. Therefore, $(u_\varepsilon^1)' \equiv 1$ in $(\alpha_1^\varepsilon, 1)$. From the Dirichlet condition $u_\varepsilon^1(1) = H_0$, $u_\varepsilon^1(t) = t - (1-H_0)$ $(t \in (\alpha_1^\varepsilon, 1))$, and from $u_\varepsilon^1(\alpha_1^\varepsilon) = \varepsilon$ we have $\alpha_1^\varepsilon = (1-H_0)+\varepsilon$. Hence, we obtain $\alpha_0^\varepsilon = (1-H_0)-\varepsilon$. Finally, we have

$$u_\varepsilon^1(t) = \begin{cases} 0 & \text{for } t \in [0, (1-H_0)-\varepsilon], \\ \dfrac{1}{4\varepsilon}\{t-(1-H_0)+\varepsilon\}^2 & \text{for } t \in ((1-H_0)-\varepsilon, (1-H_0)+\varepsilon), \\ t-1+H_0 & \text{for } t \in ((1-H_0)+\varepsilon, 1). \end{cases} \tag{78}$$

Remarking that $\chi_\varepsilon(u_\varepsilon^1) = \frac{1}{\varepsilon}u_\varepsilon^1$ in $((1-H_0)-\varepsilon, (1-H_0)+\varepsilon)$, we can compute $\mathbb{F}_\varepsilon(u_\varepsilon^1)$ as follows:

$$\begin{aligned} \mathbb{F}_\varepsilon(u_\varepsilon^1) &= \int_{(1-H_0)-\varepsilon}^{1} \big(((u_\varepsilon^1)')^2 + \chi_\varepsilon(u_\varepsilon^1)\big)dt \\ &= \int_{(1-H_0)-\varepsilon}^{(1-H_0)+\varepsilon} \Big\{\frac{1}{4\varepsilon^2}(t-(1-H_0)+\varepsilon)^2 + \frac{1}{4\varepsilon^2}(t-(1-H_0)+\varepsilon)^2\Big\}dt \end{aligned}$$

$$+\int_{(1-H_0)+\varepsilon}^{1}(1+1)dt = 2H_0 + \frac{2}{3}\varepsilon.$$

We denote by w^+ the $\mathbb{F}^1_\varepsilon$-minimizer under the hypothesis $\alpha_0^\varepsilon > 0$. Then

$$\lim_{\varepsilon\downarrow 0}\mathbb{F}^1_\varepsilon(u_\varepsilon^+) = 2H_0. \tag{79}$$

Now, from (76) and (79), we obtain

$$\varliminf_{\varepsilon\to 0}\mathbb{F}^1_\varepsilon(w^0) \geqq 1 + H_0^2 \gneqq 2H_0 = \lim_{\varepsilon\to 0}\mathbb{F}^1_\varepsilon(w^+).$$

This tells us that there exists a positive number ε_0 such that $\varepsilon < \varepsilon_0$ implies the inequality $\mathbb{F}^1_\varepsilon(w^0) > \mathbb{F}^1_\varepsilon(w^+)$. Namely, we accomplish the proof of the Fact (67). In particular, if $\varepsilon < \varepsilon_0$ then $\mathbb{F}^1_\varepsilon$-minimizer is w^+, which is given by (78). So we establish the conclusion of theorem. □

Example 33. (An Example of Corollary 23) Let us show that the assertion of Corollary 23 holds for the whole of the sequence (u_ε) consisting of $\mathbb{F}_\varepsilon$-minimizer.

Example of (i). We first notice that open subset $U \subset (0,1)$ satisfying the assumption of (i) must satisfy the inclusion $U \subset (0, 1-H_0]$. Otherwise, it holds that $U \cap (1-H_0, 1) \neq \emptyset$. Then, for any $\xi \in U \cap (1-H_0, 1)$ and for sufficiently small $\varepsilon > 0$, it holds that $u^1_\varepsilon(\xi) = u^1(\xi) = \xi - 1 + H_0 > 0$, where u^1 is as in Proposition 28. So when ε is small enough, it does not hold that $u_\varepsilon < \varepsilon$ in U. Now for any open subset $U \subset (0, 1-H_0]$ and for sufficiently small $\varepsilon > 0$, it holds that $u^1_\varepsilon = u^1 \equiv 0$ in U, and so the conclusion $\lim_{\varepsilon\to 0}\mathcal{L}^n\big(U(u^1_\varepsilon \neq 0)\big) = 0$ surely holds.

Example of (ii). Let C_0 be a positive constant. Then by the same reason as in (i), $\Omega_\varrho(x_0)$ in the assumption must satisfy the inclusion $\Omega_\varrho(x_0) \subset (0, 1-H_0]$. For any $\Omega_\varrho(x_0) \subset (0, 1-H_0]$, the conclusion holds in the same reason as for (i).

5.3. On the $\widetilde{\mathbb{F}}^1_\varepsilon$-Minimizer

We start from stating the result.

Theorem 34. ($\widetilde{\mathbb{F}}_\varepsilon$-Minimizer) *Let $\tilde{u}^1_\varepsilon$ be a minimizer of the variational problem $(\tilde{P}^1_\varepsilon)$. Then it holds that*

$$F_{\tilde{u}^1_\varepsilon} = \left(0,\ 1 + \frac{\varepsilon - H_0}{\sqrt{1+B_\varepsilon}}\right) \quad \Big(\supset (0, 1-H_0)\Big),$$

where B_ε is a positive constant with $B_\varepsilon \to 0$ as $\varepsilon \to 0$.

Remark. The inclusion of the statement of Theorem 34 tells us that the approximated free boundary $F_{\tilde{u}^1_\varepsilon}$ does not converge to $\partial(0,1)(u^1 > 0) = \{1 - H_0\}$ in any sense.

Proof. Since $\widetilde{\chi}_\varepsilon(\cdot)$ is non-decreasing, the proof of Lemma 27 remains valid if we replace χ with $\widetilde{\chi}_\varepsilon$. Therefore, $\tilde{u}^1_\varepsilon$ turns out to be non-decreasing in $(0,1)$, and so there exist $\tilde{\alpha}^\varepsilon_0$ and $\tilde{\alpha}^\varepsilon_0$ with $0 \leqq \tilde{\alpha}^\varepsilon_0 < \tilde{\alpha}^\varepsilon_1 < 1$ such that

$$\begin{cases} \tilde{u}^1_\varepsilon \equiv 0 & \text{in } [0, \tilde{\alpha}^\varepsilon_0], \\ 0 < \tilde{u}^1_\varepsilon < \varepsilon & \text{in } [\tilde{\alpha}^\varepsilon_1, \tilde{\alpha}^\varepsilon_1), \\ \tilde{u}^1_\varepsilon \geqq \varepsilon & \text{in } [\tilde{\alpha}^\varepsilon_1, 1]. \end{cases} \tag{80}$$

$\tilde{u}^1_\varepsilon$ satisfies the Euler-Lagrange equation

$$(\tilde{u}^1_\varepsilon)'' = \frac{1}{2}\widetilde{\chi}_\varepsilon(\tilde{u}^1_\varepsilon) \quad \text{in } (0,1). \tag{81}$$

Since $\widetilde{\chi}_\varepsilon(\cdot)$ is continuous in $\mathbb{R}$ and $\tilde{u}^1_\varepsilon \in C^0[0,1]$ by Remark 26, we have $\tilde{u}^1_\varepsilon \in C^2[0,1]$ from (81). Now, multiplying $(\tilde{u}^1_\varepsilon)'$ on the both sides of (81), we have $2(\tilde{u}^1_\varepsilon)'(\tilde{u}^1_\varepsilon)'' = (\tilde{u}^1_\varepsilon)'\widetilde{\chi}'_\varepsilon(\tilde{u}^1_\varepsilon)$. Therefore, $\big\{((\tilde{u}^1_\varepsilon)')^2\big\}' = \big\{\widetilde{\chi}_\varepsilon(\tilde{u}^1_\varepsilon)\big\}'$. Let us integrate from 0 to x, where $x \in [0,1]$, set $B_\varepsilon := \{(\tilde{u}^1_\varepsilon)'(0)\}^2$. Then we have

$$\Big\{(\tilde{u}^1_\varepsilon)'(x)\Big\}^2 = \widetilde{\chi}_\varepsilon(\tilde{u}^1_\varepsilon(x)) + B_\varepsilon \quad \text{for } x \in (0,1). \tag{82}$$

Therefore

$$(\tilde{u}^1_\varepsilon)' = \sqrt{\widetilde{\chi}_\varepsilon(\tilde{u}^1_\varepsilon) + B_\varepsilon} \;=: S_\varepsilon(\tilde{u}^1_\varepsilon) \quad \text{in } (0,1). \tag{83}$$

Here we remark that the function $S_\varepsilon : \tau \mapsto \sqrt{\widetilde{\chi}_\varepsilon(\tau) + B_\varepsilon}$ is globally Lipschitz continuous in $\mathbb{R}$. Using this, we shall show that $\tilde{\alpha}^\varepsilon_0 = 0$, where $\tilde{\alpha}^\varepsilon_0$ is as in (80). Otherwise, define $v(t) := \tilde{u}^1_\varepsilon(t + \tilde{\alpha}^\varepsilon_0)$ for $t \geqq 0$, and set $\sigma = 1 - \tilde{\alpha}^\varepsilon_0$. Then both $\tilde{u}^1_\varepsilon$ and v are solution of the initial value problem of autonomous ODE:

$$\begin{cases} u(0) = 0, \\ u' = S_\varepsilon(u) \quad \text{in } (0,\sigma). \end{cases}$$

However, since S_ε is Lipschitz in $\mathbb{R}$ as stated above, the solution is known to be unique. In the interval $(0, \min(\tilde{\alpha}^\varepsilon_0, \sigma))$, $\tilde{u}^1_\varepsilon = 0$ and $v > 0$. This contradicts the uniqueness. Thus, there exists $\tilde{\alpha}^\varepsilon_1$ with $0 < \exists\tilde{\alpha}^\varepsilon_1 < 1$ such that

$$\begin{cases} \tilde{u}^1_\varepsilon(0) = 0, \\ 0 < \tilde{u}^1_\varepsilon < \varepsilon \quad \text{in } (0, \tilde{\alpha}^\varepsilon_1), \\ \tilde{u}^1_\varepsilon > \varepsilon \quad \text{in } [\tilde{\alpha}^\varepsilon_1, 1), \end{cases}$$

and hence we obtain

$$F_{\tilde{u}^1_\varepsilon} = (0, \tilde{\alpha}^\varepsilon_1). \tag{84}$$

In the interval $(\tilde{\alpha}_1^\varepsilon, 1)$, the Euler-Lagrange equation is $(\tilde{u}_\varepsilon^1)'' = 0$. From (82) we have $(\tilde{u}_\varepsilon^1)' = \sqrt{1+B_\varepsilon}$. Moreover, considering the Dirichlet condition $\tilde{u}_\varepsilon^1(1) = H_0$, we can determine the for of $\tilde{u}_\varepsilon^1$ in the interval $(\tilde{\alpha}_1^\varepsilon, 1)$ as follows:

$$\tilde{u}_\varepsilon^1(t) = \sqrt{1+B_\varepsilon}(t-1) + H_0 \quad \text{for } t \in (\tilde{\alpha}_1^\varepsilon, 1)\,. \tag{85}$$

Since $\tilde{u}_\varepsilon^1(\tilde{\alpha}_1^\varepsilon) = \varepsilon$, we infer $\varepsilon = \tilde{u}_\varepsilon^1(\tilde{\alpha}_1^\varepsilon) = \sqrt{1+B_\varepsilon}(\tilde{\alpha}_1^\varepsilon - 1) + H_0$. Therefore

$$\tilde{\alpha}_1^\varepsilon = 1 + \frac{\varepsilon - H_0}{\sqrt{1+B_\varepsilon}}. \tag{86}$$

Having proved (84) and (86), in order to complete the proof we have only show that $B_\varepsilon > 0$ and $B_\varepsilon \to 0$ as $\varepsilon \to 0$.

— Let us show $B_\varepsilon > 0$. Assume that $B_\varepsilon = 0$. Then from (83) we have

$$\int \frac{1}{\sqrt{\widetilde{\chi}_\varepsilon(\tilde{u}_\varepsilon^1)}} d\tilde{u}_\varepsilon^1 = \int dt. \tag{87}$$

Let γ_ε be a real number in the interval $(0,1)$ such that $\tilde{u}_\varepsilon^1(\gamma_\varepsilon) = \frac{\varepsilon}{2}$, and let $\delta \in (0, \gamma_\varepsilon)$. We integrate the right side of (87) in the interval $(\delta, \gamma_\varepsilon)$ with respect to the variable t, and the left side $\left(\tilde{u}_\varepsilon^1(\delta), \frac{\varepsilon}{2}\right)$ with respect to the variable $\tilde{u}_\varepsilon^1$. In the interval $(0, \gamma_\varepsilon)$, it holds that $\tilde{u}_\varepsilon^1 \in (0, \frac{\varepsilon}{2})$, $\widetilde{\chi}_\varepsilon(\tilde{u}_\varepsilon^1) = \frac{2}{\varepsilon^2}(\tilde{u}_\varepsilon^1)^2$, and so $\sqrt{\widetilde{\chi}_\varepsilon(\tilde{u}_\varepsilon^1)} = \sqrt{\frac{2}{\varepsilon^2}(\tilde{u}_\varepsilon^1)^2} = \frac{\sqrt{2}}{\varepsilon}\tilde{u}_\varepsilon^1$. Therefore,

$$\frac{\varepsilon}{\sqrt{2}} \int_{u(\delta)}^{\frac{\varepsilon}{2}} \frac{1}{u} du = \int_\delta^{\gamma_\varepsilon} dt.$$

As a result, we obtain

$$\tilde{u}_\varepsilon^1(\delta) = \frac{\varepsilon}{2} e^{\frac{\sqrt{2}}{\varepsilon}(\delta - \gamma_\varepsilon)} \quad \text{for } \delta \in (0, \gamma_\varepsilon). \tag{88}$$

If we let $\delta \downarrow 0$ in (88), then the left side converges to 0 although the right side converges to a positive number. This is a contradiction.

— Let us show $B_\varepsilon \to 0$ as $\varepsilon \to 0$. Let $v_0(x) := H_0 x$ for $x \in [0,1]$. Then by the minimality of $\tilde{u}_\varepsilon$, we have $\widetilde{\mathbb{F}}_\varepsilon^1(\tilde{u}_\varepsilon) \leqq \widetilde{\mathbb{F}}_\varepsilon^1(v_0)$. The right side is estimated as follows:

$$\widetilde{\mathbb{F}}_\varepsilon^1(v_0) \leqq \int_0^1 \Big(|v_0'|^2 + 1\Big) d\mathcal{L}^1 = 1 + H_0^2.$$

Moreover, the left side is estimated as follows:

$$\widetilde{\mathbb{F}}_\varepsilon^1(\tilde{u}_\varepsilon^1) \geqq \int_{\tilde{\alpha}_1^\varepsilon}^1 |(\tilde{u}_\varepsilon^1)'|^2 d\mathcal{L}^1 = (1+B_\varepsilon) \times (1 - \tilde{\alpha}_1^\varepsilon) = (H_0 - \varepsilon)\sqrt{1+B_\varepsilon},$$

where the first equality follows from (85), and the second one follows from (86). Therefore, we have $(1+B_\varepsilon) \times (1 - \tilde{\alpha}_1^\varepsilon) = (H_0 - \varepsilon)\sqrt{1+B_\varepsilon} \leqq 1 + H_0^2$. Recalling the restriction $\varepsilon < \frac{H_0}{2}$, we obtain

$$B_\varepsilon \leqq 1 + B_\varepsilon \leqq \frac{2(1+H_0^2)}{H_0}. \tag{89}$$

Hence,

$$\sup_{0<\varepsilon<\frac{H_0}{2}} B_\varepsilon < +\infty. \tag{90}$$

Now since $(\tilde{u}_\varepsilon^1)'' = \frac{1}{2}\widetilde{\chi}_\varepsilon'(\tilde{u}_\varepsilon^1) \geqq 0$, the function $(\tilde{u}_\varepsilon^1)'$ is non-decreasing in $[0,1]$. From this with $(\tilde{u}_\varepsilon^1)' \equiv \sqrt{1+B_\varepsilon}$, we derive $\sup_{(0,1)} |(\tilde{u}_\varepsilon^1)'| \leqq \sqrt{1+B_\varepsilon}$. Combining (89) with (90), we have

$$\sup_{0<\varepsilon<\frac{H_0}{2}} ||(\tilde{u}_\varepsilon^1)'||_{\infty(0,1)} \leqq \sup_{0<\varepsilon<\frac{H_0}{2}} \sqrt{1+B_\varepsilon} < +\infty.$$

Therefore, by Ascoli-Arzelà Theorem, we can select if needed a subsequence $\tilde{u}_{\varepsilon_j}^1$ and $u \in C^0[0,1]$ such that

$$\tilde{u}_{\varepsilon_j}^1 \rightrightarrows u \quad \text{uniformly in } [0,1] \quad \text{as } j \to \infty. \tag{91}$$

Here the convergence $\tilde{\mathbb{F}}_\varepsilon^1 \to \mathbb{F}^1$ as $\varepsilon \downarrow 0$ in the sense of $\Gamma(L^2(0,1))$ can be shown in the same way as Proposition 10. Hence, from the $L^2(0,1)$-convergence $\tilde{u}_{\varepsilon_j}^1 \to u$ as $j \to \infty$, which follows especially from (91), the limit function u must be an $\mathbb{F}^1$-minimizer which is unique and equals to u^1 as in Proposition 28. By the uniqueness of the limit function, the uniform convergence (91) holds for the whole of the sequence (u_ε). We thus arrive at our goal that $B_\varepsilon \to 0$ as $\varepsilon \to 0$, because, in the neighbourhood of $x = 1$, the gradient of $\tilde{u}_\varepsilon^1$ is identically equal to $\sqrt{1+B_\varepsilon}$, whereas the gradient of u^1 is identically equal to 1. □

Through the proof of Theorem 34, we have obtained proved also the following fact.

Corollary 35. *Let $\tilde{u}_\varepsilon^1$ be as in Theorem 34. Then $\sup_{\varepsilon<\frac{H_0}{2}} ||(\tilde{u}_\varepsilon^1)'||_{L^\infty(0,1)} < \infty$ and the following convergence holds:*

$$\tilde{u}_\varepsilon^1 \rightrightarrows u^1 \quad \textit{uniformly in } [0,1],$$

where u^1 is the $\mathbb{F}^1$-minimizer.

Example 36. (A Counter-Example of Corollary 23) The assertion of Corollary 23 does not generally hold if we replace χ_ε by a C^1-approximations. We construct in this remark a counter-example to the assertion of Corollary 23 with a C^1-approximation.

Counter-Example of (i). Let U be a non-empty open subset of $(0, 1-H_0)$. Then since $(0, 1-H_0) \subset (0, \tilde{\alpha}_1^\varepsilon) = (0,1)(0 < \tilde{u}_\varepsilon^1 < \varepsilon)$, $\tilde{u}_\varepsilon^1 < \varepsilon$ in U. Furthermore,

$U(\tilde{u}_\varepsilon^1 \neq 0) = U$, and so

$$\mathcal{L}^1\Big(U(\tilde{u}_\varepsilon^1 \neq 0)\Big) = \mathcal{L}^1(U) \not\to 0 \quad \text{as } \varepsilon \to 0.$$

Therefore, it turns out that the sequence $(\tilde{u}_\varepsilon^1)$ is a counter-example.

Counter-Example of (ii). Let $x_0 = 0$ and $0 < \varrho_0 \leqq 1 - H_0$. Then since $\Omega_{\varrho_0}(x_0) = (0, \varrho_0)$, $\sup_{(0,\varrho_0)} \tilde{u}_\varepsilon^1 < \varepsilon$. Therefore, by setting $C_0 = \frac{1}{\varrho_0}$, we have $\varrho_0 \leqq \frac{1}{C_0}$ and $\frac{1}{\varrho_0} \sup_{(0,\varrho_0)} \tilde{u}_\varepsilon^1 < \frac{\varepsilon}{\varrho_0} = C_0\varepsilon$ for $\varepsilon > 0$. On the other hand, since $(0, \varrho_0) \subset (0, 1 - H_0) \subset (0,1)(0 < \tilde{u}_\varepsilon^1 < \varepsilon)$, $(0, \varrho_0)(\tilde{u}_\varepsilon^1 \neq 0) = (0, \varrho_0)(\tilde{u}_\varepsilon^1 > 0) = (0, \varrho_0)$. We thus have

$$\mathcal{L}^1\Big((0, \varrho_0)(\tilde{u}_\varepsilon^1 \neq 0)\Big) = \mathcal{L}^1((0, \varrho_0)) = \varrho_0 \not\to 0 \quad \text{as } \varepsilon \to 0.$$

Therefore, it turns out that the sequence $(\tilde{u}_\varepsilon^1)$ is a counter-example.

References

[1] H.W. Alt, L.A. Caffarelli, Existence and regularity for a minimum problem with free boundary, *J. Reine Angew. Math.*, **325** (1981), 105-144.

[2] H.W. Alt, L.A. Caffarelli, A.Friedman, A free boundary problem for quasi linear elliptic equation, *Ann. Scuola Norm. Sup. Pisa, Cl. Sci.*, **11**, No. 4 (1984), 1-44.

[3] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Science Publication (2000).

[4] L. Ambrosio, V.M. Tortorelli, Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence, *Comm. Pure Appl. Math.*, **XLIII** (1990), 999-1036.

[5] L. Ambrosio, V.M. Tortorelli, On the approximation of free discontinuity problems, *Bollettino U.M.I.*, **6-B**, No. 7 (1992), 105-123.

[6] S. Baldo, G. Bellettini, Γ-convergence and numerical analysis: an application to the minimal partition problem, *Richerche Math.*, **XL**, No. 1 (1991), 33-64.

[7] H. Berestycki, L.A. Caffarelli, L. Nirenberg, Uniform estimate for regularisations of free boundary problems, *Publ. du Laboratorie D'Analyse Numerique*, **89022** (1989), 1-53.

[8] H. Brezis, *Analyse Fonctionnelle*, Masson, Editeur, Paris (1983), Japanese Translated.

[9] L.A. Caffarelli, C. Lederman, N. Wolanski, Uniform estimates and limits for a two phase parabolic singular perturbation problem, *Indiana Univ. Math. J.*, **46** (1997), 453-489.

[10] L.A. Caffarelli, J.L. Vázquez, A free-boundary problem for the heat equation arising in flame propagation, *Trans. AMS,* **347** (1995), 411-441.

[11] S. Conti, I. Fonseca, G. Leoni, A Γ-Convergence resut for the two-gradient theory of phase transitions, *Communications on Pure and Applied Mathematics*, **LV** (2002), 0857-0963.

[12] E. De Giorgi, T. Franzoni, Su un tipo di convergenza variazionale, *Atti Accad. Naz, Lincei Rend., Cl. Sci. Fis. Mat. Natur.*, **58** (1975), 842-850.

[13] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monographs Math., **80**, Birkhäuser, Boston-Basel-Stuttgart (1984).

[14] L. Evans, R. Gariepy, *Measure Theory and Fine Propties of Functions*, Monographs Math., **80**, Birkhäuser, Boston-Basel-Stuttgart (1984).

[15] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Ann. of Math. Stud., **105**, Princeton Univ. Press (1983).

[16] M. Gobbino, Gradient flow for the one-dimensional Mumford-Shah functional, *Ann. Scuola Norm. Sup. Pisa, Cl. Sci.*, **27**, No. 4 (1998), 145-193.

[17] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Second Edition, Grund. Math. Wiss., **224**, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo (1983).

[18] L. Simon, *Lecture on Geometric Measure Theory*, Proc. Centre for Math. Anal. Austral. Nat. Univ., **3** (1983).

[19] O.A. Ladyzhenskaya, N.N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York-London-Toronto-Sydney-San Francisco (1968).

[20] R. March, Visual reconstruction with discontinuities using variational methods, *Image and Vision Computing*, **10**, No. 1 (1992), 30-38.

[21] L. Modica, The gradient theory of phase transitions and the minimal interface criterion, *Arch. Rat. Mech. Anal.*, **98** (1987), 123-142.

[22] S. Omata, Y. Yamaura, A free boundary problem for quasilinear elliptic equations, Part II: $C^{1,\alpha}$-regularity of free boundary, *Funkcialaj Ekvacioj*, **42**, No. 1 (Aprilo, 1999), 9-70.

[23] G.S. Weiss, A singular limit arising in combustion theory: Fine properties of the free boundary, *Calc. Var.*, **17** (2003), 311-340.

[24] Y. Yamaura, A construction of a Lipschitz continuous minimizer of a free boundary problem, *Nonlinear Analysis*, **54** (2003), 1175-1191.

[25] Y. Yamaura, Gradient flow for the one-dimensional Alt-Caffarelli functional, *Mediterranean Journal of Mathematics*, **4** (2007), To Appear.