**International Journal of Pure and Applied Mathematics**

**Volume 40 No. 4 2007, 519-541**

# ALGEBRAIC PROPERTIES OF GRIDS OF FAT LINES

M. Guida[1], F. Orecchia[2 §]

[12]Department of Mathematics and Applications "R. Caccioppoli"
University of Naples "Federico II"
Via Cintia, Naples, 80126, ITALY
[1]e-mail: maguida@unina.it
[2]e-mail: orecchia@unina.it

**Abstract:** In this paper we consider the generators of the ideal $I$ defining a grid $X$ of fat lines in the projective space $\mathbb{P}^n$. We compute a minimal set of generators in the homogeneous and quasi-homogeneous cases in $\mathbb{P}^3$. We show also that in these cases the homogeneous coordinate ring of $X$ is always Cohen-Macaulay.

**AMS Subject Classification:** 14M05, 14H50, 14N05
**Key Words:** grids, fat lines, generators, Cohen-Macaulay rings

## 1. Introduction

In [3] a (complete) *grid* of projective lines was defined as the projective closure in $\mathbb{P}^n_{\mathbf{K}}$ of lines in the affine space $\mathbb{A}^n_{\mathbf{K}}$, which are parallel to the coordinate axes, and pass through a *lattice* of points $\{(a_{i_1 1}, \ldots, a_{i_n n}) : i_1 = 1, \ldots, l_1, \ldots, i_n = 1, \ldots, l_n\} \subset \mathbb{A}^n_{\mathbf{K}}$. In that paper the generators, the Hilbert function and the Hilbert polynomial of grids were computed. Also the Cohen-Macaulay property of their coordinate ring was discussed. In many papers questions on finite sets of projective points were extended to fat points that is powers of the prime ideals defining the points. In particular many author focalized their attention to the case of points in $\mathbb{P}^2_{\mathbf{K}}$, see [4]. So it is natural to ask if the results of [3] extend to a finite union of fat lines (a fat line is the scheme corresponding to a power of the prime ideals that defines the line) which form a grid in $\mathbb{P}^3_{\mathbf{K}}$. These questions were considered in [5] for a thetraedal set of fat lines and the results of that paper show that the situation in that case is very complicated and it

Received: September 14, 2007


§Correspondence author

is almost impossible to write a formula that covers all the cases. So since the case of grids involves more lines than the case of thetraedal curves it becomes clear that in general it is pratically impossible to have formulas for the number of generators, the Hilbert function and the Hilbert polynomial of fat grids. It is also very difficult to state when the coordinate ring is Cohen-Macaulay. In this paper we give an answer to these questions in the homogeneous and quasi-homogeneous case in $\mathbb{P}^3$. We prove that in these cases the coordinate ring is Cohen-Macaulay.

## 2. Grids of Fat Lines in $\mathbb{P}^n_{\mathbf{K}}$

Let $\mathbf{K}$ be an infinite field.

**Definition 1.** Let $h, m_1, \ldots, m_h$ be positive integers. A union of fat lines in $\mathbb{P}^n_{\mathbf{K}}$ is the projective scheme $\mathrm{Proj}(\frac{K[x_0,\ldots,x_n]}{\mathcal{P}_1^{m_1}\cap\ldots\cap\mathcal{P}_h^{m_h}})$ whose support is a union of lines $V(\mathcal{P}_1)\cup\ldots\cup V(\mathcal{P}_h)$.

**Remark 2.** We note that since $\mathcal{P}_1, \ldots, \mathcal{P}_h$ are generated by linear forms, the powers $\mathcal{P}_i^{m_i}$ are primary ideals and then $\mathcal{P}_i^{m_i}$ is equal to the symbolic power $\mathcal{P}_i^{(m_i)}$ (see [1], Example 8 and Example 13, Chapter 4).

Let $Y$ be a finite union of $\sigma$ fat lines of $\mathbb{P}^n_{\mathbf{K}}$. We denote with $I_Y$ the ideal of $Y$,that is, $I_Y = \cap_{i=1}^{\sigma}\mathcal{P}_i^{m_i}$. We call $A = K[x_0, \ldots, x_n]/I_Y$ the homogeneous coordinate ring of $Y$. We consider a linear form $H$ of $K[x_0, \ldots, x_n]$ such that $H \notin \cup_{i=1}^{\sigma}\mathcal{P}_i$, $H \notin \mathcal{P}_i + \mathcal{P}_j$ for any $i, j \in \{1, \ldots, \sigma\}$ such that $\mathcal{P}_i + \mathcal{P}_j \subsetneqq \mathcal{M}$. Denoting with $h = \overline{H}$ the class of $H$ in $A$, we have that $h$ is a non-zero divisor of $A$. In the following Lemma, we set $B = A/(h)$ and we denote the Hilbert function of $B$, $\dim_{\mathbf{K}}(\mathcal{M}^t/\mathcal{M}^{t+1})$ ($\mathcal{M}$ maximal homogeneous ideal of $B$), with $H_B(t)$.

**Lemma 3.** *The following relations hold:*

*a) If* $(\cap_{i=1}^{\sigma}\mathcal{P}_i^{m_i}, H) = \cap_{i=1}^{\sigma}(\mathcal{P}_i^{m_i}, H)$*, then* $A$ *is Cohen-Macaulay;*

*b) If* $H_B(t) > H_B(t+1)$ *for some integer* $t$*, then* $A$ *is not Cohen-Macaulay.*

*Proof.* a) By the assumption we have that $B = K[x_0, \ldots, x_n]/(\cap_{i=1}^{\sigma}\mathcal{P}_i^{m_i}, H) = K[x_0, \ldots, x_n]/\cap_{i=1}^{\sigma}(\mathcal{P}_i^{m_i}, H)$. But $(\mathcal{P}_i^{m_i}, H)$ is a primary ideal for any $i$ (see [1]). Then $(0) = \cap_{i=1}^{\sigma}\overline{(\mathcal{P}_i^{m_i}, H)}$ is a primary decomposition of the null ideal of $B$.

Moreover, $\sqrt{\mathcal{P}_i^{m_i} + H} \neq \overline{\mathfrak{M}}$ (maximal homogeneous ideal of $B$) for any $i$.

Then, $\overline{\mathfrak{M}} - \cup_{i=1}^{\sigma}\sqrt{(\mathcal{P}_i^{m_i}, H)} \neq \emptyset$, thus $B$ contains a non-zero divisor. Since $B$ is one-dimensional, $B$ is Cohen-Macaulay and recalling that $h$ is a non-zero divisor $A$ is Cohen-Macaulay.

b) If we suppose that $A$ is Cohen-Macaulay, then $B$ is Cohen-Macaulay and one dimensional, hence there exists a non-zero divisor $x$ in $B$. But $H^0_{B/(x)}(t) = H^1_{B/(x)}(t)$ and $\Delta H^0_B(t) = H^0_{B/(x)}(t) \geq 0$ ([6]) hence $H_B$ is not decreasing. □

**Definition 4.** Let $\{a_{11}, \ldots, a_{l_1 1}\}, \ldots, \{a_{1n}, \ldots, a_{l_n n}\}$ be $n$ finite subsets of elements of $K$. The finite set of points $X = \{(1, a_{i_1 1}, \ldots, a_{i_n n}), i_1 = 1, \ldots, l_1, i_2 = 1, \ldots, l_2, \ldots, i_n = 1, \ldots, l_n\}$ of $\mathbb{P}^n_{\mathbf{K}}$ is called a lattice of type $(l_1, \ldots, l_n)$, where $l_i \geq 2$ for any $i \in \{1, \ldots, n\}$.

By Lemma 3 of [3] the defining ideal $I_X$ of a lattice of type $(l_1, \ldots, l_n)$ is:

$$I_X = (\textstyle\prod_{i=1}^{l_1}(x_1 - a_{i1}x_0), \ldots, \prod_{i=1}^{l_n}(x_n - a_{in}x_0)).$$

**Definition 5.** Let $X$ be a lattice of type $(l_1, \ldots, l_n)$ of points of $\mathbb{P}^n_{\mathbf{K}}$. A finite set $Y$ of $(l_2 \ldots l_n + l_1 l_3 \ldots l_n + l_1 l_2 l_4 \ldots l_n + \ldots + l_1 \ldots l_{n-1})$ lines of $\mathbb{P}^n_{\mathbf{K}}$ is called a complete grid of type $(l_1, \ldots, l_n)$, with basis $X$, if it satisfies the two following conditions:

a) each line of $Y$ contains one of the points $(0{:}\ldots 1 \ldots{:}0)$, where 1 is the $i$-element, where $i = \{2, \ldots, n+1\}$;

b) any line of $Y$ contains at least one point of $X$.

A grid line is a line of a complete grid.

**Definition 6.** A complete grid $Y$ of type (2,2,2) is called a simple grid.

**Definition 7.** Let $m_1, \ldots, m_h$ be positive integers, a fat complete grid of type $(l_1, \ldots, l_n)$ is the projective scheme $\mathrm{Proj}(\frac{K[x_0,\ldots,x_n]}{\mathcal{P}_1^{m_1} \cap \ldots \cap \mathcal{P}_h^{m_h}})$ whose support is a complete grid of lines $V(\mathcal{P}_1) \cup \ldots \cup V(\mathcal{P}_h)$ (where $h = l_2 \ldots l_n + \ldots + l_1 \ldots l_{n-1}$).

**Definition 8.** A fat complete grid of type $(l_1, \ldots, l_n)$ is a $m$-homogeneous fat complete grid of type $(l_1, \ldots, l_n)$, if $m_1 = \ldots = m_h = m$.

**Definition 9.** A fat complete grid of type $(l_1, \ldots, l_n)$ is a $m$ quasi-homogeneous fat complete grid of type $(l_1, \ldots, l_n)$, if there exists an $i \in \{1, \ldots, h\}$ such that $m_1 = m_{i-1} = m_{i+1} = m_h = m$ and $m_i \in \{m-1, m+1\}$.

We were not able to compute in general the generators of fat grids of projective lines in $\mathbb{P}^n_{\mathbf{K}}$, and also the Cohen-Macaulay property. In the following sections we will discuss the situation in $\mathbb{P}^3_{\mathbf{K}}$. By using computer algebra systems like *CoCoA* it is possible to work examples whose points of the lattice are in $\mathbb{A}^n_{\mathbb{Q}}$ ($\mathbb{Q}$ field of rational numbers) thus providing examples over any algebraically closed field $K$ of characteristic zero.

**Example 10.** If in Definition 4 we set $\{a_{11}, a_{21}\} = \ldots = \{a_{14}, a_{24}\} = \{0, 1\}$, we have a lattice in $\mathbb{P}^4$ of type $(2,2,2,2)$ of 16 points $\{P_1, \ldots, P_{16}\}$, and we can consider the following fat complete grid $Y = \mathrm{Proj}(\frac{\mathbb{Q}[x_0,\ldots,x_4]}{\mathcal{P}_1^3 \cap \mathcal{P}_2 \cap \ldots \cap \mathcal{P}_{32}})$ of type $(2,2,2,2)$. By using the computer algebra system *CoCoA*, see [2], we have calculated the generators of $Y$, we have found that is generated by 10 polynomials of degree 5 and 7 polynomials of degree 6. Moreover, we have computed the Hilbert function of $B$ which is: $H_B(0) = 0, H_B(1) = 4, H_B(2) = 10, H_B(3) = 20, H_B(4) = 35, H_B(5) = 44, H_B(t) = 41$ for $t \geq 6$, so by b) of Lemma 3 we have that $Y$ is not Cohen-Macaulay.

**Example 11.** We consider a lattice in $\mathbb{P}^4$ of type $(2,2,2,2)$ as in Example 10 and a 2-homogeneous fat complete grid $Y_1 = \mathrm{Proj}(\frac{\mathbb{Q}[x_0,\ldots,x_4]}{\mathcal{P}_1^2 \cap \ldots \cap \mathcal{P}_{32}^2})$ of type $(2,2,2,2)$. By using *CoCoA*, see [2], we have calculated that $Y_1$ is generated by 4 polynomials of degree 6 and 6 polynomials of degree 8. Moreover, by using a) of Lemma 3 we have checked that $Y_1$ is Cohen-Macaulay.

## 3. The Homogeneous Case in $\mathbb{P}^3_{\mathbf{K}}$

Let $\mathbf{K}$ be an algebraically closed field. In this section we compute the generators and we study the Cohen-Macaulay property of a m-homogeneous fat complete grid of lines of $\mathbb{P}^3_{\mathbf{K}}$. Let $Y \subseteq \mathbb{P}^3_{\mathbf{K}}$ be a $m$-homogeneous fat complete grid of type $(l_1, l_2, l_3)$, $I_Y$ be its defining ideal and $A$ its homogeneous coordinate ring. We set $L_{ij} = (x_j - a_{ij}x_0)$, where $j \in \{1,2,3\}$, $i \in \{l_1, \ldots, l_q\}$ for any $q \in \{1,2,3\}$ and we assume also that $l_1 \geq l_2 \geq l_3$.

**Lemma 12.** *Let $S = \{L_1, \ldots, L_t\}$ be a finite set of distinct linear forms in $K[x_0, \ldots, x_n]$ such that $L_i \notin (L_j, L_k)$ for any $i \neq j$, $i \neq k$, $j \neq k$. Let $S = S_1 \cup S_2 \cup \ldots \cup S_n$ ($n \geq 3$) be a partition of the set $S$. Let $F_i$ a finite product of distinct elements of $S_i$ $i = 1, 2, \ldots, n$. We have the following relations:*

*a)* $(F_1, F_2)^m = (F_1^m, F_2^m, F_1F_2) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m+2}{2}}, F_1^{\frac{m}{2}}F_2^{\frac{m}{2}})$, *when $m$ is even;*

*b)* $(F_1, F_2)^m = (F_1^m, F_2^m, F_1F_2) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_2^{\frac{m+3}{2}}, F_1^{\frac{m-1}{2}}F_2^{\frac{m-1}{2}}) \cap (F_1^{\frac{m+1}{2}},$

$F_2^{\frac{m+1}{2}}$), *when* $m$ *is odd;*

*c)* $(F_1, F_2)^m \cap \ldots \cap (F_1, F_n)^m = (F_1, F_2 \ldots F_n)^m$.

*Proof.* a) We set $I = (F_1^m, F_2^m, F_1F_2) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m+2}{2}}, F_1^{\frac{m}{2}} F_2^{\frac{m}{2}})$ and we suppose that $F_1, F_2$ are linear forms. It is enough to prove that $I \subseteq (F_1, F_2)^m$, because the other inclusion is obvious.

By using the modular law (see [1], p. 6) we have that $I \subseteq (F_1^m, F_2^m, F_1^{m-1}F_2, F_1F_2^{m-1}, F_1^2F_2^2) \cap (F_1^{m-2}, F_2^{m-2}, F_1^3F_2^3) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m+2}{2}}, F_1^{\frac{m}{2}} F_2^{\frac{m}{2}})$.

By applying the modular law, after a finite number of steps we obtain: $I \subseteq (F_1^m, F_2^m, F_1^{m-1}F_2, F_1F_2^{m-1}, \ldots, F_1^{\frac{m}{2}} F_2^{\frac{m}{2}}) \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m+2}{2}}, F_1^{\frac{m}{2}} F_2^{\frac{m}{2}}) \subseteq (F_1^m, F_2^m, F_1^{m-1}F_2, F_1F_2^{m-1}, \ldots, F_1^{\frac{m}{2}} F_2^{\frac{m}{2}}, F_1^{\frac{m-2}{2}} F_2^{\frac{m+2}{2}} F_1^{\frac{m+2}{2}}, F_2^{\frac{m-2}{2}}) \subseteq (F_1, F_2)^m$.

b) We set $J = (F_1^m, F_2^m, F_1F_2) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_2^{\frac{m+3}{2}}, F_1^{\frac{m-1}{2}} F_2^{\frac{m-1}{2}}) \cap (F_1^{\frac{m+1}{2}}, F_2^{\frac{m+1}{2}})$. It is enough to prove that $J \subseteq (F_1, F_2)^m$.

By using the modular law we have that $J \subseteq (F_1^m, F_2^m, F_1^{m-1}F_2, F_1F_2^{m-1}, F_1^2F_2^2) \cap (F_1^{m-2}, F_2^{m-2}, F_1^3F_2^3) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_2^{\frac{m+3}{2}}, F_1^{\frac{m-1}{2}} F_2^{\frac{m-1}{2}}) \cap (F_1^{\frac{m+1}{2}}, F_2^{\frac{m+1}{2}})$.

By applying the modular law, after a finite number of steps we obtain: $I \subseteq (F_1^m, F_1^{m-1}F_2, \ldots, F_1^{\frac{m+3}{2}} F_2^{\frac{m-3}{2}}, F_1^{\frac{m+1}{2}} F_2^{\frac{m-1}{2}}, F_1^{\frac{m-1}{2}} F_2^{\frac{m+1}{2}}, F_1^{\frac{m-3}{2}} F_2^{\frac{m+3}{2}}, F_1 F_2^{m-1}, F_2^m) \subseteq (F_1, F_2)^m$.

Moreover, since we have proved a) and b) for linear forms, it follows easily for $F_1 = \prod_{i=1}^s L_{ij}, F_2 = \prod_{q=1}^t L_{qr}$, where $L_{ij} \in S_1, L_{qr} \in S_2$.

c) By induction, it is enough to prove the case $n = 3$, that is, $(F_1, F_2)^m \cap (F_1, F_3)^m = (F_1, F_2F_3)^m$. We set $I = (F_1, F_2)^m \cap (F_1, F_3)^m$. Suppose that $F_1, F_2, F_3$ are linear forms. We distinguish the case $m$ even and $m$ odd.

If $m$ is even, by using the property a) we have that $I = (F_1^m, F_2^m, F_1F_2) \cap$

$(F_1^{m-1}, F_2^{m-1}, F_1^2 F_2^2) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m+2}{2}}, F_1^{\frac{m}{2}} F_2^{\frac{m}{2}}) \cap (F_1^m, F_3^m, F_1 F_3) \cap (F_1^{m-1}, F_3^{m-1}, F_1^2 F_3^2) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_3^{\frac{m+2}{2}}, F_1^{\frac{m}{2}} F_3^{\frac{m}{2}})$.

By using the modular law we obtain: $I = (F_1^m, F_2) \cap (F_1, F_2^m) \cap (F_1^{m-1}, F_2^2) \cap (F_1^2, F_2^{m-1}) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m}{2}}) \cap (F_1^{\frac{m}{2}}, F_2^{\frac{m+2}{2}}) \cap (F_1^m, F_3) \cap (F_1, F_3^m) \cap (F_1^{m-1}, F_3^2) \cap (F_1^2, F_3^{m-1}) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_3^{\frac{m}{2}}) \cap (F_1^{\frac{m}{2}}, F_3^{\frac{m+2}{2}})$.

By the modular law, we have that: $I = (F_1^m, F_2 F_3) \cap (F_1, F_2^m F_3^m) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m}{2}} F_3^{\frac{m}{2}}) \cap (F_1^{\frac{m}{2}}, F_2^{\frac{m+2}{2}} F_3^{\frac{m+2}{2}})$.

Again by the modular law: $I = (F_1^m, F_1 F_2 F_3, F_2^m F_3^m) \cap (F_1^{m-1}, F_2^{m-1} F_3^{m-1}, F_1^2 F_2^2 F_3^2) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m+2}{2}} F_3^{\frac{m+2}{2}}, F_1^{\frac{m}{2}} F_2^{\frac{m}{2}} F_3^{\frac{m}{2}}) = (F_1^m, F_1^{m-1} F_2 F_3, F_1^{m-2} F_2^2 F_3^2, \ldots, F_1^{\frac{m+2}{2}} F_2^{\frac{m-2}{2}} F_3^{\frac{m-2}{2}}, F_1^{\frac{m}{2}} F_2^{\frac{m}{2}} F_3^{\frac{m}{2}}, F_1^{\frac{m-2}{2}} F_2^{\frac{m+2}{2}} F_3^{\frac{m+2}{2}}, \ldots, F_1 F_2^{m-1} F_3^{m-1}, F_2^m F_3^m) = (F_1, F_2 F_3)^m$.

If $m$ is odd, by using the property b) we have that $I = (F_1^m, F_2^m, F_1 F_2) \cap (F_1^{m-1}, F_2^{m-1}, F_1^2 F_2^2) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_2^{\frac{m+3}{2}}, F_1^{\frac{m-1}{2}} F_2^{\frac{m-1}{2}}) \cap (F_1^{\frac{m+1}{2}}, F_2^{\frac{m+1}{2}}) \cap \cap (F_1^m, F_3^m, F_1 F_3) \cap (F_1^{m-1}, F_3^{m-1}, F_1^2 F_3^2) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_3^{\frac{m+3}{2}}, F_1^{\frac{m-1}{2}} F_3^{\frac{m-1}{2}}) \cap (F_1^{\frac{m+1}{2}}, F_3^{\frac{m+1}{2}})$.

By using the modular law we obtain: $I = (F_1^m, F_2) \cap (F_1, F_2^m) \cap (F_1^{m-1}, F_2^2) \cap (F_1^2, F_2^{m-1}) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_2^{\frac{m-1}{2}}) \cap (F_1^{\frac{m-1}{2}}, F_2^{\frac{m+3}{2}}) \cap (F_1^{\frac{m+1}{2}}, F_2^{\frac{m+1}{2}}) \cap (F_1^m, F_3) \cap (F_1, F_3^m) \cap (F_1^{m-1}, F_3^2) \cap (F_1^2, F_3^{m-1}) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_3^{\frac{m-1}{2}}) \cap (F_1^{\frac{m-1}{2}}, F_3^{\frac{m+3}{2}}) \cap (F_1^{\frac{m+1}{2}}, F_3^{\frac{m+1}{2}})$.

By the modular law, we have that: $I = (F_1^m, F_2 F_3) \cap (F_1, F_2^m F_3^m) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_2^{\frac{m-1}{2}} F_3^{\frac{m-1}{2}}) \cap (F_1^{\frac{m-1}{2}}, F_2^{\frac{m+3}{2}} F_3^{\frac{m+3}{2}}) \cap (F_1^{\frac{m+1}{2}}, F_2^{\frac{m+1}{2}} F_3^{\frac{m+1}{2}})$.

Again by the modular law: $I = (F_1^m, F_1 F_2 F_3, F_2^m F_3^m) \cap (F_1^{m-1}, F_2^{m-1} F_3^{m-1}, F_1^2 F_2^2 F_3^2) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_2^{\frac{m+3}{2}} F_3^{\frac{m+3}{2}}, F_1^{\frac{m-1}{2}} F_2^{\frac{m-1}{2}} F_3^{\frac{m-1}{2}}) \cap (F_1^{\frac{m+1}{2}}, F_2^{\frac{m+1}{2}} F_3^{\frac{m+1}{2}}) = (F_1^m, F_1^{m-1} F_2 F_3, F_1^{m-2} F_2^2 F_3^2, \ldots, F_1^{\frac{m-3}{2}} F_2^{\frac{m+3}{2}} F_3^{\frac{m+3}{2}}, \ldots, F_1 F_2^{m-1} F_3^{m-1}, F_2^m F_3^m) = (F_1, F_2 F_3)^m$.

Moreover, since we have proved the relation for linear forms, it follows easily for $F_1 = \prod_{i=1}^{s} L_{ij}, F_2 = \prod_{q=1}^{t} L_{qr}$, where $L_{ij} \in S_1, L_{qr} \in S_2$. $\square$

**Theorem 13.** *If $m$ is even, $I_Y$ is minimally generated by $\frac{3m}{2}+1$ polynomials, precisely: one polynomial $G$ of degree $(l_1+l_2+l_3)(\frac{m}{2})$ and for any $q = 0, \dots, \frac{m-2}{2}$ one polynomial $G_{1q}$ of degree $ql_1 + (l_2+l_3)(m-q)$, one polynomial $G_{2q}$ of degree $ql_2 + (l_1+l_3)(m-q)$, one polynomial $G_{3q}$ of degree $ql_3 + (l_1+l_2)(m-q)$, where: $G = L_{11}^{\frac{m}{2}} L_{l_1 1}^{\frac{m}{2}} L_{12}^{\frac{m}{2}} L_{l_2 2}^{\frac{m}{2}} L_{13}^{\frac{m}{2}} L_{l_3 3}^{\frac{m}{2}}$; $G_{1q} = L_{11}^{q} L_{l_1 1}^{q} L_{12}^{m-q} L_{l_2 2}^{m-q} L_{13}^{m-q} L_{l_3 3}^{m-q}$; $G_{2q} = L_{11}^{m-q} L_{l_1 1}^{m-q} L_{12}^{q} L_{l_2 2}^{q} L_{13}^{m-q} L_{l_3 3}^{m-q}$; $G_{3q} = L_{11}^{m-q} L_{l_1 1}^{m-q} L_{12}^{m-q} L_{l_2 2}^{m-q} L_{13}^{q} L_{l_3 3}^{q}$.*

*Proof.* By the definition of $m$-homogeneous fat complete grid of type $(l_1, l_2, l_3)$, the equality follows easily: $I_Y = \bigcap_{i=1}^{l_1} (L_{i1}, L_{12})^m \bigcap \dots \bigcap_{i=1}^{l_1} (L_{i1}, L_{l_2 2})^m \bigcap_{i=1}^{l_1} (L_{i1}, L_{13})^m \bigcap \dots \bigcap_{i=1}^{l_1} (L_{i1}, L_{l_3 3})^m \bigcap_{i=1}^{l_2} (L_{i2}, L_{13})^m \bigcap \dots \bigcap_{i=1}^{l_2} (L_{i2}, L_{l_3 3})^m$.

If we apply c) of Lemma 12,we obtain that: $I_Y = (L_{11} L_{l_1 1}, L_{12} L_{l_2 2} L_{13} L_{l_3 3})^m \cap (L_{12} L_{l_2 2}, L_{13} L_{l_3 3})^m$.

By a) of Lemma 12 we obtain: $I_Y = (L_{11}^m L_{l_1 1}^m, L_{12}^m L_{l_2 2}^m L_{13}^m L_{l_3 3}^m, L_{11} \dots L_{l_1 1} L_{12} \dots L_{l_2 2} L_{13} \dots L_{l_3 3}) \cap (L_{11}^{m-1} L_{l_1 1}^{m-1}, L_{12}^{m-1} L_{l_2 2}^{m-1} L_{13}^{m-1} L_{l_3 3}^{m-1}, L_{11}^2 L_{l_1 1}^2 L_{12}^2 L_{l_2 2}^2 L_{13}^2 L_{l_3 3}^2) \cap \dots \cap (L_{11}^{\frac{m+2}{2}} L_{l_1 1}^{\frac{m+2}{2}}, L_{12}^{\frac{m+2}{2}} L_{l_2 2}^{\frac{m+2}{2}} L_{13}^{\frac{m+2}{2}} L_{l_3 3}^{\frac{m+2}{2}}, L_{11}^{\frac{m}{2}} L_{l_1 1}^{\frac{m}{2}} L_{12}^{\frac{m}{2}} L_{l_2 2}^{\frac{m}{2}} L_{13}^{\frac{m}{2}} L_{l_3 3}^{\frac{m}{2}}) \cap (L_{12}^m L_{l_2 2}^m, L_{13}^m L_{l_3 3}^m, L_{12} \dots L_{l_2 2} L_{13} \dots L_{l_3 3}) \cap (L_{12}^{m-1} L_{l_2 2}^{m-1}, L_{13}^{m-1} L_{l_3 3}^{m-1}, L_{12}^2 L_{l_2 2}^2 L_{13}^2 L_{l_3 3}^2) \cap \dots \cap (L_{12}^{\frac{m+2}{2}} L_{l_2 2}^{\frac{m+2}{2}}, L_{13}^{\frac{m+2}{2}} L_{l_3 3}^{\frac{m+2}{2}}, L_{12}^{\frac{m}{2}} L_{l_2 2}^{\frac{m}{2}} L_{13}^{\frac{m}{2}} L_{l_3 3}^{\frac{m}{2}})$.

By using the modular law we have that: $I_Y = (L_{11}^m L_{l_1 1}^m L_{12}^m L_{l_2 2}^m, L_{11}^m L_{l_1 1}^m L_{13}^m L_{l_3 3}^m, L_{12}^m L_{l_2 2}^m L_{13}^m L_{l_3 3}^m, L_{11} \dots L_{l_1 1} L_{12}^{m-1} L_{l_2 2}^{m-1} L_{13}^{m-1} L_{l_3 3}^{m-1}, L_{11}^{m-1} L_{l_1 1}^{m-1} L_{12} \dots L_{l_2 2} L_{13}^{m-1} L_{l_3 3}^{m-1}, L_{11}^{m-1} L_{l_1 1}^{m-1} L_{12}^{m-1} L_{l_2 2}^{m-1} L_{13} \dots L_{l_3 3}, L_{11}^2 L_{l_1 1}^2 L_{12}^2 L_{l_2 2}^2 L_{13}^2 L_{l_3 3}^2) \cap (L_{11}^{m-2} L_{l_1 1}^{m-2} L_{12}^{m-2} L_{l_2 2}^{m-2}, L_{11}^{m-2} L_{l_1 1}^{m-2} L_{13}^{m-2} L_{l_3 3}^{m-2}, L_{12}^{m-2} L_{l_2 2}^{m-2} L_{13}^{m-2} L_{l_3 3}^{m-2}, L_{11}^3 L_{l_1 1}^3 L_{12}^3 L_{l_2 2}^3 L_{13}^3 L_{l_3 3}^3) \cap \dots \cap (L_{11}^{\frac{m+2}{2}} L_{l_1 1}^{\frac{m+2}{2}} L_{12}^{\frac{m+2}{2}} L_{l_2 2}^{\frac{m+2}{2}}, L_{11}^{\frac{m+2}{2}} L_{l_1 1}^{\frac{m+2}{2}} L_{13}^{\frac{m+2}{2}} L_{l_3 3}^{\frac{m+2}{2}}, L_{12}^{\frac{m+2}{2}} L_{l_2 2}^{\frac{m+2}{2}} L_{13}^{\frac{m+2}{2}} L_{l_3 3}^{\frac{m+2}{2}}, L_{11}^{\frac{m}{2}} L_{l_1 1}^{\frac{m}{2}} L_{12}^{\frac{m}{2}} L_{l_2 2}^{\frac{m}{2}} L_{13}^{\frac{m}{2}} L_{l_3 3}^{\frac{m}{2}})$.

By applying the modular law, after a finite number of steps we obtain: $I_Y = (L_{11}^m\ L_{l_11}^m\ L_{12}^m\ L_{l_22}^m,\ L_{11}^m\ L_{l_11}^m\ L_{13}^m\ L_{l_33}^m,\ L_{12}^m\ L_{l_22}^m\ L_{13}^m\ L_{l_33}^m,\ L_{11}\dots L_{l_11}\ L_{12}^{m-1}\ L_{l_22}^{m-1}\ L_{13}^{m-1}\ L_{l_33}^{m-1},\ L_{11}^{m-1}\ L_{l_11}^{m-1}\ L_{12}\dots L_{l_22}\ L_{13}^{m-1}\ L_{l_33}^{m-1},\ L_{11}^{m-1}\ L_{l_11}^{m-1}\ L_{12}^{m-1}\ L_{l_22}^{m-1}\ L_{13}\dots L_{l_33},\dots, L_{11}^{\frac{m-2}{2}}\ L_{l_11}^{\frac{m-2}{2}}\ L_{12}^{\frac{m+2}{2}}\ L_{l_22}^{\frac{m+2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}},\ L_{11}^{\frac{m+2}{2}}\ L_{l_11}^{\frac{m+2}{2}}\ L_{12}^{\frac{m-2}{2}}\ L_{l_22}^{\frac{m-2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}}, L_{11}^{\frac{m+2}{2}}\ L_{l_11}^{\frac{m+2}{2}}\ L_{12}^{\frac{m+2}{2}}\ L_{l_22}^{\frac{m+2}{2}}\ L_{13}^{\frac{m-2}{2}}\ L_{l_33}^{\frac{m-2}{2}}, L_{11}^{\frac{m}{2}}\ L_{l_11}^{\frac{m}{2}}\ L_{12}^{\frac{m}{2}}\ L_{l_22}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_33}^{\frac{m}{2}})$. □

**Theorem 14.** *If $m$ is odd, $I_Y$ is minimally generated by $\frac{3}{2}(m+1)$ polynomials, precisely: for any $q = 0,\dots,\frac{m-1}{2}$ one polynomial $G_{1q}$ of degree $ql_1 + (l_2+l_3)(m-q)$, one polynomial $G_{2q}$ of degree $ql_2 + (l_1+l_3)(m-q)$, one polynomial $G_{3q}$ of degree $ql_3 + (l_1+l_2)(m-q)$, where: $G_{1q} = L_{11}^q\ L_{l_11}^q\ L_{12}^{m-q}\ L_{l_22}^{m-q}\ L_{13}^{m-q}\ L_{l_33}^{m-q}$; $G_{2q} = L_{11}^{m-q}\ L_{l_11}^{m-q}\ L_{12}^q\ L_{l_22}^q\ L_{13}^{m-q}\ L_{l_33}^{m-q}$; $G_{3q} = L_{11}^{m-q}\ L_{l_11}^{m-q}\ L_{12}^{m-q}\ L_{l_22}^{m-q}\ L_{13}^q\ L_{l_33}^q$.*

*Proof.* By the definition of $m$-homogeneous fat complete grid of type $(l_1,l_2,l_3)$, the equality follows easily: $I_Y = \bigcap_{i=1}^{l_1}(L_{i1}, L_{12})^m \bigcap\dots\bigcap_{i=1}^{l_1}(L_{i1}, L_{l_22})^m \bigcap_{i=1}^{l_1}(L_{i1}, L_{13})^m \bigcap\dots\bigcap_{i=1}^{l_1}(L_{i1}, L_{l_33})^m \bigcap_{i=1}^{l_2}(L_{i2}, L_{13})^m \bigcap\dots\bigcap_{i=1}^{l_2}(L_{i2}, L_{l_33})^m$.

If we apply c) of Lemma 12,we obtain that: $I_Y = (L_{11}\ L_{l_11}, L_{12}\ L_{l_22}\ L_{13}\ L_{l_33})^m \cap (L_{12}\ L_{l_22}, L_{13}\ L_{l_33})^m$.

By b) of Lemma 12 we obtain: $I_Y = (L_{11}^m\ L_{l_11}^m, L_{12}^m\ L_{l_22}^m\ L_{13}^m\ L_{l_33}^m, L_{11}\dots L_{l_11}\ L_{12}\dots L_{l_22}\ L_{13}\dots L_{l_33}) \cap (L_{11}^{m-1}\ L_{l_11}^{m-1}, L_{12}^{m-1}\ L_{l_22}^{m-1}\ L_{13}^{m-1}\ L_{l_33}^{m-1}, L_{11}^2\ L_{l_11}^2\ L_{12}^2\ L_{l_22}^2\ L_{13}^2\ L_{l_33}^2) \cap \dots \cap (L_{11}^{\frac{m+3}{2}}\ L_{l_11}^{\frac{m+3}{2}}, L_{12}^{\frac{m+3}{2}}\ L_{l_22}^{\frac{m+3}{2}}\ L_{13}^{\frac{m+3}{2}}\ L_{l_33}^{\frac{m+3}{2}}, L_{11}^{\frac{m-1}{2}}\dots L_{l_11}^{\frac{m-1}{2}}\ L_{12}^{\frac{m-1}{2}}\ L_{l_22}^{\frac{m-1}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_33}^{\frac{m-1}{2}}) \cap (L_{11}^{\frac{m+1}{2}}\ L_{l_11}^{\frac{m+1}{2}}, L_{12}^{\frac{m+1}{2}}\ L_{l_22}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_33}^{\frac{m+1}{2}}) \cap (L_{12}^m\ L_{l_22}^m, L_{13}^m\ L_{l_33}^m, L_{12}\dots L_{l_22}\ L_{13}\dots L_{l_33}) \cap (L_{12}^{m-1}\ L_{l_22}^{m-1}, L_{13}^{m-1}\ L_{l_33}^{m-1}, L_{12}^2\ L_{l_22}^2\ L_{13}^2\ L_{l_33}^2) \cap\dots\cap (L_{12}^{\frac{m+3}{2}}\ L_{l_22}^{\frac{m+3}{2}}, L_{13}^{\frac{m+3}{2}}\ L_{l_33}^{\frac{m+3}{2}}, L_{12}^{\frac{m-1}{2}}\ L_{l_22}^{\frac{m-1}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_33}^{\frac{m-1}{2}}) \cap (L_{12}^{\frac{m+1}{2}}\ L_{l_22}^{\frac{m+1}{2}}, L_{13}^{\frac{m+1}{2}}\ L_{l_33}^{\frac{m+1}{2}})$.

By using the modular law we have that: $I_Y = (L_{11}^m\ L_{l_11}^m\ L_{12}^m\ L_{l_22}^m, L_{11}^m\ L_{l_11}^m\ L_{13}^m\ L_{l_33}^m, L_{12}^m\ L_{l_22}^m\ L_{13}^m\ L_{l_33}^m, L_{11}\dots L_{l_11}\ L_{12}^{m-1}\ L_{l_22}^{m-1}\ L_{13}^{m-1}\ L_{l_33}^{m-1}, L_{11}^{m-1}\ L_{l_11}^{m-1}\ L_{12}\dots L_{l_22}\ L_{13}^{m-1}\ L_{l_33}^{m-1}, L_{11}^{m-1}\ L_{l_11}^{m-1}\ L_{12}^{m-1}\ L_{l_22}^{m-1}\ L_{13}\dots L_{l_33}, L_{11}^2\ L_{l_11}^2\ L_{12}^2\ L_{l_22}^2\ L_{13}^2$

$L_{l_3 3}^{2}) \cap \ldots \cap (L_{11}^{\frac{m+3}{2}} L_{l_1 1}^{\frac{m+3}{2}} L_{12}^{\frac{m+3}{2}} L_{l_2 2}^{\frac{m+3}{2}}, L_{11}^{\frac{m+3}{2}} L_{l_1 1}^{\frac{m+3}{2}} L_{13}^{\frac{m+3}{2}} L_{l_3 3}^{\frac{m+3}{2}}, L_{12}^{\frac{m+3}{2}} L_{l_2 2}^{\frac{m+3}{2}} L_{13}^{\frac{m+3}{2}} L_{l_3 3}^{\frac{m+3}{2}}, L_{11}^{\frac{m-1}{2}} L_{l_1 1}^{\frac{m-1}{2}} L_{12}^{\frac{m-1}{2}} L_{l_2 2}^{\frac{m-1}{2}} L_{13}^{\frac{m-1}{2}} L_{l_3 3}^{\frac{m-1}{2}}) \cap (L_{11}^{\frac{m+1}{2}} L_{l_1 1}^{\frac{m+1}{2}} L_{12}^{\frac{m+1}{2}} L_{l_2 2}^{\frac{m+1}{2}}, L_{11}^{\frac{m+1}{2}} L_{l_1 1}^{\frac{m+1}{2}} L_{13}^{\frac{m+1}{2}} L_{l_3 3}^{\frac{m+1}{2}}, L_{12}^{\frac{m+1}{2}} L_{l_2 2}^{\frac{m+1}{2}} L_{13}^{\frac{m+1}{2}} L_{l_3 3}^{\frac{m+1}{2}})$.

By applying the modular law, after a finite number of steps we obtain: $I_Y = (L_{11}^{m} L_{l_1 1}^{m} L_{12}^{m} L_{l_2 2}^{m}, L_{11}^{m} L_{l_1 1}^{m} L_{13}^{m} L_{l_3 3}^{m}, L_{12}^{m} L_{l_2 2}^{m} L_{13}^{m} L_{l_3 3}^{m}, L_{11} \ldots L_{l_1 1} L_{12}^{m-1} L_{l_2 2}^{m-1} L_{13}^{m-1} L_{l_3 3}^{m-1}, L_{11}^{m-1} L_{l_1 1}^{m-1} L_{12} \ldots L_{l_2 2} L_{13}^{m-1} L_{l_3 3}^{m-1}, L_{11}^{m-1} L_{l_1 1}^{m-1} L_{12}^{m-1} L_{l_2 2}^{m-1} L_{13} \ldots L_{l_3 3}, \ldots, L_{11}^{\frac{m-1}{2}} L_{l_1 1}^{\frac{m-1}{2}} L_{12}^{\frac{m+1}{2}} L_{l_2 2}^{\frac{m+1}{2}} L_{13}^{\frac{m+1}{2}} L_{l_3 3}^{\frac{m+1}{2}}, L_{11}^{\frac{m+1}{2}} L_{l_1 1}^{\frac{m+1}{2}} L_{12}^{\frac{m-1}{2}} L_{l_2 2}^{\frac{m-1}{2}} L_{13}^{\frac{m+1}{2}} L_{l_3 3}^{\frac{m+1}{2}}, L_{11}^{\frac{m+1}{2}} L_{l_1 1}^{\frac{m+1}{2}} L_{12}^{\frac{m+1}{2}} L_{l_2 2}^{\frac{m+1}{2}} L_{13}^{\frac{m-1}{2}} L_{l_3 3}^{\frac{m-1}{2}})$. □

Now we consider the Cohen-Macaulay property. In order to apply Lemma 3 we need the following result.

**Lemma 15.** *Let $S = \{L_1, \ldots, L_t\}$ be a finite set of distinct linear forms in $K[x_0, \ldots, x_n]$ such that $L_i \notin (L_j, L_k)$ for any $i \neq j$, $i \neq k$, $j \neq k$. Let $S = S_1 \cup S_2 \cup \ldots \cup S_n$ ($n \geq 3$) be a partition of the set $S$. Let $F_i$ a finite product of distinct elements of $S_i$ $i = 1, 2, \ldots, n$. Moreover, let $H$ be a linear form such that $L_i \notin (L_j, L_k, H)$ for any $i \neq j$, $i \neq k$, $j \neq k$. We have the following relations:*

*a)* $((F_1, F_2)^m, H) = (F_1^{m}, F_2^{m}, F_1 F_2, H) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m+2}{2}}, F_1^{\frac{m}{2}} F_2^{\frac{m}{2}}, H)$, *when $m$ is even;*

*b)* $((F_1, F_2)^m, H) = (F_1^{m}, F_2^{m}, F_1 F_2, H) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_2^{\frac{m+3}{2}}, F_1^{\frac{m-1}{2}} F_2^{\frac{m-1}{2}}, H) \cap (F_1^{\frac{m+1}{2}}, F_2^{\frac{m+1}{2}}, H)$, *when $m$ is odd;*

*c)* $((F_1, F_2)^m, H) \cap \ldots \cap ((F_1, F_n)^m, H) = ((F_1, F_2 \ldots F_n)^m, H)$.

*Proof.* a) We set $I = (F_1^{m}, F_2^{m}, F_1 F_2, H) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m+2}{2}}, F_1^{\frac{m}{2}} F_2^{\frac{m}{2}}, H)$ and we suppose that $F_1$, $F_2$ are linear forms. It is enough to prove that $I \subseteq ((F_1, F_2)^m, H)$, because the other inclusion is obvious.

By using the modular law we have that $I \subseteq (F_1^{m}, F_2^{m}, F_1^{m-1} F_2, F_1 F_2^{m-1},$

$F_1^2\ F_2^2, H) \cap (F_1^{m-2}, F_2^{m-2}, F_1^3\ F_2^3, H) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m+2}{2}}, F_1^{\frac{m}{2}}\ F_2^{\frac{m}{2}}, H)$.

By applying the modular law, after a finite number of steps we obtain: $I \subseteq (F_1^m, F_2^m, F_1^{m-1}\ F_2, F_1\ F_2^{m-1}, \ldots, F_1^{\frac{m}{2}}\ F_2^{\frac{m}{2}}, H) \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m+2}{2}}, F_1^{\frac{m}{2}}\ F_2^{\frac{m}{2}}, H) \subseteq (F_1^m, F_2^m, F_1^{m-1}\ F_2, F_1\ F_2^{m-1}, \ldots, F_1^{\frac{m}{2}}\ F_2^{\frac{m}{2}}, F_1^{\frac{m-2}{2}}\ F_2^{\frac{m+2}{2}}, F_1^{\frac{m+2}{2}}\ F_2^{\frac{m-2}{2}}, H) \subseteq ((F_1, F_2)^m, H)$.

b) We set $J = (F_1^m, F_2^m, F_1\ F_2, H) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_2^{\frac{m+3}{2}}, F_1^{\frac{m-1}{2}}\ F_2^{\frac{m-1}{2}}, H) \cap (F_1^{\frac{m+1}{2}}, F_2^{\frac{m+1}{2}}, H)$. It is enough to prove that $J \subseteq ((F_1, F_2)^m, H)$.

By using the modular law we have that $J \subseteq (F_1^m, F_2^m, F_1^{m-1}\ F_2, F_1\ F_2^{m-1}, F_1^2\ F_2^2, H) \cap (F_1^{m-2}, F_2^{m-2}, F_1^3\ F_2^3, H) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_2^{\frac{m+3}{2}}, F_1^{\frac{m-1}{2}}\ F_2^{\frac{m-1}{2}}, H) \cap (F_1^{\frac{m+1}{2}}, F_2^{\frac{m+1}{2}}, H)$.

By applying the modular law, after a finite number of steps we obtain: $I \subseteq (F_1^m, F_2^m, F_1^{m-1}\ F_2, F_1\ F_2^{m-1}, \ldots, F_1^{\frac{m+3}{2}}\ F_2^{\frac{m-3}{2}}, F_1^{\frac{m-3}{2}}\ F_2^{\frac{m+3}{2}}, F_1^{\frac{m-1}{2}}\ F_2^{\frac{m+1}{2}}, F_1^{\frac{m+1}{2}}\ F_2^{\frac{m-1}{2}}, H) \subseteq ((F_1, F_2)^m, H)$.

Moreover, since we have proved a) and b) for linear forms, it follows easily for $F_1 = \prod_{i=1}^{s} L_{ij}, F_2 = \prod_{q=1}^{t} L_{qr}$, where $L_{ij} \in S_1, L_{qr} \in S_2$.

c) By induction, it is enough to prove the case $n = 3$, that is, $((F_1, F_2)^m, H) \cap ((F_1, F_3)^m, H) = ((F_1, F_2F_3)^m, H)$. We set $I = ((F_1, F_2)^m, H) \cap ((F_1, F_3)^m, H)$. Suppose that $F_1, F_2, F_3$ are linear forms. We distinguish the case $m$ even and $m$ odd.

If $m$ is even, by using the property a) we have that $I = (F_1^m, F_2^m, F_1\ F_2, H) \cap (F_1^{m-1}, F_2^{m-1}, F_1^2\ F_2^2, H) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m+2}{2}}, F_1^{\frac{m}{2}}\ F_2^{\frac{m}{2}}, H) \cap (F_1^m, F_3^m, F_1\ F_3, H) \cap (F_1^{m-1}, F_3^{m-1}, F_1^2\ F_3^2, H) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_3^{\frac{m+2}{2}}, F_1^{\frac{m}{2}}\ F_3^{\frac{m}{2}}, H)$.

By using the modular law we obtain: $I = (F_1^m, F_2, H) \cap (F_1, F_2^m, H) \cap (F_1^{m-1}, F_2^2, H) \cap (F_1^2, F_2^{m-1}, H) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m}{2}}, H) \cap (F_1^{\frac{m}{2}}, F_2^{\frac{m+2}{2}}, H) \cap (F_1^m, F_3, H) \cap (F_1, F_3^m, H) \cap (F_1^{m-1}, F_3^2, H) \cap (F_1^2, F_3^{m-1}, H) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_3^{\frac{m}{2}}, H) \cap (F_1^{\frac{m}{2}},$

$F_3^{\frac{m+2}{2}}, H)$.

By [3], Lemma 7, a), we have that: $I = (F_1^m, F_2 F_3, H) \cap (F_1, F_2^m F_3^m, H) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m}{2}} F_3^{\frac{m}{2}}, H) \cap (F_1^{\frac{m}{2}}, F_2^{\frac{m+2}{2}} F_3^{\frac{m+2}{2}}, H)$.

Again by the modular law: $I = (F_1^m, F_1 F_2 F_3, F_2^m F_3^m, H) \cap (F_1^{m-1}, F_2^{m-1} F_3^{m-1}, F_1^2 F_2^2 F_3^2, H) \cap \ldots \cap (F_1^{\frac{m+2}{2}}, F_2^{\frac{m+2}{2}} F_3^{\frac{m+2}{2}}, F_1^{\frac{m}{2}} F_2^{\frac{m}{2}} F_3^{\frac{m}{2}}, H) = (F_1^m, F_1^{m-1} F_2 F_3, F_1^{m-2} F_2^2 F_3^2, \ldots, F_1^{\frac{m+2}{2}} F_2^{\frac{m-2}{2}} F_3^{\frac{m-2}{2}}, F_1^{\frac{m}{2}} F_2^{\frac{m}{2}} F_3^{\frac{m}{2}}, F_1^{\frac{m-2}{2}} F_2^{\frac{m+2}{2}} F_3^{\frac{m+2}{2}}, \ldots, F_1 F_2^{m-1} F_3^{m-1}, F_2^m F_3^m, H) = ((F_1, F_2 F_3)^m, H)$.

If $m$ is odd, by using the property b) we have that: $I = (F_1^m, F_2^m, F_1 F_2, H) \cap (F_1^{m-1}, F_2^{m-1}, F_1^2 F_2^2, H) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_2^{\frac{m+3}{2}}, F_1^{\frac{m-1}{2}} F_2^{\frac{m-1}{2}}, H) \cap (F_1^{\frac{m+1}{2}}, F_2^{\frac{m+1}{2}}, H) \cap (F_1^m, F_3^m, F_1 F_3, H) \cap (F_1^{m-1}, F_3^{m-1}, F_1^2 F_3^2, H) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_3^{\frac{m+3}{2}}, F_1^{\frac{m-1}{2}} F_3^{\frac{m-1}{2}}, H) \cap (F_1^{\frac{m+1}{2}}, F_3^{\frac{m+1}{2}}, H)$.

By using the modular law we obtain: $I = (F_1^m, F_2, H) \cap (F_1, F_2^m, H) \cap (F_1^{m-1}, F_2^2, H) \cap (F_1^2, F_2^{m-1}, H) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_2^{\frac{m-1}{2}}, H) \cap (F_1^{\frac{m-1}{2}}, F_2^{\frac{m+3}{2}}, H) \cap (F_1^{\frac{m+1}{2}}, F_2^{\frac{m+1}{2}}, H) \cap (F_1^m, F_3, H) \cap (F_1, F_3^m, H) \cap (F_1^{m-1}, F_3^2, H) \cap (F_1^2, F_3^{m-1}, H) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_3^{\frac{m-1}{2}}, H) \cap (F_1^{\frac{m-1}{2}}, F_3^{\frac{m+3}{2}}, H) \cap (F_1^{\frac{m+1}{2}}, F_3^{\frac{m+1}{2}}, H)$.

By [3], Lemma 7, a), we have that: $I = (F_1^m, F_2 F_3, H) \cap (F_1, F_2^m F_3^m, H) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_2^{\frac{m-1}{2}} F_3^{\frac{m-1}{2}}, H) \cap (F_1^{\frac{m-1}{2}}, F_2^{\frac{m+3}{2}} F_3^{\frac{m+3}{2}}, H) \cap (F_1^{\frac{m+1}{2}}, F_2^{\frac{m+1}{2}} F_3^{\frac{m+1}{2}}, H)$.

Again by the modular law: $I = (F_1^m, F_1 F_2 F_3, F_2^m F_3^m, H) \cap (F_1^{m-1}, F_2^{m-1} F_3^{m-1}, F_1^2 F_2^2 F_3^2, H) \cap \ldots \cap (F_1^{\frac{m+3}{2}}, F_2^{\frac{m+3}{2}} F_3^{\frac{m+3}{2}}, F_1^{\frac{m-1}{2}} F_2^{\frac{m-1}{2}} F_3^{\frac{m-1}{2}}, H) \cap (F_1^{\frac{m+1}{2}}, F_2^{\frac{m+1}{2}} F_3^{\frac{m+1}{2}}, H) = (F_1^m, F_1^{m-1} F_2 F_3, F_1^{m-2} F_2^2 F_3^2, \ldots, F_1^{\frac{m-3}{2}} F_2^{\frac{m+3}{2}} F_3^{\frac{m+3}{2}}, \ldots, F_1 F_2^{m-1} F_3^{m-1}, F_2^m F_3^m, H) = ((F_1, F_2 F_3)^m, H)$.

Moreover, since we have proved the relation for linear forms, it follows easily for $F_1 = \prod_{i=1}^{s} L_{ij}, F_2 = \prod_{q=1}^{t} L_{qr}$, where $L_{ij} \in S_1, L_{qr} \in S_2$. □

Let $Y$ be a finite union of $\sigma$ fat lines of $\mathbb{P}^n_{\mathbf{K}}$. We denote with $I_Y$ the ideal of $Y$, that is, $I_Y = \cap_{i=1}^{\sigma} \mathcal{P}_i^{m_i}$. We call $A = K[x_0, \ldots, x_n]/I_Y$ the homogeneous coordinate ring of $Y$. We consider a linear form $H$ of $K[x_0, \ldots, x_n]$ such that $H \notin \cup_{i=1}^{\sigma} \mathcal{P}_i$, $H \notin \mathcal{P}_i + \mathcal{P}_j$ for any $i, j \in \{1, \ldots, \sigma\}$ such that $\mathcal{P}_i + \mathcal{P}_j \subsetneq \mathcal{M}$. Denoting with $h = \overline{H}$ the class of $H$ in $A$, we have that $h$ is a non-zerodivisor of $A$. In the following theorem, we set $B = A/(h)$ and we denote with $H_B(t)$ the Hilbert function of $B$.

**Theorem 16.** *The ring $A$ is Cohen-Macaulay.*

*Proof.* We denote by $I_Y$ the defining ideal of $Y$. We note that $I_Y$ is the intersection of $(l_1l_2 + l_2l_3 + l_1l_3)$ ideals of fat lines, which we denote by $I_i^m$, i.e., $I_Y = \cap_{i=1}^{\sigma} I_i^m$, where $\sigma = (l_1l_2 + l_2l_3 + l_1l_3)$. We distinguish the case $m$ even and $m$ odd.

If $m$ is even by Theorem13 we have that: $I_Y = (L_{11}^{m}\ L_{l_11}^{m}\ L_{12}^{m}\ L_{l_22}^{m},\ L_{11}^{m}\ L_{l_11}^{m}\ L_{13}^{m}\ L_{l_33}^{m},\ L_{12}^{m}\ L_{l_22}^{m}\ L_{13}^{m}\ L_{l_33}^{m},\ L_{11} \ldots L_{l_11}\ L_{12}^{m-1}\ L_{l_22}^{m-1}\ L_{13}^{m-1}\ L_{l_33}^{m-1},\ L_{11}^{m-1}\ L_{l_11}^{m-1}\ L_{12} \ldots L_{l_22}\ L_{13}^{m-1}\ L_{l_33}^{m-1},\ L_{11}^{m-1}\ L_{l_11}^{m-1}\ L_{12}^{m-1}\ L_{l_22}^{m-1}\ L_{13} \ldots L_{l_33}, \ldots, L_{11}^{\frac{m-2}{2}}\ L_{l_11}^{\frac{m-2}{2}}\ L_{12}^{\frac{m+2}{2}}\ L_{l_22}^{\frac{m+2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}},\ L_{11}^{\frac{m+2}{2}}\ L_{l_11}^{\frac{m+2}{2}}\ L_{12}^{\frac{m-2}{2}}\ L_{l_22}^{\frac{m-2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}},\ L_{11}^{\frac{m+2}{2}}\ L_{l_11}^{\frac{m+2}{2}}\ L_{12}^{\frac{m+2}{2}}\ L_{l_22}^{\frac{m+2}{2}}\ L_{13}^{\frac{m-2}{2}}\ L_{l_33}^{\frac{m-2}{2}},\ L_{11}^{\frac{m}{2}}\ L_{l_11}^{\frac{m}{2}}\ L_{12}^{\frac{m}{2}}\ L_{l_22}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_33}^{\frac{m}{2}})$.

If we consider a linear form $H$ as in Lemma 3 and we prove that $(I_Y, H) = \cap_{i=1}^{\sigma}(I_i^m, H)$, then by Lemma 3 we have that $A$ is Cohen-Macaulay. We set $J = \cap_{i=1}^{\sigma}(I_i^m, H)$, then: $J = \bigcap_{i=1}^{l_1}((L_{i1}, L_{12})^m, H) \bigcap \ldots \bigcap_{i=1}^{l_1}((L_{i1}, L_{l_22})^m, H) \bigcap_{i=1}^{l_1}((L_{i1}, L_{13})^m, H) \bigcap \ldots \bigcap_{i=1}^{l_1}((L_{i1}, L_{l_33})^m, H) \bigcap_{i=1}^{l_2}((L_{i2}, L_{13})^m, H) \bigcap \ldots \bigcap_{i=1}^{l_2}((L_{i2}, L_{l_33})^m, H)$.

If we apply c) of Lemma 15,we obtain that: $J = ((L_{11}\ L_{l_11},\ L_{12}\ L_{l_22}\ L_{13}\ L_{l_33})^{m}, H) \cap ((L_{12}\ L_{l_22},\ L_{13}\ L_{l_33})^{m}, H)$.

By a) of Lemma 15 we obtain: $J = (L_{11}^{m}\ L_{l_11}^{m},\ L_{12}^{m}\ L_{l_22}^{m}\ L_{13}^{m}\ L_{l_33}^{m},\ L_{11} \ldots L_{l_11}\ L_{12} \ldots L_{l_22}\ L_{13} \ldots L_{l_33}, H) \cap (L_{11}^{m-1}\ L_{l_11}^{m-1},\ L_{12}^{m-1}\ L_{l_22}^{m-1}\ L_{13}^{m-1}\ L_{l_33}^{m-1},\ L_{11}^{2}\ L_{l_11}^{2}\ L_{12}^{2}\ L_{l_22}^{2}\ L_{13}^{2}\ L_{l_33}^{2}, H) \cap \ldots \cap (L_{11}^{\frac{m+2}{2}} \ldots L_{l_11}^{\frac{m+2}{2}},\ L_{12}^{\frac{m+2}{2}}\ L_{l_22}^{\frac{m+2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}},\ L_{11}^{\frac{m}{2}}\ L_{l_11}^{\frac{m}{2}}\ L_{12}^{\frac{m}{2}}\ L_{l_22}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_33}^{\frac{m}{2}}, H) \cap (L_{12}^{m}\ L_{l_22}^{m},\ L_{13}^{m}\ L_{l_33}^{m},\ L_{12} \ldots L_{l_22}\ L_{13} \ldots L_{l_33}, H) \cap (L_{12}^{m-1}\ L_{l_22}^{m-1},$

$L_{13}^{m-1}\ L_{l_33}^{m-1}, L_{12}^{2}\ L_{l_22}^{2}\ L_{13}^{2}\ L_{l_33}^{2}, H) \cap \ldots \cap (L_{12}^{\frac{m+2}{2}}\ L_{l_22}^{\frac{m+2}{2}}, L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}}, L_{12}^{\frac{m}{2}}\ L_{l_22}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_33}^{\frac{m}{2}}, H)$.

By using the modular law we have that: $J = (L_{11}^{m}\ L_{l_11}^{m}\ L_{12}^{m}\ L_{l_22}^{m}, L_{11}^{m}\ L_{l_11}^{m}\ L_{13}^{m}\ L_{l_33}^{m}, L_{12}^{m}\ L_{l_22}^{m}\ L_{13}^{m}\ L_{l_33}^{m}, L_{11}\ldots L_{l_11}\ L_{12}^{m-1}\ L_{l_22}^{m-1}\ L_{13}^{m-1}\ L_{l_33}^{m-1}, L_{11}^{m-1}\ L_{l_11}^{m-1}\ L_{12}\ldots L_{l_22}\ L_{13}^{m-1}\ L_{l_33}^{m-1}, L_{11}^{m-1}\ L_{l_11}^{m-1}\ L_{12}^{m-1}\ L_{l_22}^{m-1}\ L_{13}\ldots L_{l_33}, L_{11}^{2}\ L_{l_11}^{2}\ L_{12}^{2}\ L_{l_22}^{2}\ L_{13}^{2}\ L_{l_33}^{2}, H) \cap (L_{11}^{m-2}\ L_{l_11}^{m-2}\ L_{12}^{m-2}\ L_{l_22}^{m-2}, L_{11}^{m-2}\ L_{l_11}^{m-2}\ L_{13}^{m-2}\ L_{l_33}^{m-2}, L_{12}^{m-2}\ L_{l_22}^{m-2}\ L_{13}^{m-2}\ L_{l_33}^{m-2}, L_{11}^{3}\ L_{l_11}^{3}\ L_{12}^{3}\ L_{l_22}^{3}\ L_{13}^{3}\ L_{l_33}^{3}, H) \cap \ldots \cap (L_{11}^{\frac{m+2}{2}} \ldots L_{l_11}^{\frac{m+2}{2}}\ L_{12}^{\frac{m+2}{2}}\ L_{l_22}^{\frac{m+2}{2}}, L_{11}^{\frac{m+2}{2}}\ L_{l_11}^{\frac{m+2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}}, L_{12}^{\frac{m+2}{2}}\ L_{l_22}^{\frac{m+2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}}, L_{11}^{\frac{m}{2}}\ L_{l_11}^{\frac{m}{2}}\ L_{12}^{\frac{m}{2}}\ L_{l_22}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_33}^{\frac{m}{2}}, H)$.

By applying the modular law, after a finite number of steps we obtain: $J = (L_{11}^{m}\ L_{l_11}^{m}\ L_{12}^{m}\ L_{l_22}^{m}, L_{11}^{m}\ L_{l_11}^{m}\ L_{13}^{m}\ L_{l_33}^{m}, L_{12}^{m}\ L_{l_22}^{m}\ L_{13}^{m}\ L_{l_33}^{m}, L_{11}\ldots L_{l_11}\ L_{12}^{m-1}\ L_{l_22}^{m-1}\ L_{13}^{m-1}\ L_{l_33}^{m-1}, L_{11}^{m-1}\ L_{l_11}^{m-1}\ L_{12}\ldots L_{l_22}\ L_{13}^{m-1}\ L_{l_33}^{m-1}, L_{11}^{m-1}\ L_{l_11}^{m-1}\ L_{12}^{m-1}\ L_{l_22}^{m-1}\ L_{13}\ldots L_{l_33}, \ldots, L_{11}^{\frac{m-2}{2}}\ L_{l_11}^{\frac{m-2}{2}}\ L_{12}^{\frac{m+2}{2}}\ L_{l_22}^{\frac{m+2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}}, L_{11}^{\frac{m+2}{2}}\ L_{l_11}^{\frac{m+2}{2}}\ L_{12}^{\frac{m-2}{2}}\ L_{l_22}^{\frac{m-2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}}, L_{11}^{\frac{m+2}{2}}\ L_{l_11}^{\frac{m+2}{2}}\ L_{12}^{\frac{m+2}{2}}\ L_{l_22}^{\frac{m+2}{2}}\ L_{13}^{\frac{m-2}{2}}\ L_{l_33}^{\frac{m-2}{2}}, L_{11}^{\frac{m}{2}}\ L_{l_11}^{\frac{m}{2}}\ L_{12}^{\frac{m}{2}}\ L_{l_22}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_33}^{\frac{m}{2}}, H)$.

Hence we have that $\cap_{i=1}^{\sigma}(I_i^{m_i}, H) = (I_Y, H)$.

If $m$ is odd by Theorem14 we have that: $I_Y = (L_{11}^{m}\ L_{l_11}^{m}\ L_{12}^{m}\ L_{l_22}^{m}, L_{11}^{m}\ L_{l_11}^{m}\ L_{13}^{m}\ L_{l_33}^{m}, L_{12}^{m}\ L_{l_22}^{m}\ L_{13}^{m}\ L_{l_33}^{m}, L_{11}\ldots L_{l_11}\ L_{12}^{m-1}\ L_{l_22}^{m-1}\ L_{13}^{m-1}\ L_{l_33}^{m-1}, L_{11}^{m-1}\ L_{l_11}^{m-1}\ L_{12}\ldots L_{l_22}\ L_{13}^{m-1}\ L_{l_33}^{m-1}, L_{11}^{m-1}\ L_{l_11}^{m-1}\ L_{12}^{m-1}\ L_{l_22}^{m-1}\ L_{13}\ldots L_{l_33}, \ldots, L_{11}^{\frac{m-1}{2}}\ L_{l_11}^{\frac{m-1}{2}}\ L_{12}^{\frac{m+1}{2}}\ L_{l_22}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_33}^{\frac{m+1}{2}}, L_{11}^{\frac{m+1}{2}}\ L_{l_11}^{\frac{m+1}{2}}\ L_{12}^{\frac{m-1}{2}}\ L_{l_22}^{\frac{m-1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_33}^{\frac{m+1}{2}}, L_{11}^{\frac{m+1}{2}}\ L_{l_11}^{\frac{m+1}{2}}\ L_{12}^{\frac{m+1}{2}}\ L_{l_22}^{\frac{m+1}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_33}^{\frac{m-1}{2}})$.

If we consider a linear form $H$ as in Lemma 3 and we prove that $(I_Y, H) = \cap_{i=1}^{\sigma}(I_i^m, H)$, then by Lemma 3 we have that $A$ is Cohen-Macaulay. We set $J = \cap_{i=1}^{\sigma}(I_i^m, H)$, then: $J = \bigcap_{i=1}^{l_1}((L_{i1}, L_{12})^m, H) \bigcap \ldots \bigcap_{i=1}^{l_1}((L_{i1}, L_{l_22})^m, H) \bigcap_{i=1}^{l_1}((L_{i1}, L_{13})^m, H) \bigcap \ldots \bigcap_{i=1}^{l_1}((L_{i1}, L_{l_33})^m, H) \bigcap_{i=1}^{l_2}((L_{i2}, L_{13})^m, H) \bigcap \ldots \bigcap_{i=1}^{l_2}((L_{i2}, L_{l_33})^m, H)$.

If we apply c) of Lemma 15,we obtain that: $J = ((L_{11}\ L_{l_11}, L_{12}\ L_{l_22}\ L_{13}$

$L_{l_3 3})^m, H) \cap ((L_{12}\ L_{l_2 2}, L_{13}\ L_{l_3 3})^m, H)$.

By b) of Lemma 15 we obtain: $J = (L_{11}^m\ L_{l_1 1}^m, L_{12}^m\ L_{l_2 2}^m\ L_{13}^m\ L_{l_3 3}^m, L_{11} \dots L_{l_1 1}\ L_{12} \dots L_{l_2 2}\ L_{13} \dots L_{l_3 3}, H) \cap (L_{11}^{m-1}\ L_{l_1 1}^{m-1}, L_{12}^{m-1}\ L_{l_2 2}^{m-1}\ L_{13}^{m-1}\ L_{l_3 3}^{m-1}, L_{11}^2\ L_{l_1 1}^2\ L_{12}^2\ L_{l_2 2}^2\ L_{13}^2\ L_{l_3 3}^2, H) \cap \dots \cap (L_{11}^{\frac{m+3}{2}}\ L_{l_1 1}^{\frac{m+3}{2}}, L_{12}^{\frac{m+3}{2}}\ L_{l_2 2}^{\frac{m+3}{2}}\ L_{13}^{\frac{m+3}{2}}\ L_{l_3 3}^{\frac{m+3}{2}}, L_{11}^{\frac{m-1}{2}}\ L_{l_1 1}^{\frac{m-1}{2}}\ L_{12}^{\frac{m-1}{2}}\ L_{l_2 2}^{\frac{m-1}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_3 3}^{\frac{m-1}{2}}, H) \cap (L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}, L_{12}^{\frac{m+1}{2}}\ L_{l_2 2}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}}, H) \cap (L_{12}^m\ L_{l_2 2}^m, L_{13}^m\ L_{l_3 3}^m, L_{12} \dots L_{l_2 2}\ L_{13} \dots L_{l_3 3}, H) \cap (L_{12}^{m-1}\ L_{l_2 2}^{m-1}, L_{13}^{m-1}\ L_{l_3 3}^{m-1}, L_{12}^2\ L_{l_2 2}^2\ L_{13}^2\ L_{l_3 3}^2, H) \cap \dots \cap (L_{12}^{\frac{m+3}{2}}\ L_{l_2 2}^{\frac{m+3}{2}}, L_{13}^{\frac{m+3}{2}}\ L_{l_3 3}^{\frac{m+3}{2}}, L_{12}^{\frac{m-1}{2}}\ L_{l_2 2}^{\frac{m-1}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_3 3}^{\frac{m-1}{2}}, H) \cap (L_{12}^{\frac{m+1}{2}}\ L_{l_2 2}^{\frac{m+1}{2}}, L_{13}^{\frac{m+1}{2}} \dots L_{l_3 3}^{\frac{m+1}{2}}, H)$.

By using the modular law we have that: $J = (L_{11}^m\ L_{l_1 1}^m\ L_{12}^m\ L_{l_2 2}^m, L_{11}^m\ L_{l_1 1}^m\ L_{13}^m\ L_{l_3 3}^m, L_{12}^m\ L_{l_2 2}^m\ L_{13}^m\ L_{l_3 3}^m, L_{11} \dots L_{l_1 1}\ L_{12}^{m-1}\ L_{l_2 2}^{m-1}\ L_{13}^{m-1}\ L_{l_3 3}^{m-1}, L_{11}^{m-1}\ L_{l_1 1}^{m-1}\ L_{12} \dots L_{l_2 2}\ L_{13}^{m-1}\ L_{l_3 3}^{m-1}, L_{11}^{m-1}\ L_{l_1 1}^{m-1}\ L_{12}^{m-1}\ L_{l_2 2}^{m-1}\ L_{13} \dots L_{l_3 3}, L_{11}^2\ L_{l_1 1}^2\ L_{12}^2\ L_{l_2 2}^2\ L_{13}^2\ L_{l_3 3}^2, H) \cap \dots \cap (L_{11}^{\frac{m+3}{2}} \dots L_{l_1 1}^{\frac{m+3}{2}}\ L_{12}^{\frac{m+3}{2}}\ L_{l_2 2}^{\frac{m+3}{2}}, L_{11}^{\frac{m+3}{2}}\ L_{l_1 1}^{\frac{m+3}{2}}\ L_{13}^{\frac{m+3}{2}}\ L_{l_3 3}^{\frac{m+3}{2}}, L_{12}^{\frac{m+3}{2}}\ L_{l_2 2}^{\frac{m+3}{2}} L_{13}^{\frac{m+3}{2}}\ L_{l_3 3}^{\frac{m+3}{2}}, L_{11}^{\frac{m-1}{2}}\ L_{l_1 1}^{\frac{m-1}{2}}\ L_{12}^{\frac{m-1}{2}}\ L_{l_2 2}^{\frac{m-1}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_3 3}^{\frac{m-1}{2}}, H) \cap (L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}\ L_{12}^{\frac{m+1}{2}}\ L_{l_2 2}^{\frac{m+1}{2}}, L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}}, L_{12}^{\frac{m+1}{2}}\ L_{l_2 2}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}}, H)$.

By applying the modular law, after a finite number of steps we obtain: $J = (L_{11}^m\ L_{l_1 1}^m\ L_{12}^m\ L_{l_2 2}^m, L_{11}^m\ L_{l_1 1}^m\ L_{13}^m\ L_{l_3 3}^m, L_{12}^m\ L_{l_2 2}^m\ L_{13}^m\ L_{l_3 3}^m, L_{11} \dots L_{l_1 1}\ L_{12}^{m-1}\ L_{l_2 2}^{m-1}\ L_{13}^{m-1}\ L_{l_3 3}^{m-1}, L_{11}^{m-1}\ L_{l_1 1}^{m-1}\ L_{12} \dots L_{l_2 2}\ L_{13}^{m-1}\ L_{l_3 3}^{m-1}, L_{11}^{m-1}\ L_{l_1 1}^{m-1}\ L_{12}^{m-1}\ L_{l_2 2}^{m-1}\ L_{13} \dots L_{l_3 3}, \dots, L_{11}^{\frac{m-1}{2}}\ L_{l_1 1}^{\frac{m-1}{2}}\ L_{12}^{\frac{m+1}{2}}\ L_{l_2 2}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}}, L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}\ L_{12}^{\frac{m-1}{2}}\ L_{l_2 2}^{\frac{m-1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}}, L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}\ L_{12}^{\frac{m+1}{2}}\ L_{l_2 2}^{\frac{m+1}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_3 3}^{\frac{m-1}{2}}, H)$, and hence the required result. □

## 4. The Quasi-Homogeneous Case in $\mathbb{P}^3_{\mathbf{K}}$

Let $\mathbf{K}$ be an algebraically closed field. In this section we compute the generators and we study the Cohen-Macaulay property of a $m$-quasi-homogeneous fat complete grid of lines of $\mathbb{P}^3_{\mathbf{K}}$(Definition 9). Let $Y \subseteq \mathbb{P}^3_{\mathbf{K}}$ be a $m$ quasi-homogeneous fat complete grid of type $(l_1, l_2, l_3)$, $I_Y$ its defining ideal and $A$ its homogeneous coordinate ring.We set $L_{ij} = (x_j - a_{ij}x_0)$, where $j \in \{1,2,3\}$, $i \in \{l_1, \dots, l_q\}$ for any $q \in \{1,2,3\}$ and we assume also that $l_1 \geq l_2 \geq l_3$. Moreover, we consider $(l_1l_2 + l_2l_3 + l_1l_3) - 1$ lines with defining ideal of power $m$ and one line with defining ideal of power $p \in \{m-1, m+1\}$. Precisely: $I_Y = \bigcap_{i=1}^{l_1}(L_{i1}, L_{12})^m \bigcap \dots \bigcap_{i=1}^{l_1}(L_{i1}, L_{l_22})^m \bigcap_{i=1}^{l_1}(L_{i1}, L_{13})^m \bigcap \dots \bigcap_{i=1}^{l_1}(L_{i1}, L_{l_33})^m \bigcap_{i=1}^{l_2}(L_{i2}, L_{13})^m \bigcap \dots \bigcap_{i=1}^{l_2}(L_{i2}, L_{l_3-13})^m \bigcap_{i=1}^{l_2-1}(L_{i2}, L_{l_33})^m \bigcap (L_{l_22}, L_{l_33})^p$.

**Theorem 17.** *If $p = m+1$ and $m$ is even, $I_Y$ is minimally generated by $\frac{5m}{2}+2$ polynomials, precisely: for any $q = 0, \dots, \frac{m-2}{2}$ one polynomial $G_{1q}$ of degree $(l_2+l_3)(m-q)+ql_1$, one polynomial $G_{2q}$ of degree $(l_1+l_3)(m-q)+ql_2+1$, one polynomial $G_{3q}$ of degree $(l_1+l_3)(m-q)+ql_2+1$, for any $q = 0, \dots, \frac{m}{2}$ one polynomial $G_{4q}$ of degree $(l_1+l_2)(m-q)+ql_3+1$, one polynomial $G_{5q}$ of degree $(l_1+l_2)(m-q)+ql_3+1$, where: $G_{1q} = L_{11}^{q} L_{l_11}^{q} L_{12}^{m-q} L_{l_22}^{m-q} L_{13}^{m-q} L_{l_33}^{m-q}$; $G_{2q} = L_{11}^{m-q} L_{l_11}^{m-q} L_{12}^{q} L_{l_22}^{q} L_{13}^{m-q} L_{l_3-13}^{m-q} L_{l_33}^{m+1-q}$; $G_{3q} = L_{11}^{m-q} L_{l_11}^{m-q} L_{12}^{q} L_{l_2-12}^{q} L_{l_22}^{q+1} L_{13}^{m-q} L_{l_33}^{m-q}$; $G_{4q} = L_{11}^{m-q} L_{l_11}^{m-q} L_{12}^{m-q} L_{l_2-12}^{m-q} L_{l_22}^{m-q+1} L_{13}^{q} L_{l_33}^{q}$; $G_{5q} = L_{11}^{m-q} L_{l_11}^{m-q} L_{12}^{m-q} L_{l_22}^{m-q} L_{13}^{q} L_{l_3-13}^{q} L_{l_33}^{q+1}$;*

*Proof.* By the definition given in this section $I_Y = \bigcap_{i=1}^{l_1}(L_{i1}, L_{12})^m \bigcap \dots \bigcap_{i=1}^{l_1}(L_{i1}, L_{l_22})^m \bigcap_{i=1}^{l_1}(L_{i1}, L_{13})^m \bigcap \ \bigcap_{i=1}^{l_1}(L_{i1}, L_{l_33})^m \bigcap_{i=1}^{l_2}(L_{i2}, L_{13})^m \bigcap \dots \bigcap_{i=1}^{l_2}(L_{i2}, L_{l_3-13})^m \bigcap_{i=1}^{l_2-1}(L_{i2}, L_{l_33})^m \bigcap (L_{l_22}, L_{l_33})^{m+1}$.

If we apply c) of Lemma 12 we obtain that: $I_Y = (L_{11} L_{l_11}, L_{12} L_{l_22} L_{13} L_{l_33})^m \cap (L_{12} L_{l_2-12}, L_{13} L_{l_33})^m \cap \cap (L_{l_22}, L_{13} L_{l_3-13})^m \cap (L_{l_22}, L_{l_33})^{m+1}$.

By a) of Lemma 12 we have that: $I_Y = (L_{11}^m L_{l_11}^m, L_{12}^m L_{l_22}^m L_{13}^m L_{l_33}^m, L_{11} \dots L_{l_11} L_{12} \dots L_{l_22} L_{13} \dots L_{l_33}) \cap (L_{12}^m L_{l_2-12}^m, L_{13}^m L_{l_33}^m, L_{12} L_{l_2-12} L_{13} L_{l_33}) \cap \dots \cap (L_{11}^{\frac{m+2}{2}} L_{l_11}^{\frac{m+2}{2}}, L_{12}^{\frac{m+2}{2}} L_{l_22}^{\frac{m+2}{2}} L_{13}^{\frac{m+2}{2}} L_{l_33}^{\frac{m+2}{2}}, L_{11}^{\frac{m}{2}} L_{l_11}^{\frac{m}{2}} L_{12}^{\frac{m}{2}} L_{l_22}^{\frac{m}{2}} L_{13}^{\frac{m}{2}} L_{l_33}^{\frac{m}{2}}) \cap (L_{12}^{\frac{m+2}{2}} L_{l_2-12}^{\frac{m+2}{2}}, L_{13}^{\frac{m+2}{2}} L_{l_33}^{\frac{m+2}{2}}, L_{12}^{\frac{m}{2}} L_{l_2-12}^{\frac{m}{2}} L_{13}^{\frac{m}{2}} L_{l_33}^{\frac{m}{2}}) \cap (L_{l_22}, L_{13} L_{l_3-13})^m \cap (L_{l_22}, L_{l_33})^{m+1}$.

By using the modular law we obtain: $I_Y = (L_{11}^m \dots L_{l_11}^m L_{12}^m L_{l_2-12}^m, L_{11}^m L_{l_11}^m L_{13}^m L_{l_33}^m, L_{12}^m L_{l_22}^m L_{13}^m L_{l_33}^m, L_{11} L_{l_11} L_{12} L_{l_22} L_{13} L_{l_33}, L_{11}^m L_{l_11}^m L_{12} L_{l_2-12} L_{13}$

$L_{l_3 3}) \cap \ldots \cap (L_{11}^{\frac{m+2}{2}} \ldots L_{l_1 1}^{\frac{m+2}{2}} \, L_{12}^{\frac{m+2}{2}} \, L_{l_2-1 2}^{\frac{m+2}{2}}, L_{11}^{\frac{m+2}{2}} \, L_{l_1 1}^{\frac{m+2}{2}} \, L_{13}^{\frac{m+2}{2}} \, L_{l_3 3}^{\frac{m+2}{2}}, L_{12}^{\frac{m+2}{2}} \, L_{l_2 2}^{\frac{m+2}{2}} \, L_{13}^{\frac{m+2}{2}} \, L_{l_3 3}^{\frac{m+2}{2}}, L_{11}^{\frac{m}{2}} \, L_{l_1 1}^{\frac{m}{2}} \, L_{12}^{\frac{m}{2}} \, L_{l_2 2}^{\frac{m}{2}} \, L_{13}^{\frac{m}{2}} \, L_{l_3 3}^{\frac{m}{2}}, L_{11}^{\frac{m+2}{2}} \, L_{l_1 1}^{\frac{m+2}{2}} \, L_{12}^{\frac{m}{2}} \, L_{l_2-1 2}^{\frac{m}{2}} \, L_{13}^{\frac{m}{2}} \, L_{l_3 3}^{\frac{m}{2}}) \cap \cap (L_{l_2 2}, L_{13} \, L_{l_3-1 3})^m \cap (L_{l_2 2}, L_{l_3 3})^{m+1}.$

By Lemma 12 we have that: $I_Y = (L_{11}^{m} \, L_{l_1 1}^{m} \, L_{12}^{m} \, L_{l_2-1 2}^{m}, L_{11}^{m} \, L_{l_1 1}^{m} \, L_{13}^{m} \, L_{l_3 3}^{m}, L_{12}^{m} \, L_{l_2 2}^{m} \, L_{13}^{m} \, L_{l_3 3}^{m}, L_{11} \, L_{l_1 1} \, L_{12} \, L_{l_2 2} \, L_{13} \, L_{l_3 3}, L_{11}^{m} \, L_{l_1 1}^{m} \, L_{12} \, L_{l_2-1 2} \, L_{13} \, L_{l_3 3}) \cap \ldots \cap (L_{11}^{\frac{m+2}{2}} \ldots L_{l_1 1}^{\frac{m+2}{2}} \, L_{12}^{\frac{m+2}{2}} \, L_{l_2-1 2}^{\frac{m+2}{2}}, L_{11}^{\frac{m+2}{2}} \, L_{l_1 1}^{\frac{m+2}{2}} \, L_{13}^{\frac{m+2}{2}} \, L_{l_3 3}^{\frac{m+2}{2}}, L_{12}^{\frac{m+2}{2}} \, L_{l_2 2}^{\frac{m+2}{2}} \, L_{13}^{\frac{m+2}{2}} \, L_{l_3 3}^{\frac{m+2}{2}}, L_{11}^{\frac{m}{2}} \, L_{l_1 1}^{\frac{m}{2}} \, L_{12}^{\frac{m}{2}} \, L_{l_2 2}^{\frac{m}{2}} \, L_{13}^{\frac{m}{2}} \, L_{l_3 3}^{\frac{m}{2}}, L_{11}^{\frac{m+2}{2}} \, L_{l_1 1}^{\frac{m+2}{2}} \, L_{12}^{\frac{m}{2}} \, L_{l_2-1 2}^{\frac{m}{2}} \, L_{13}^{\frac{m}{2}} \, L_{l_3 3}^{\frac{m}{2}}) \cap (L_{l_2 2}^{m}, L_{13}^{m} \, L_{l_3-1 3}^{m}, L_{l_2 2} \, L_{13} \, L_{l_3-1 3}) \cap (L_{l_2 2}^{m+1}, L_{l_3 3}^{m+1}, L_{l_2 2} \, L_{l_3 3}) \cap \ldots \cap (L_{l_2 2}^{\frac{m+2}{2}}, L_{13}^{\frac{m+2}{2}} \, L_{l_3-1 3}^{\frac{m+2}{2}}, L_{l_2 2}^{\frac{m}{2}} \, L_{13}^{\frac{m}{2}} \, L_{l_3-1 3}^{\frac{m}{2}}) \cap (L_{l_2 2}^{\frac{m+4}{2}}, L_{l_3 3}^{\frac{m+4}{2}}, L_{l_2 2}^{\frac{m}{2}} \, L_{l_3 3}^{\frac{m}{2}}) \cap (L_{l_2 2}^{\frac{m+2}{2}}, L_{l_3 3}^{\frac{m+2}{2}}).$

By using the modular law we have that: $I_Y = (L_{11}^{m} \, L_{l_1 1}^{m} \, L_{12}^{m} \, L_{l_2-1 2}^{m}, L_{11}^{m} \, L_{l_1 1}^{m} \, L_{13}^{m} \, L_{l_3 3}^{m}, L_{12}^{m} \, L_{l_2 2}^{m} \, L_{13}^{m} \, L_{l_3 3}^{m}, L_{11} \, L_{l_1 1} \, L_{12} \, L_{l_2 2} \, L_{13} \, L_{l_3 3}, L_{11}^{m} \, L_{l_1 1}^{m} \, L_{12} \, L_{l_2-1 2} \, L_{13} \, L_{l_3 3}) \cap \ldots \cap (L_{11}^{\frac{m+2}{2}} \ldots L_{l_1 1}^{\frac{m+2}{2}} \, L_{12}^{\frac{m+2}{2}} \, L_{l_2-1 2}^{\frac{m+2}{2}}, L_{11}^{\frac{m+2}{2}} \, L_{l_1 1}^{\frac{m+2}{2}} \, L_{13}^{\frac{m+2}{2}} \, L_{l_3 3}^{\frac{m+2}{2}}, L_{12}^{\frac{m+2}{2}} \, L_{l_2 2}^{\frac{m+2}{2}} \, L_{13}^{\frac{m+2}{2}} \, L_{l_3 3}^{\frac{m+2}{2}}, L_{11}^{\frac{m}{2}} \, L_{l_1 1}^{\frac{m}{2}} \, L_{12}^{\frac{m}{2}} \, L_{l_2 2}^{\frac{m}{2}} \, L_{13}^{\frac{m}{2}} \, L_{l_3 3}^{\frac{m}{2}}, L_{11}^{\frac{m+2}{2}} \, L_{l_1 1}^{\frac{m+2}{2}} \, L_{12}^{\frac{m}{2}} \, L_{l_2-1 2}^{\frac{m}{2}} \, L_{13}^{\frac{m}{2}} \, L_{l_3 3}^{\frac{m}{2}}) \cap (L_{l_2 2}^{m+1}, L_{l_2 2}^{m} \, L_{l_3 3}, L_{l_2 2} \, L_{13} \, L_{l_3 3}, L_{13}^{m} \, L_{l_3-1 3}^{m} \, L_{l_3 3}^{m+1}) \cap \ldots \cap (L_{l_2 2}^{\frac{m+4}{2}}, L_{l_2 2}^{\frac{m+2}{2}} \, L_{l_3 3}^{\frac{m}{2}}, L_{l_2 2}^{\frac{m}{2}} \, L_{13}^{\frac{m}{2}} \, L_{l_3 3}^{\frac{m}{2}}, L_{13}^{\frac{m+2}{2}} \, L_{l_3-1 3}^{\frac{m+2}{2}} \, L_{l_3 3}^{\frac{m+4}{2}}) \cap (L_{l_2 2}^{\frac{m+2}{2}}, L_{l_3 3}^{\frac{m+2}{2}}).$

Using again the modular law we obtain: $I_Y = (L_{12}^{m} \, L_{l_2 2}^{m} \, L_{13}^{m} \, L_{l_3 3}^{m}, L_{11}^{m} \, L_{l_1 1}^{m} \, L_{13}^{m} \, L_{l_3-1 3}^{m} \, L_{l_3 3}^{m+1}, L_{11}^{m} \, L_{l_1 1}^{m} \, L_{l_2 2} \, L_{13}^{m} \, L_{l_3 3}^{m}, L_{11}^{m} \, L_{l_1 1}^{m} \, L_{12}^{m} \, L_{l_2-1 2}^{m} \, L_{l_2 2}^{m+1}, L_{11}^{m} \, L_{l_1 1}^{m} \, L_{12}^{m} \, L_{l_2 2}^{m} \, L_{l_3 3}, \ldots\ldots, L_{11}^{\frac{m-2}{2}} \, L_{l_1 1}^{\frac{m-2}{2}} \, L_{12}^{\frac{m+2}{2}} \, L_{l_2 2}^{\frac{m+2}{2}} \, L_{13}^{\frac{m+2}{2}} \, L_{l_3 3}^{\frac{m+2}{2}}, L_{11}^{\frac{m+2}{2}} \, L_{l_1 1}^{\frac{m+2}{2}} \, L_{12}^{\frac{m-2}{2}} \, L_{l_2 2}^{\frac{m-2}{2}} \, L_{13}^{\frac{m+2}{2}} \, L_{l_3-1 3}^{\frac{m+2}{2}} \, L_{l_3 3}^{\frac{m+4}{2}}, L_{11}^{\frac{m+2}{2}} \, L_{l_1 1}^{\frac{m+2}{2}} \, L_{12}^{\frac{m-2}{2}} \, L_{l_2-1 2}^{\frac{m-2}{2}} \, L_{l_2 2}^{\frac{m}{2}} \, L_{13}^{\frac{m+2}{2}} \, L_{l_3 3}^{\frac{m+2}{2}}, L_{11}^{\frac{m}{2}} \, L_{l_1 1}^{\frac{m}{2}} \, L_{12}^{\frac{m}{2}} \, L_{l_2 2}^{\frac{m}{2}} \, L_{13}^{\frac{m}{2}} \, L_{l_3-1 3}^{\frac{m}{2}} \, L_{l_3 3}^{\frac{m+2}{2}}, L_{11}^{\frac{m}{2}} \, L_{l_1 1}^{\frac{m}{2}} \, L_{12}^{\frac{m}{2}} \, L_{l_2-1 2}^{\frac{m}{2}} \, L_{l_2 2}^{\frac{m+2}{2}} \, L_{13}^{\frac{m}{2}} \, L_{l_3 3}^{\frac{m}{2}}).$ □

**Theorem 18.** *If $p = m + 1$ and $m$ is odd, $I_Y$ is minimally generated by $\frac{5m+5}{2}$ polynomials, precisely: for any $q = 0, \ldots, \frac{m-1}{2}$ one polynomial $G_{1q}$ of degree $(l_2+l_3)(m-q)+ql_1$, one polynomial $G_{2q}$ of degree $(l_1+l_3)(m-q)+ql_2+1$, one polynomial $G_{3q}$ of degree $(l_1 + l_2)(m - q) + ql_3 + 1$, one polynomial $G_{4q}$ of degree $(l_1+l_3)(m-q)+ql_2+1$, one polynomial $G_{5q}$ of degree $(l_1+l_2)(m-q)+$*

$ql_3+1$, *where:* $G_{1q} = L_{11}^{q}\, L_{l_11}^{q}\, L_{12}^{m-q}\, L_{l_22}^{m-q}\, L_{13}^{m-q}\, L_{l_33}^{m-q}$; $G_{2q} = L_{11}^{m-q}\, L_{l_11}^{m-q}\, L_{12}^{q}\, L_{l_22}^{q}\, L_{13}^{m-q}\, L_{l_3-13}^{m-q}\, L_{l_33}^{m+1-q}$; $G_{3q} = L_{11}^{m-q}\, L_{l_11}^{m-q}\, L_{12}^{m-q}\, L_{l_2-12}^{m-q}\, L_{l_22}^{m-q+1}\, L_{13}^{q}\, L_{l_33}^{q}$; $G_{4q} = L_{11}^{m-q}\, L_{l_11}^{m-q}\, L_{12}^{q}\, L_{l_2-12}^{q}\, L_{l_22}^{q+1}\, L_{13}^{m-q}\, L_{l_33}^{m-q}$; $G_{5q} = L_{11}^{m-q}\, L_{l_11}^{m-q}\, L_{12}^{m-q}\, L_{l_22}^{m-q}\, L_{13}^{q}\, L_{l_3-13}^{q}\, L_{l_33}^{q+1}$;

*Proof.* By the definition given in this section $I_Y = \bigcap_{i=1}^{l_1}(L_{i1},\, L_{12})^m \bigcap \ldots \bigcap_{i=1}^{l_1}(L_{i1},\, L_{l_22})^m \bigcap_{i=1}^{l_1}(L_{i1},\, L_{13})^m \bigcap\ \bigcap_{i=1}^{l_1}(L_{i1},\, L_{l_33})^m \bigcap_{i=1}^{l_2}(L_{i2},\, L_{13})^m \bigcap \ldots \bigcap_{i=1}^{l_2}(L_{i2},\, L_{l_3-13})^m \bigcap_{i=1}^{l_2-1}(L_{i2},\, L_{l_33})^m \bigcap (L_{l_22},\, L_{l_33})^{m+1}$.

If we apply c) of Lemma 12 we obtain that: $I_Y = (L_{11}\, L_{l_11},\, L_{12}\, L_{l_22}\, L_{13}\, L_{l_33})^m \cap (L_{12}\, L_{l_2-12},\, L_{13}\, L_{l_33})^m \cap \cap (L_{l_22},\, L_{13}\, L_{l_3-13})^m \cap (L_{l_22},\, L_{l_33})^{m+1}$.

By b) of Lemma 12 we have that: $I_Y = (L_{11}^{m}\, L_{l_11}^{m},\, L_{12}^{m}\, L_{l_22}^{m}\, L_{13}^{m}\, L_{l_33}^{m},\, L_{11}\, L_{l_11}\, L_{12}\, L_{l_22}\, L_{13}\, L_{l_33}) \cap (L_{12}^{m}\, L_{l_2-12}^{m},\, L_{13}^{m}\, L_{l_33}^{m},\, L_{12}\, L_{l_2-12}\, L_{13}\, L_{l_33}) \cap \ldots \cap (L_{11}^{\frac{m+3}{2}}\, L_{l_11}^{\frac{m+3}{2}}, L_{12}^{\frac{m+3}{2}}\, L_{l_22}^{\frac{m+3}{2}}\, L_{13}^{\frac{m+3}{2}}\, L_{l_33}^{\frac{m+3}{2}}, L_{11}^{\frac{m-1}{2}} \ldots L_{l_11}^{\frac{m-1}{2}}\, L_{12}^{\frac{m-1}{2}}\, L_{l_22}^{\frac{m-1}{2}}\, L_{13}^{\frac{m-1}{2}}\, L_{l_33}^{\frac{m-1}{2}}) \cap (L_{12}^{\frac{m+3}{2}} \ldots L_{l_2-12}^{\frac{m+3}{2}}, L_{13}^{\frac{m+3}{2}}\, L_{l_33}^{\frac{m+3}{2}}, L_{12}^{\frac{m-1}{2}} \ldots L_{l_2-12}^{\frac{m-1}{2}}\, L_{13}^{\frac{m-1}{2}}\, L_{l_33}^{\frac{m-1}{2}}) \cap (L_{11}^{\frac{m+1}{2}} \ldots L_{l_11}^{\frac{m+1}{2}}, L_{12}^{\frac{m+1}{2}}\, L_{l_22}^{\frac{m+1}{2}}\, L_{13}^{\frac{m+1}{2}}\, L_{l_33}^{\frac{m+1}{2}}) \cap (L_{12}^{\frac{m+1}{2}} \ldots L_{l_2-12}^{\frac{m+1}{2}}, L_{13}^{\frac{m+1}{2}}\, L_{l_33}^{\frac{m+1}{2}}) \cap (L_{l_22}, L_{13}\, L_{l_3-13})^m \cap (L_{l_22}, L_{l_33})^{m+1}$.

By using the modular law we obtain: $I_Y = (L_{11}^{m} \ldots L_{l_11}^{m}\, L_{12}^{m}\, L_{l_2-12}^{m},\, L_{11}^{m}\, L_{l_11}^{m}\, L_{13}^{m}\, L_{l_33}^{m},\, L_{12}^{m}\, L_{l_22}^{m}\, L_{13}^{m}\, L_{l_33}^{m},\, L_{11}\, L_{l_11}\, L_{12}\, L_{l_22}\, L_{13}\, L_{l_33},\, L_{11}^{m}\, L_{l_11}^{m}\, L_{12}\, L_{l_2-12}\, L_{13}\, L_{l_33}) \cap \ldots \cap (L_{11}^{\frac{m+3}{2}} \ldots L_{l_11}^{\frac{m+3}{2}}\, L_{12}^{\frac{m+3}{2}}\, L_{l_2-12}^{\frac{m+3}{2}}, L_{11}^{\frac{m+3}{2}}\, L_{l_11}^{\frac{m+3}{2}}\, L_{13}^{\frac{m+3}{2}}\, L_{l_33}^{\frac{m+3}{2}}, L_{12}^{\frac{m+3}{2}}\, L_{l_22}^{\frac{m+3}{2}}\, L_{13}^{\frac{m+3}{2}}\, L_{l_33}^{\frac{m+3}{2}}, L_{11}^{\frac{m-1}{2}}\, L_{l_11}^{\frac{m-1}{2}}\, L_{12}^{\frac{m-1}{2}}\, L_{l_22}^{\frac{m-1}{2}}\, L_{13}^{\frac{m-1}{2}}\, L_{l_33}^{\frac{m-1}{2}}, L_{11}^{\frac{m+3}{2}}\, L_{l_11}^{\frac{m+3}{2}}\, L_{12}^{\frac{m-1}{2}}\, L_{l_2-12}^{\frac{m-1}{2}}\, L_{13}^{\frac{m-1}{2}}\, L_{l_33}^{\frac{m-1}{2}}) \cap (L_{11}^{\frac{m+1}{2}} \ldots L_{l_11}^{\frac{m+1}{2}}\, L_{12}^{\frac{m+1}{2}}\, L_{l_2-12}^{\frac{m+1}{2}}, L_{11}^{\frac{m+1}{2}}\, L_{l_11}^{\frac{m+1}{2}}\, L_{13}^{\frac{m+1}{2}}\, L_{l_33}^{\frac{m+1}{2}}, L_{12}^{\frac{m+1}{2}}\, L_{l_22}^{\frac{m+1}{2}}\, L_{13}^{\frac{m+1}{2}}\, L_{l_33}^{\frac{m+1}{2}}) \cap (L_{l_22},\, L_{13}\, L_{l_3-13})^m \cap (L_{l_22}, L_{l_33})^{m+1}$.

By Lemma 12 we have that: $I_Y = (L_{11}^{m} \ldots L_{l_11}^{m}\, L_{12}^{m}\, L_{l_2-12}^{m},\, L_{11}^{m}\, L_{l_11}^{m}\, L_{13}^{m}\, L_{l_33}^{m},\, L_{12}^{m}\, L_{l_22}^{m}\, L_{13}^{m}\, L_{l_33}^{m},\, L_{11}\, L_{l_11}\, L_{12}\, L_{l_22}\, L_{13}\, L_{l_33},\, L_{11}^{m}\, L_{l_11}^{m}\, L_{12}\, L_{l_2-12}\, L_{13}\, L_{l_33}) \cap \ldots \cap (L_{11}^{\frac{m+3}{2}} \ldots L_{l_11}^{\frac{m+3}{2}}\, L_{12}^{\frac{m+3}{2}}\, L_{l_2-12}^{\frac{m+3}{2}}, L_{11}^{\frac{m+3}{2}}\, L_{l_11}^{\frac{m+3}{2}}\, L_{13}^{\frac{m+3}{2}}\, L_{l_33}^{\frac{m+3}{2}}, L_{12}^{\frac{m+3}{2}}\, L_{l_22}^{\frac{m+3}{2}}\, L_{13}^{\frac{m+3}{2}}\, L_{l_33}^{\frac{m+3}{2}}, L_{11}^{\frac{m-1}{2}}\, L_{l_11}^{\frac{m-1}{2}}\, L_{12}^{\frac{m-1}{2}}\, L_{l_22}^{\frac{m-1}{2}}\, L_{13}^{\frac{m-1}{2}}\, L_{l_33}^{\frac{m-1}{2}}, L_{11}^{\frac{m+3}{2}}\, L_{l_11}^{\frac{m+3}{2}}\, L_{12}^{\frac{m-1}{2}}\, L_{l_2-12}^{\frac{m-1}{2}}\, L_{13}^{\frac{m-1}{2}}\, L_{l_33}^{\frac{m-1}{2}}) \cap (L_{11}^{\frac{m+1}{2}} \ldots L_{l_11}^{\frac{m+1}{2}}\, L_{12}^{\frac{m+1}{2}}\, L_{l_2-12}^{\frac{m+1}{2}}, L_{11}^{\frac{m+1}{2}}\, L_{l_11}^{\frac{m+1}{2}}\, L_{13}^{\frac{m+1}{2}}\, L_{l_33}^{\frac{m+1}{2}}, L_{12}^{\frac{m+1}{2}}\, L_{l_22}^{\frac{m+1}{2}}\, L_{13}^{\frac{m+1}{2}}\, L_{l_33}^{\frac{m+1}{2}}) \cap (L_{l_22}^{m}, L_{13}^{m}\, L_{l_3-13}^{m}, L_{l_22}\, L_{13}\, L_{l_3-13}) \cap (L_{l_22}^{m+1}, L_{l_33}^{m+1}, L_{l_22}\, L_{l_33}) \cap \ldots \cap (L_{l_22}^{\frac{m+1}{2}},$

$L_{13}^{\frac{m+1}{2}}\ L_{l_3-1\,3}^{\frac{m+1}{2}}) \cap (L_{l_2 2}^{\frac{m+3}{2}},\ L_{l_3 3}^{\frac{m+3}{2}},\ L_{l_2 2}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}})$.

By using the modular law we have that: $I_Y = (L_{11}^{m}\ L_{l_1 1}^{m}\ L_{12}^{m}\ L_{l_2-1\,2}^{m},\ L_{11}^{m}\ L_{l_1 1}^{m}\ L_{13}^{m}\ L_{l_3 3}^{m},\ L_{12}^{m}\ L_{l_2 2}^{m}\ L_{13}^{m}\ L_{l_3 3}^{m},\ L_{11}\ L_{l_1 1}\ L_{12}\ L_{l_2 2}\ L_{13}\ L_{l_3 3},\ L_{11}^{m}\ L_{l_1 1}^{m}\ L_{12}\ L_{l_2-1\,2}\ L_{13}\ L_{l_3 3}) \cap \ldots \cap (L_{11}^{\frac{m+3}{2}} \ldots L_{l_1 1}^{\frac{m+3}{2}}\ L_{12}^{\frac{m+3}{2}}\ L_{l_2-1\,2}^{\frac{m+3}{2}},\ L_{11}^{\frac{m+3}{2}}\ L_{l_1 1}^{\frac{m+3}{2}}\ L_{13}^{\frac{m+3}{2}}\ L_{l_3 3}^{\frac{m+3}{2}},\ L_{12}^{\frac{m+3}{2}}\ L_{l_2 2}^{\frac{m+3}{2}}\ L_{13}^{\frac{m+3}{2}}\ L_{l_3 3}^{\frac{m+3}{2}},\ L_{11}^{\frac{m-1}{2}}\ L_{l_1 1}^{\frac{m-1}{2}}\ L_{12}^{\frac{m-1}{2}}\ L_{l_2 2}^{\frac{m-1}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_3 3}^{\frac{m-1}{2}},\ L_{11}^{\frac{m+3}{2}}\ L_{l_1 1}^{\frac{m+3}{2}}\ L_{12}^{\frac{m-1}{2}}\ L_{l_2-1\,2}^{\frac{m-1}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_3 3}^{\frac{m-1}{2}}) \cap (L_{11}^{\frac{m+1}{2}} \ldots L_{l_1 1}^{\frac{m+1}{2}}\ L_{12}^{\frac{m+1}{2}}\ L_{l_2-1\,2}^{\frac{m+1}{2}},\ L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}},\ L_{12}^{\frac{m+1}{2}}\ L_{l_2 2}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}}) \cap (L_{l_2 2}^{m+1},\ L_{l_2 2}^{m}\ L_{l_3 3},\ L_{l_2 2}\ L_{13}\ L_{l_3 3},\ L_{13}^{m}\ L_{l_3-1\,3}^{m}\ L_{l_3 3}^{m+1}) \cap \ldots \cap (L_{l_2 2}^{\frac{m+3}{2}},\ L_{l_2 2}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}},\ L_{13}^{\frac{m+1}{2}}\ L_{l_3-1\,3}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+3}{2}})$.

By using again the modular law we obtain: $I_Y = (L_{12}^{m}\ L_{l_2 2}^{m}\ L_{13}^{m}\ L_{l_3 3}^{m},\ L_{11}^{m}\ L_{l_1 1}^{m}\ L_{13}^{m}\ L_{l_3-1\,3}^{m}\ L_{l_3 3}^{m+1},\ L_{11}^{m}\ L_{l_1 1}^{m}\ L_{12}^{m}\ L_{l_2-1\,2}^{m}\ L_{l_2 2}^{m+1},\ L_{11}^{m}\ L_{l_1 1}^{m}\ L_{l_2 2}\ L_{13}^{m}\ L_{l_3 3}^{m},\ L_{11}^{m}\ L_{l_1 1}^{m}\ L_{12}^{m}\ L_{l_2 2}^{m}\ L_{l_3 3}, \ldots, L_{11}^{\frac{m-1}{2}}\ L_{l_1 1}^{\frac{m-1}{2}}\ L_{12}^{\frac{m+1}{2}}\ L_{l_2 2}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}},\ L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}\ L_{12}^{\frac{m-1}{2}}\ L_{l_2 2}^{\frac{m-1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3-1\,3}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+3}{2}},\ L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}\ L_{12}^{\frac{m+1}{2}}\ L_{l_2-1\,2}^{\frac{m+1}{2}}\ L_{l_2 2}^{\frac{m+3}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_3 3}^{\frac{m-1}{2}},\ L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}\ L_{12}^{\frac{m-1}{2}}\ L_{l_2-1\,2}^{\frac{m-1}{2}}\ L_{l_2 2}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}},\ L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}\ L_{12}^{\frac{m+1}{2}} \ldots L_{l_2 2}^{\frac{m+1}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_3-1\,3}^{\frac{m-1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}})$. $\square$

**Theorem 19.** *If $p = m + 1$, then the ring $A$ is Cohen-Macaulay.*

*Proof.* We recall that: $I_Y = \bigcap_{i=1}^{l_1}(L_{i1},\ L_{12})^m \bigcap \ldots \bigcap_{i=1}^{l_1}(L_{i1},\ L_{l_2 2})^m \bigcap_{i=1}^{l_1}(L_{i1},\ L_{13})^m \bigcap \bigcap_{i=1}^{l_1}(L_{i1},\ L_{l_3 3})^m \bigcap_{i=1}^{l_2}(L_{i2},\ L_{13})^m \bigcap \ldots \bigcap_{i=1}^{l_2}(L_{i2},\ L_{l_3-1\,3})^m \bigcap_{i=1}^{l_2-1}(L_{i2},\ L_{l_3 3})^m \bigcap (L_{l_2 2},\ L_{l_3 3})^{m+1}$.

We distinguish the case $m$ even and $m$ odd. If $m$ is even by Theorem 17 we have that: $I_Y = (L_{12}^{m}\ L_{l_2 2}^{m}\ L_{13}^{m}\ L_{l_3 3}^{m},\ L_{11}^{m}\ L_{l_1 1}^{m}\ L_{13}^{m}\ L_{l_3-1\,3}^{m}\ L_{l_3 3}^{m+1},\ L_{11}^{m}\ L_{l_1 1}^{m}\ L_{l_2 2}\ L_{13}^{m}\ L_{l_3 3}^{m},\ L_{11}^{m}\ L_{l_1 1}^{m}\ L_{12}^{m}\ L_{l_2-1\,2}^{m}\ L_{l_2 2}^{m+1},\ L_{11}^{m}\ L_{l_1 1}^{m}\ L_{12}^{m}\ L_{l_2 2}^{m}\ L_{l_3 3}, \ldots\ldots, L_{11}^{\frac{m-2}{2}}\ L_{l_1 1}^{\frac{m-2}{2}}\ L_{12}^{\frac{m+2}{2}}\ L_{l_2 2}^{\frac{m+2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_3 3}^{\frac{m+2}{2}},\ L_{11}^{\frac{m+2}{2}}\ L_{l_1 1}^{\frac{m+2}{2}}\ L_{12}^{\frac{m-2}{2}}\ L_{l_2 2}^{\frac{m-2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_3-1\,3}^{\frac{m+2}{2}}\ L_{l_3 3}^{\frac{m+4}{2}},\ L_{11}^{\frac{m+2}{2}}\ L_{l_1 1}^{\frac{m+2}{2}}\ L_{12}^{\frac{m-2}{2}}\ L_{l_2-1\,2}^{\frac{m-2}{2}}\ L_{l_2 2}^{\frac{m}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_3 3}^{\frac{m+2}{2}},\ L_{11}^{\frac{m}{2}}\ L_{l_1 1}^{\frac{m}{2}}\ L_{12}^{\frac{m}{2}}\ L_{l_2 2}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_3-1\,3}^{\frac{m}{2}}\ L_{l_3 3}^{\frac{m+2}{2}},\ L_{11}^{\frac{m}{2}}\ L_{l_1 1}^{\frac{m}{2}}\ L_{12}^{\frac{m}{2}}\ L_{l_2-1\,2}^{\frac{m}{2}}\ L_{l_2 2}^{\frac{m+2}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_3 3}^{\frac{m}{2}})$.

If we consider a linear form $H$ as in Lemma 15 and we prove that $(I_Y, H) =$

$\cap_{i=1}^{\sigma}(I_i^{m_i}, H)$, by Lemma 3 we have that $A$ is Cohen-Macaulay.

We set $J = \cap_{i=1}^{\sigma}(I_i^{m_i}, H)$, then: $J = \bigcap\ {}_{i=1}^{l_1}((L_{i1},\ L_{12})^m, H) \bigcap \ldots \bigcap\ {}_{i=1}^{l_1}((L_{i1}, L_{l_22})^m, H) \bigcap{}_{i=1}^{l_1}((L_{i1}, L_{13})^m, H) \bigcap \bigcap{}_{i=1}^{l_1}((L_{i1}, L_{l_33})^m, H) \bigcap{}_{i=1}^{l_2}((L_{i2}, L_{13})^m, H) \bigcap \ldots \bigcap\ {}_{i=1}^{l_2}((L_{i2},\ L_{l_3-13})^m, H) \bigcap\ {}_{i=1}^{l_2-1}((L_{i2},\ L_{l_33})^m, H) \bigcap ((L_{l_22}, L_{l_33})^{m+1}, H)$.

If we apply Lemma 15 we obtain that: $J = ((L_{11}\ L_{l_11}, L_{12}\ L_{l_22}\ L_{13}\ L_{l_33})^m, H)\cap ((L_{12}\ L_{l_2-12}, L_{13}\ L_{l_33})^m, H) \cap ((L_{l_22}, L_{13}\ L_{l_3-13})^m, H) \cap ((L_{l_22}, L_{l_33})^{m+1}, H)$.

By Lemma 15 we have that: $J = (L_{11}^m\ L_{l_11}^m,\ L_{12}^m\ L_{l_22}^m\ L_{13}^m\ L_{l_33}^m,\ L_{11}\ldots L_{l_11}\ L_{12}\ldots L_{l_22}\ L_{13}\ldots L_{l_33}, H) \cap (L_{12}^m\ L_{l_2-12}^m,\ L_{13}^m\ L_{l_33}^m,\ L_{12}\ L_{l_2-12}\ L_{13}\ L_{l_33}, H) \cap \ldots \cap (L_{11}^{\frac{m+2}{2}} \ldots L_{l_11}^{\frac{m+2}{2}}, L_{12}^{\frac{m+2}{2}}\ L_{l_22}^{\frac{m+2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}}, L_{11}^{\frac{m}{2}}\ L_{l_11}^{\frac{m}{2}}\ L_{12}^{\frac{m}{2}}\ L_{l_22}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_33}^{\frac{m}{2}}, H)\cap(L_{12}^{\frac{m+2}{2}} \ldots L_{l_2-12}^{\frac{m+2}{2}}, L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}}, L_{12}^{\frac{m}{2}}\ L_{l_2-12}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_33}^{\frac{m}{2}}, H)\cap((L_{l_22}, L_{13}\ L_{l_3-13})^m, H)\cap((L_{22}, L_{l_33})^{m+1}, H)$.

By using the modular law we obtain: $J = (L_{11}^m \ldots L_{l_11}^m\ L_{12}^m\ L_{l_2-12}^m, L_{11}^m\ L_{l_11}^m\ L_{13}^m\ L_{l_33}^m, L_{12}^m\ L_{l_22}^m\ L_{13}^m\ L_{l_33}^m, L_{11}\ L_{l_11}\ L_{12}\ L_{l_22}\ L_{13}\ L_{l_33}, L_{11}^m\ L_{l_11}^m\ L_{12}\ L_{l_2-12}\ L_{13}\ L_{l_33}, H)\cap \ldots \cap (L_{11}^{\frac{m+2}{2}} \ldots L_{l_11}^{\frac{m+2}{2}}\ L_{12}^{\frac{m+2}{2}}\ L_{l_2-12}^{\frac{m+2}{2}}, L_{11}^{\frac{m+2}{2}}\ L_{l_11}^{\frac{m+2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}}, L_{12}^{\frac{m+2}{2}}\ L_{l_22}^{\frac{m+2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}}, L_{11}^{\frac{m}{2}}\ L_{l_11}^{\frac{m}{2}}\ L_{12}^{\frac{m}{2}}\ L_{l_22}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_33}^{\frac{m}{2}}, L_{11}^{\frac{m+2}{2}}\ L_{l_11}^{\frac{m+2}{2}}\ L_{12}^{\frac{m}{2}}\ L_{l_2-12}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_33}^{\frac{m}{2}}, H) \cap\cap((L_{l_22}, L_{13}\ L_{l_3-13})^m, H) \cap ((L_{22}, L_{l_33})^{m+1}, H)$.

By Lemma 15 we have that: $J = (L_{11}^m\ L_{l_11}^m\ L_{12}^m\ L_{l_2-12}^m, L_{11}^m\ L_{l_11}^m\ L_{13}^m\ L_{l_33}^m, L_{12}^m\ L_{l_22}^m\ L_{13}^m\ L_{l_33}^m, L_{11}\ L_{l_11}\ L_{12}\ L_{l_22}\ L_{13}\ L_{l_33}, L_{11}^m\ L_{l_11}^m\ L_{12}\ L_{l_2-12}\ L_{13}\ L_{l_33}, H) \cap \ldots \cap (L_{11}^{\frac{m+2}{2}} \ldots L_{l_11}^{\frac{m+2}{2}}\ L_{12}^{\frac{m+2}{2}}\ L_{l_2-12}^{\frac{m+2}{2}}, L_{11}^{\frac{m+2}{2}}\ L_{l_11}^{\frac{m+2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}}, L_{12}^{\frac{m+2}{2}}\ L_{l_22}^{\frac{m+2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}}, L_{11}^{\frac{m}{2}}\ L_{l_11}^{\frac{m}{2}}\ L_{12}^{\frac{m}{2}}\ L_{l_22}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_33}^{\frac{m}{2}}, L_{11}^{\frac{m+2}{2}}\ L_{l_11}^{\frac{m+2}{2}}\ L_{12}^{\frac{m}{2}}\ L_{l_2-12}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_33}^{\frac{m}{2}}, H)\cap(L_{l_22}^m, L_{13}^m\ L_{l_3-13}^m, L_{l_22}\ L_{13}\ L_{l_3-13}, H) \cap (L_{l_22}^{m+1}, L_{l_33}^{m+1}, L_{l_22}\ L_{l_33}, H) \cap \ldots \cap (L_{l_22}^{\frac{m+2}{2}}, L_{13}^{\frac{m+2}{2}}\ L_{l_3-13}^{\frac{m+2}{2}}, L_{l_22}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_3-13}^{\frac{m}{2}}, H) \cap (L_{l_22}^{\frac{m+4}{2}}, L_{l_33}^{\frac{m+4}{2}}, L_{l_22}^{\frac{m}{2}}\ L_{l_33}^{\frac{m}{2}}, H) \cap (L_{l_22}^{\frac{m+2}{2}}, L_{l_33}^{\frac{m+2}{2}}, H)$.

By using the modular law we have that: $J = (L_{11}^m\ L_{l_11}^m\ L_{12}^m\ L_{l_2-12}^m, L_{11}^m\ L_{l_11}^m\ L_{13}^m\ L_{l_33}^m, L_{12}^m\ L_{l_22}^m\ L_{13}^m\ L_{l_33}^m, L_{11}\ L_{l_11}\ L_{12}\ L_{l_22}\ L_{13}\ L_{l_33}, L_{11}^m\ L_{l_11}^m\ L_{12}\ L_{l_2-12}\ L_{13}\ L_{l_33}, H)\cap\ldots\cap(L_{11}^{\frac{m+2}{2}} \ldots L_{l_11}^{\frac{m+2}{2}}\ L_{12}^{\frac{m+2}{2}}\ L_{l_2-12}^{\frac{m+2}{2}}, L_{11}^{\frac{m+2}{2}}\ L_{l_11}^{\frac{m+2}{2}}\ L_{13}^{\frac{m+2}{2}}\ L_{l_33}^{\frac{m+2}{2}}, L_{12}^{\frac{m+2}{2}}\ L_{l_22}^{\frac{m+2}{2}}$

$L_{13}^{\frac{m+2}{2}}\ L_{l_3 3}^{\frac{m+2}{2}}, L_{11}^{\frac{m}{2}}\ L_{l_1 1}^{\frac{m}{2}}\ L_{12}^{\frac{m}{2}}\ L_{l_2 2}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_3 3}^{\frac{m}{2}}, L_{11}^{\frac{m+2}{2}}\ L_{l_1 1}^{\frac{m+2}{2}}\ L_{12}^{\frac{m}{2}}\ L_{l_2-12}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_3 3}^{\frac{m}{2}}, H) \cap (L_{l_2 2}^{m+1}, L_{l_2 2}^{m}\ L_{l_3 3}, L_{l_2 2}\ L_{13}\ L_{l_3 3}, L_{13}^{m}\ L_{l_3-13}^{m}\ L_{l_3 3}^{m+1}, H) \cap \ldots \cap (L_{l_2 2}^{\frac{m+4}{2}}, L_{l_2 2}^{\frac{m+2}{2}}\ L_{l_3 3}^{\frac{m}{2}}, L_{l_2 2}^{\frac{m}{2}}\ L_{13}^{\frac{m}{2}}\ L_{l_3 3}^{\frac{m}{2}}, L_{13}^{\frac{m+2}{2}}\ L_{l_3-13}^{\frac{m+2}{2}}\ L_{l_3 3}^{\frac{m+4}{2}}, H) \cap (L_{l_2 2}^{\frac{m+2}{2}}, L_{l_3 3}^{\frac{m+2}{2}}, H)$.

By applying the modular law, after a finite number of steps we obtain $J = (I_Y, H)$ and hence the result.

If $m$ is odd by Theorem 18 we have that: $I_Y = (L_{12}^{m}\ L_{l_2 2}^{m}\ L_{13}^{m}\ L_{l_3 3}^{m}, L_{11}^{m}\ L_{l_1 1}^{m}\ L_{13}^{m}\ L_{l_3-13}^{m}\ L_{l_3 3}^{m+1}, L_{11}^{m}\ L_{l_1 1}^{m}\ L_{12}^{m}\ L_{l_2-12}^{m}\ L_{l_2 2}^{m+1}, L_{11}^{m}\ L_{l_1 1}^{m}\ L_{l_2 2}\ L_{13}^{m}\ L_{l_3 3}^{m}, L_{11}^{m}\ L_{l_1 1}^{m}\ L_{12}^{m}\ L_{l_2 2}^{m}\ L_{l_3 3}, \ldots, L_{11}^{\frac{m-1}{2}} \ldots L_{l_1 1}^{\frac{m-1}{2}}\ L_{12}^{\frac{m+1}{2}}\ L_{l_2 2}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}}, L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}\ L_{12}^{\frac{m-1}{2}}\ L_{l_2 2}^{\frac{m-1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3-13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+3}{2}}, L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}\ L_{12}^{\frac{m+1}{2}}\ L_{l_2-12}^{\frac{m+1}{2}}\ L_{l_2 2}^{\frac{m+3}{2}}, L_{13}^{\frac{m-1}{2}}\ L_{l_3 3}^{\frac{m-1}{2}}, L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}\ L_{12}^{\frac{m-1}{2}} \ldots L_{l_2-12}^{\frac{m-1}{2}}\ L_{l_2 2}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}}, L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}\ L_{12}^{\frac{m+1}{2}}\ L_{l_2 2}^{\frac{m+1}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_3-13}^{\frac{m-1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}})$.

Moreover, if we consider $J = \cap_{i=1}^{\sigma}(I_i^{m_i}, H)$, and we apply Lemma 15, we obtain that: $J = ((L_{11}\ L_{l_1 1}, L_{12}\ L_{l_2 2}\ L_{13}\ L_{l_3 3})^{m}, H) \cap ((L_{12}\ L_{l_2-12}, L_{13}\ L_{l_3 3})^{m}, H) \cap \cap ((L_{l_2 2}, L_{13}\ L_{l_3-13})^{m}, H) \cap ((L_{l_2 2}, L_{l_3 3})^{m+1}, H)$.

By Lemma 15 we have that: $J = (L_{11}^{m}\ L_{l_1 1}^{m}, L_{12}^{m}\ L_{l_2 2}^{m}\ L_{13}^{m}\ L_{l_3 3}^{m}, L_{11} \ldots L_{l_1 1}\ L_{12} \ldots L_{l_2 2}\ L_{13} \ldots L_{l_3 3}, H) \cap (L_{12}^{m}\ L_{l_2-12}^{m}, L_{13}^{m}\ L_{l_3 3}^{m}, L_{12}\ L_{l_2-12}\ L_{13}\ L_{l_3 3}, H) \cap \ldots \cap (L_{11}^{\frac{m+3}{2}} \ldots L_{l_1 1}^{\frac{m+3}{2}}, L_{12}^{\frac{m+3}{2}}\ L_{l_2 2}^{\frac{m+3}{2}}\ L_{13}^{\frac{m+3}{2}}\ L_{l_3 3}^{\frac{m+3}{2}}, L_{11}^{\frac{m-1}{2}}\ L_{l_1 1}^{\frac{m-1}{2}}\ L_{12}^{\frac{m-1}{2}}\ L_{l_2 2}^{\frac{m-1}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_3 3}^{\frac{m-1}{2}}, H) \cap (L_{12}^{\frac{m+3}{2}} \ldots L_{l_2-12}^{\frac{m+3}{2}}, L_{13}^{\frac{m+3}{2}}\ L_{l_3 3}^{\frac{m+3}{2}}, L_{12}^{\frac{m-1}{2}}\ L_{l_2-12}^{\frac{m-1}{2}}\ L_{13}^{\frac{m-1}{2}} \ldots L_{l_3 3}^{\frac{m-1}{2}}, H) \cap (L_{11}^{\frac{m+1}{2}} \ldots L_{l_1 1}^{\frac{m+1}{2}}, L_{12}^{\frac{m+1}{2}}\ L_{l_2 2}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}}, H) \cap (L_{12}^{\frac{m+1}{2}} \ldots L_{l_2-12}^{\frac{m+1}{2}}, L_{13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}}, H) \cap ((L_{l_2 2}, L_{13} \ldots L_{l_3-13})^{m}, H) \cap ((L_{22}, L_{l_3 3})^{m+1}, H)$.

By using the modular law we obtain: $J = (L_{11}^{m} \ldots L_{l_1 1}^{m}\ L_{12}^{m}\ L_{l_2-12}^{m}, L_{11}^{m}\ L_{l_1 1}^{m}\ L_{13}^{m}\ L_{l_3 3}^{m}, L_{12}^{m}\ L_{l_2 2}^{m}\ L_{13}^{m}\ L_{l_3 3}^{m}, L_{11}\ L_{l_1 1}\ L_{12}\ L_{l_2 2}\ L_{13}\ L_{l_3 3}, L_{11}^{m}\ L_{l_1 1}^{m}\ L_{12}\ L_{l_2-12}\ L_{13}\ L_{l_3 3}, H) \cap \ldots \cap (L_{11}^{\frac{m+3}{2}} \ldots L_{l_1 1}^{\frac{m+3}{2}}\ L_{12}^{\frac{m+3}{2}}\ L_{l_2-12}^{\frac{m+3}{2}}, L_{11}^{\frac{m+3}{2}}\ L_{l_1 1}^{\frac{m+3}{2}}\ L_{13}^{\frac{m+3}{2}}\ L_{l_3 3}^{\frac{m+3}{2}}, L_{12}^{\frac{m+3}{2}}\ L_{l_2 2}^{\frac{m+3}{2}}\ L_{13}^{\frac{m+3}{2}}\ L_{l_3 3}^{\frac{m+3}{2}}, L_{11}^{\frac{m-1}{2}}\ L_{l_1 1}^{\frac{m-1}{2}}\ L_{12}^{\frac{m-1}{2}}\ L_{l_2 2}^{\frac{m-1}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_3 3}^{\frac{m-1}{2}}, L_{11}^{\frac{m+3}{2}}\ L_{l_1 1}^{\frac{m+3}{2}}\ L_{12}^{\frac{m-1}{2}}\ L_{l_2-12}^{\frac{m-1}{2}}\ L_{13}^{\frac{m-1}{2}}\ L_{l_3 3}^{\frac{m-1}{2}}, H) \cap (L_{11}^{\frac{m+1}{2}} \ldots L_{l_1 1}^{\frac{m+1}{2}}\ L_{12}^{\frac{m+1}{2}}\ L_{l_2-12}^{\frac{m+1}{2}}, L_{11}^{\frac{m+1}{2}}\ L_{l_1 1}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}}\ L_{l_3 3}^{\frac{m+1}{2}}, L_{12}^{\frac{m+1}{2}}\ L_{l_2 2}^{\frac{m+1}{2}}\ L_{13}^{\frac{m+1}{2}} \ldots L_{l_3 3}^{\frac{m+1}{2}}, H) \cap ((L_{l_2 2}, L_{13}\ L_{l_3-13})^{m}, H) \cap ((L_{22},$

$L_{l_3 3})^{m+1}, H)$.

By Lemma 15 we have that: $J = (L^{m}_{11} \dots L^{m}_{l_1 1}\ L^{m}_{12}\ L^{m}_{l_2-1\,2},\ L^{m}_{11}\ L^{m}_{l_1 1}\ L^{m}_{13}\ L^{m}_{l_3 3},$ $L^{m}_{12}\ L^{m}_{l_2 2}\ L^{m}_{13}\ L^{m}_{l_3 3},\ L_{11}\ L_{l_1 1}\ L_{12}\ L_{l_2 2}\ L_{13}\ L_{l_3 3},\ L^{m}_{11}\ L^{m}_{l_1 1}\ L_{12}\ L_{l_2-1\,2}\ L_{13}\ L_{l_3 3}, H) \cap$ $\dots \cap (L^{\frac{m+3}{2}}_{11} \dots L^{\frac{m+3}{2}}_{l_1 1}\ L^{\frac{m+3}{2}}_{12}\ L^{\frac{m+3}{2}}_{l_2-1\,2},\ L^{\frac{m+3}{2}}_{11} \dots L^{\frac{m+3}{2}}_{l_1 1}\ L^{\frac{m+3}{2}}_{13}\ L^{\frac{m+3}{2}}_{l_3 3},\ L^{\frac{m+3}{2}}_{12}\ L^{\frac{m+3}{2}}_{l_2 2}\ L^{\frac{m+3}{2}}_{13}$ $L^{\frac{m+3}{2}}_{l_3 3},\ L^{\frac{m-1}{2}}_{11}\ L^{\frac{m-1}{2}}_{l_1 1}\ L^{\frac{m-1}{2}}_{12} \dots L^{\frac{m-1}{2}}_{l_2 2}\ L^{\frac{m-1}{2}}_{13}\ L^{\frac{m-1}{2}}_{l_3 3},\ L^{\frac{m+3}{2}}_{11}\ L^{\frac{m+3}{2}}_{l_1 1}\ L^{\frac{m-1}{2}}_{12}\ L^{\frac{m-1}{2}}_{l_2-1\,2}\ L^{\frac{m-1}{2}}_{13}$ $L^{\frac{m-1}{2}}_{l_3 3}, H) \cap \cap (L^{\frac{m+1}{2}}_{11}\ L^{\frac{m+1}{2}}_{l_1 1}\ L^{\frac{m+1}{2}}_{12}\ L^{\frac{m+1}{2}}_{l_2-1\,2},\ L^{\frac{m+1}{2}}_{11}\ L^{\frac{m+1}{2}}_{l_1 1}\ L^{\frac{m+1}{2}}_{13}\ L^{\frac{m+1}{2}}_{l_3 3},\ L^{\frac{m+1}{2}}_{12}\ L^{\frac{m+1}{2}}_{l_2 2}$ $L^{\frac{m+1}{2}}_{13}\ L^{\frac{m+1}{2}}_{l_3 3}, H) \cap (L^{m}_{l_2 2},\ L^{m}_{13}\ L^{m}_{l_3-1\,3},\ L_{l_2 2}\ L_{13}\ L_{l_3-1\,3}, H) \cap (L^{m+1}_{l_2 2},\ L^{m+1}_{l_3 3},\ L_{l_2 2}\ L_{l_3 3}, H) \cap$ $\dots \cap (L^{\frac{m+1}{2}}_{l_2 2},\ L^{\frac{m+1}{2}}_{13}\ L^{\frac{m+1}{2}}_{l_3-1\,3}, H) \cap (L^{\frac{m+3}{2}}_{l_2 2},\ L^{\frac{m+3}{2}}_{l_3 3},\ L^{\frac{m+1}{2}}_{l_2 2}\ L^{\frac{m+1}{2}}_{l_3 3}, H)$.

By using the modular law we have that: $J = (L^{m}_{11}\ L^{m}_{l_1 1}\ L^{m}_{12}\ L^{m}_{l_2-1\,2},\ L^{m}_{11}$ $L^{m}_{l_1 1}\ L^{m}_{13}\ L^{m}_{l_3 3},\ L^{m}_{12}\ L^{m}_{l_2 2}\ L^{m}_{13}\ L^{m}_{l_3 3},\ L_{11}\ L_{l_1 1}\ L_{12}\ L_{l_2 2}\ L_{13}\ L_{l_3 3},\ L^{m}_{11}\ L^{m}_{l_1 1}\ L_{12}\ L_{l_2-1\,2}\ L_{13}$ $L_{l_3 3}, H) \cap \dots \cap (L^{\frac{m+3}{2}}_{11} \dots L^{\frac{m+3}{2}}_{l_1 1}\ L^{\frac{m+3}{2}}_{12}\ L^{\frac{m+3}{2}}_{l_2-1\,2},\ L^{\frac{m+3}{2}}_{11}\ L^{\frac{m+3}{2}}_{l_1 1}\ L^{\frac{m+3}{2}}_{13}\ L^{\frac{m+3}{2}}_{l_3 3},\ L^{\frac{m+3}{2}}_{12}\ L^{\frac{m+3}{2}}_{l_2 2}$ $L^{\frac{m+3}{2}}_{13}\ L^{\frac{m+3}{2}}_{l_3 3},\ L^{\frac{m-1}{2}}_{11}\ L^{\frac{m-1}{2}}_{l_1 1}\ L^{\frac{m-1}{2}}_{12}\ L^{\frac{m-1}{2}}_{l_2 2}\ L^{\frac{m-1}{2}}_{13}\ L^{\frac{m-1}{2}}_{l_3 3},\ L^{\frac{m+3}{2}}_{11}\ L^{\frac{m+3}{2}}_{l_1 1}\ L^{\frac{m-1}{2}}_{12}\ L^{\frac{m-1}{2}}_{l_2-1\,2}\ L^{\frac{m-1}{2}}_{13}$ $L^{\frac{m-1}{2}}_{l_3 3}, H) \cap (L^{\frac{m+1}{2}}_{11} \dots L^{\frac{m+1}{2}}_{l_1 1}\ L^{\frac{m+1}{2}}_{12}\ L^{\frac{m+1}{2}}_{l_2-1\,2},\ L^{\frac{m+1}{2}}_{11} \dots L^{\frac{m+1}{2}}_{l_1 1}\ L^{\frac{m+1}{2}}_{13}\ L^{\frac{m+1}{2}}_{l_3 3},\ L^{\frac{m+1}{2}}_{12}\ L^{\frac{m+1}{2}}_{l_2 2}$ $L^{\frac{m+1}{2}}_{13}\ L^{\frac{m+1}{2}}_{l_3 3}, H) \cap (L^{m+1}_{l_2 2},\ L^{m}_{l_2 2}\ L_{l_3 3},\ L_{l_2 2}\ L_{13}\ L_{l_3 3},\ L^{m}_{13}\ L^{m}_{l_3-1\,3}\ L^{m+1}_{l_3 3}, H) \cap (L^{\frac{m+3}{2}}_{l_2 2},$ $L^{\frac{m+1}{2}}_{l_2 2}\ L^{\frac{m+1}{2}}_{l_3 3},\ L^{\frac{m+1}{2}}_{13}\ L^{\frac{m+1}{2}}_{l_3-1\,3}\ L^{\frac{m+3}{2}}_{l_3 3}, H)$.

By using again the modular law after a finite number of steps we obtain $J = (I_Y, H)$ and hence the result. □

**Theorem 20.** *If $p = m-1$ and $m$ is even, $I_Y$ is minimally generated by $\frac{5m}{2}+1$ polynomials, precisely: for any $q = 0, \dots, \frac{m-2}{2}$ one polynomial $G_{1q}$ of degree $(l_2+l_3)(m-q)+ql_1$, one polynomial $G_{2q}$ of degree $(l_1+l_3)(m-q)+ql_2$, one polynomial $G_{3q}$ of degree $(l_1+l_2)(m-q)+ql_3$, one polynomial $F_{1q}$ of degree $(l_1+l_3)(m-q)+(q+1)l_2-2$, one polynomial $F_{2q}$ of degree $(l_1+l_2)(m-q)+(q+1)l_3-2$, one polynomial $G$ of degree $(l_1+l_2+l_3)(\frac{m}{2})$, where: $G_{1q} = L^{q}_{11}$ $L^{q}_{l_1 1}\ L^{m-q}_{12}\ L^{m-q}_{l_2 2}\ L^{m-q}_{13}\ L^{m-q}_{l_3 3}$; $G_{2q} = L^{m-q}_{11}\ L^{m-q}_{l_1 1}\ L^{q}_{12}\ L^{q}_{l_2 2}\ L^{m-q}_{13}\ L^{m-q}_{l_3 3}$; $G_{3q} = L^{m-q}_{11}$ $L^{m-q}_{l_1 1}\ L^{m-q}_{12}\ L^{m-q}_{l_2 2}\ L^{q}_{13}\ L^{q}_{l_3 3}$; $F_{1q} = L^{m-q}_{11}\ L^{m-q}_{l_1 1}\ L^{q+1}_{12}\ L^{q+1}_{l_2-1\,2}\ L^{q}_{l_2 2}\ L^{m-q}_{13}\ L^{m-q}_{l_3-1\,3}\ L^{m-q-1}_{l_3 3}$;*

$F_{2q} = L_{_{11}}^{m-q}\ L_{_{l_11}}^{m-q}\ L_{_{12}}^{m-q}\ L_{_{l_2-12}}^{m-q}\ L_{_{l_22}}^{m-q-1}\ L_{_{13}}^{q+1}\ L_{_{l_3-13}}^{q+1}\ L_{_{l_33}}^{q}$; $G = L_{_{11}}^{\frac{m}{2}}\ L_{_{l_11}}^{\frac{m}{2}}\ L_{_{12}}^{\frac{m}{2}}\ L_{_{l_22}}^{\frac{m}{2}}\ L_{_{13}}^{\frac{m}{2}}\ L_{_{l_33}}^{\frac{m}{2}}$.

*Proof.* The proof is similar to the proof of Theorem 17 and is omitted. □

**Theorem 21.** *If $p = m - 1$ and $m$ is odd, $I_Y$ is minimally generated by $\frac{5m+3}{2}$ polynomials, precisely: for any $q = 0, \ldots, \frac{m-1}{2}$ one polynomial $G_{1q}$ of degree $(l_2+l_3)(m-q)+ql_1$, one polynomial $G_{2q}$ of degree $(l_1+l_3)(m-q)+ql_2$, one polynomial $G_{3q}$ of degree $(l_1+l_2)(m-q)+ql_3$, one polynomial $G_{4q}$ of degree $(l_1+l_2)(m-q)+(q+1)l_3-2$, for any $q = 0, \ldots, \frac{m-3}{2}$ one polynomial $G_{5q}$ of degree $(l_1+l_3)(m-q)+(q+1)l_2-2$, where: $G_{1q} = L_{_{11}}^{q}\ L_{_{l_11}}^{q}\ L_{_{12}}^{m-q}\ L_{_{l_22}}^{m-q}\ L_{_{13}}^{m-q}\ L_{_{l_33}}^{m-q}$; $G_{2q} = L_{_{11}}^{m-q}\ L_{_{l_11}}^{m-q}\ L_{_{12}}^{q}\ L_{_{l_22}}^{q}\ L_{_{13}}^{m-q}\ L_{_{l_33}}^{m-q}$; $G_{3q} = L_{_{11}}^{m-q}\ L_{_{l_11}}^{m-q}\ L_{_{12}}^{m-q}\ L_{_{l_22}}^{m-q}\ L_{_{13}}^{q}\ L_{_{l_33}}^{q}$; $G_{4q} = L_{_{11}}^{m-q}\ L_{_{l_11}}^{m-q}\ L_{_{12}}^{m-q}\ L_{_{l_2-12}}^{m-q}\ L_{_{l_22}}^{m-q-1}\ L_{_{13}}^{q+1}\ L_{_{l_3-13}}^{q+1}\ L_{_{l_33}}^{q}$; $G_{5q} = L_{_{11}}^{m-q}\ L_{_{l_11}}^{m-q}\ L_{_{12}}^{q+1}\ L_{_{l_2-12}}^{q+1}\ L_{_{l_22}}^{q}\ L_{_{13}}^{m-q}\ L_{_{l_3-13}}^{m-q}\ L_{_{l_33}}^{m-q-1}$.*

*Proof.* The proof is similar to the proof of Theorem 18 and is omitted. □

**Theorem 22.** *If $p = m - 1$ the ring $A$ is Cohen-Macaulay.*

*Proof.* The proof is similar to the proof of Theorem 19 and is omitted. □

**Remark 23.** Let $Y$ be a finite union of $\sigma$ fat lines of $\mathbb{P}^3_{\mathbf{K}}$, $I_Y$ the defining ideal of $Y$, and $A = K[x_0, \ldots, x_n]/I_Y$ the homogeneous coordinate ring of $Y$. To detect the Cohen-Macaulay property of $A$ is in general very unvolved also in $\mathbb{P}^3_{\mathbf{K}}$. If $I_Y = \cap_{i=1}^{\sigma-1} \mathcal{P}_i^m \cap \mathcal{P}_\sigma^{m+q}$, and $q \in \mathbb{Z}$. We can give a conjecture which we have proved in a finite number of cases with the computer algebra system *CoCoA*.

**Conjecture.** 1. If $\mid q \mid = 2$ and $m$ is even, then $A(Y)$ is Cohen-Macaulay;

2. If $q = -2$ and $m$ is odd, then $A(Y)$ is Cohen-Macaulay;

3. If $q = 2$ and $m$ is odd, then $A(Y)$ is not Cohen-Macaulay;

4. If $\mid q \mid > 2$, then $A(Y)$ is not Cohen-Macaulay for any integer $m$.

We have proved Conjecture with *CoCoA* over the field $\mathbb{Q}$ (and then over any algebraically closed field of characteristic zero) when $m \leq 6$ and $q \leq 6$ $(1 \leq m + q)$.

## References

[1] M.F. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing (1969).

[2] *CoCoA:* A system for doing computations in commutative algebra. Available at http://cocoa.dima.unige.it.

[3] M. Guida, F. Orecchia, Algebraic properties of grids of projective lines, *Journal of Pure and Applied Algebra*, **208**, No. 2 (2007), 603-615.

[4] B. Harbourne, Problems and progress: a survey on fat points in $\mathbf{P}^2$, *Zero Dimensional Schemes and Applications*, Naples (2000), 85-132; *Queen's Papers in Pure and Appl. Math.*, **123**, Queen's Univ., Kingston, ON (2002).

[5] J. Migliore, U. Nagel, Tetrahedral curves, *Int. Math. Res. Not.*, **15** (2005), 899-939.

[6] F. Orecchia, Generalized Hilbert functions of Cohen-Macaulay varieties, *Lecture Notes in Math.*, **997**, Springer, Berlin (1983), 376-390.