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WEIGHTED BIVARIANT POLYNOMIAL INTERPOLATION

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Abstract: Here we study some extremal cases of weighted bivariant polynomial interpolation, mainly when one of the weights is much lower than the other one.

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Fix a field K and integers $b \geq a > 0$. The case $a = b = 1$ is very classical and we will just lifts the classical case to the general one which is heavely used in coding theory. Give weight a to the variable a and weight b to the varable y. For all integers d let $P(d,a,b)$ denote the K-linear subspace of $K[a,b]$ formed by the polynomials with weight degree at most d. Hence $P(d,a,b)$ is formed by the constant polynomials if $0 \leq d < a$. For any $S \subset K^2$ set $P(d,a,b)(-S) = \{f \in P(d,a,b) : f|S \equiv 0\}$. If S is finite, then $\dim(P(d,a,b)(-S)) \geq \dim(P(d,a,b)) - \sharp(S)$. Here we look at conditions on d,a,b,S such that $\dim(P(d,a,b)(-S)) = \dim(P(d,a,b)) - \sharp(S)$, i.e. S gives $\sharp(S)$ independent conditions to $P(d,a,b)$.

Remark 1. Fix $x_0, y_0 \in K$. Set $L := \{y = y_0\}$ and $D := \{x - x_0\}$. Counting monomials we see that $P(d,a,b)(-L) \cong P(d-b,a,b)$ and $P(d,a,b)(-D) \cong P(d-a,a,b)$. The restriction of $P(d,a,b)$ to L induces the linear subspace of $K[x]$ formed by all polynomials of degree at most $\lfloor d/a \rfloor$ and hence it has dimension $\lfloor d/a \rfloor + 1$ The restriction of $P(d,a,b)$ to D induces the linear subspace

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of $K[y]$ formed by all polynomials of degree at most $\lfloor d/b \rfloor$ and hence it has dimension $\lfloor d/b \rfloor + 1$. Notice that every finite subset E of K gives $\min\{\sharp(E), c+1\}$ indipendent condition to the vector space of all univariate polynomials of degree at most x. Hence $P(d,a,b)(-A) = P(d,a,b)(-L)$ and $P(d,a,b)(-B) = P(d,a,b)(-D)$ for all $A \subseteq L$ and $B \subseteq D$ such that $\sharp(A) \geq \lfloor d/a \rfloor + 1$ and $\sharp(B) \geq \lfloor d/b \rfloor + 1$.

Proposition 1. *Fix integers $d > 0$, $s > 0$ and a set $S \subset K^2$ such that $\sharp(S) = s$. Set $u := \lfloor d/b \rfloor$ and $v := \lfloor d/a \rfloor$. Assume $v > u$.*

(i) If $s \leq u + 1$, then $\dim(P(d,a,b)(-S)) = \dim(P(d,a,b)) - \sharp(S)$.

(ii) If $s = u + 2$, then $\dim(P(d,a,b)(-S)) > \dim(P(d,a,b)) - \sharp(S)$ if and only S is contained in a line $\{x = x_0\}$ for some $x_0 \in K$.

Proof. There is a minimal positive integer $m \leq s$ and $x_1, \dots, x_m \in K$, $x_i \neq x_j$ for all $i \neq j$ such that $S \subseteq D_1 \cup \cdots \cup D_m$, where $D_i := \{x = x_i\}$. Set $B_i := S \cap D_i$ and $b_i := \sharp(B_i)$. Hence $\sum_{i=1}^{m} b_i = s$. Up to a permutation of the lines D_i we may assume $b_1 \geq \cdots \geq b_m > 0$. Assume $b_1 \leq u + 1$. This inequality is always satisfied if $s \leq u+1$. Since $b_1 \leq u+1$, Remark 1 shows that B_1 imposes b_1 independent conditions to $P(d,a,b)$. If $m = 1$, we just checked part (i). Assume $m \geq 2$. Notice that S imposes s independent conditions to $P(d,a,b)$ if $S \backslash B_1$ imposes $s - b_1$ independent conditions to $P(d-b,a,b)$. Since $b_2 \leq b_1$, $b_2 \leq u$ if $s \leq u+1$. Hence applying again Remark 1 to $S \backslash B_1$ we get (i) if $m = 2$. If $m \geq 3$, we apply Remark 1 to the set $S \backslash (B_1 \cup B_2)$ and the line D_3. And so on, if $m \geq 4$. To be able to do the last step we need to be sure that $S \cap D_m$ imposes independent conditions to $P(d-(m-1)a,a,b)$. Here we use that $b \geq a$ and hence $v \geq u$ and hence $m - 1 \leq v$. Fix a finite subset A of a line $\{x = x_0\}$. Since $\dim(P(d,a,b))(-A) \geq \dim(P(d-b,a,b)$, Remark 1 gives $\dim(P(d,a,b)) \geq \dim(P(d,a,b)) - u - 1$. This inequality proves the "if" part of (ii). Assume the non-existence of a line $\{x = x_0\}$ containing S. We repeat the first part of the proof. The non-existence of a line $\{x = x_0\}$ containing S implies $m \geq 2$. Hence $b_1 < s = u+1$. The proof just given works verbatim. $\square$

Remark 2. Fix integers $d > 0$, $d \geq b \geq a > 0$, $m > 0$. Set $u := \lfloor d/b \rfloor$ and $v := \lfloor d/a \rfloor$. Fix m integers $b_1 \geq b_2 \geq \cdots \geq b_m > 0$ and assume $b_i \leq u + 2 - i$ for all i and Assume $m \leq v - 1$. Fix $x_1, \dots, x_m \in K$, $x_i \neq x_j$ for all $i \neq j$ and set $D_i := \{x = x_i\}$. Fix any $S_i \subseteq D_i$, $1 \leq i \leq m$, such that $\sharp(S_i) = b_i$ and set $S := \cup_{i=1}^{m} S_i$. The proof of Proposition 1 shows that S imposes $b_1 + \cdots + b_m$ independent conditions to $P(d,a,b)$.

Now we impose a strong restriction on the weights a, b.

Proposition 2. *Set* $u := \lfloor d/b \rfloor$ *and* $v := \lfloor d/b \rfloor$. *Fix an integer* s *such that* $u+2 \le s \le v$ *and* $S \subset K^2$ *such that* $\sharp(S) = s$. S *does not impose independent conditions to* $P(d,a,b)$ *if and only if there is an integer* $e > 0$ *and* $x_i \in K$, $1 \le i \le e$, $x_i \ne x_j$ *for all* $i \ne j$, *such that* $\sum_{i=1}^{e} \sharp(S \cap \{x = x_i\}) > \sum_{i=1}^{e}(u+2-i)$.

Proof. Take $m, b_1, \dots, b_m, D_1, \dots, D_m$ as in the proof of Proposition 1. If $b_1 \ge u+2$, then set $e := 1$, if $b_1 \le u+1$, but $b_1 + b_2 \ge 2u+2$ (i.e. if $b_1 = b_2 = u+1$) then set $e := 2$. And so on. We stop and find e if S does not impose independent conditions to $P(d,a,b)$, because $m \le s$ and hence $m \le v$. □

In the case $a = b$ the proof of Proposition 1 gives the following result whose easy proof is left to the reader.

Proposition 3. *Assume* $b = a$. *Fix integers* $d > 0$, $s > 0$ *and a set* $S \subset K^2$ *such that* $\sharp(S) = s$. *Set* $u := \lfloor d/b \rfloor$.

(i) If $s \le u+1$, *then* $\dim(P(d,a,b)(-S)) = \dim(P(d,a,b)) - \sharp(S)$.

(ii) If $s = u+2$, *then* $\dim(P(d,a,b)(-S)) > \dim(P(d,a,b)) - \sharp(S)$ *if and only there is* $u \in K$ *such that either* S *is contained in a line* $\{x = u\}$ *or it is contained in a line* $\{y = u\}$.

For evaluation codes it is not only important to know if a certain S gives independent conditions to $P(d,a,b)$, but to enumerate all S which give very few independent conditions to $P(d,a,b)$. The first problem is related to the computation of the minimum distance of an evaluation code, while the second problem is related to the list of all codewords with low weight. Concerning this problem we will prove the following result.

Theorem 1. *Fix integers* $d > b \ge a > 0$, $s > 0$ *and set* $u := \lfloor d/b \rfloor$ *and* $v := \lfloor d/a \rfloor$. *Fix* $S \subset K^2$ *such that* $\sharp(S) = s$ *and set* $\epsilon := \dim(P(d,a,b)) - \dim(P(d,a,b)(-S))$.

(i) Assume $v \ge u+2$ *and* $\epsilon \ge s-u-2$. *Then there is* $c \in K$ *such that either* $\sharp(S \cap \{x = c\}) = epsilon$.

(ii) Assume $u+2 \le s \le v$ *and* $\epsilon > 0$. *Then there is an integer* $e > 0$ *and* $x_i \in K$, $1 \le i \le e$, $x_i \ne x_j$ *for all* $i \ne j$, *such that* $\sum_{i=1}^{e} \sharp(S \cap \{x = x_i\}) = \epsilon + \sum_{i=1}^{e}(u+2-i)$.

Proof. The "if" parts follows from Remarks 1 and 2.

(a) Here we prove the "only if" part of (i). Let m be the minimal positive integer $m \le s$ such that $x_1, \dots, x_m \in K$, $x_i \ne x_j$ for all $i \ne j$ and $S \subseteq D_1 \cup \dots \cup D_m$, where $D_i := \{x = x_i\}$. Set $B_i := S \cap D_i$ and $b_i := \sharp(B_i)$. Hence $\sum_{i=1}^{m} b_i = s$. Up to a permutation of the lines D_i we may assume

$b_1 \geq \cdots \geq b_m > 0$. If $b_1 \geq u+2$, then every $f \in P(d,a,b)(-S)$ vanishes on D. If either $b_2 \geq 2$ or $m \geq 3$, we easily get $\epsilon \leq s-u-3$, contradiction. If $m = 2$ and $b_2 = 1$, then take $c := x_1$. Now assume $b_1 \leq u+1$. Let μ denote the maximal integer i such that $2 \leq \mu \leq m$ and $b_i \leq i(u+3-i)$. The proof of Proposition 1 shows that if $m \leq v+1$, then $\cup_{i=1}^{\mu} B_i$ gives $\sum_{i=1}^{\mu} b_i$ independent conditions to $P(d,a,b)$ and that $\epsilon \leq \sum_{i=\mu+1}^{m} \max\{0, u+3-i-b_i\}$. We get a contradiction.

(b) Copy the proof of Proposition 2 to get the "only if" part of (ii). □

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