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# POLYNOMIAL INTERPOLATION WITH RESTRICTED PARTIAL SUPPORT

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

**Abstract:** Here we prove a polynomial interpolation problems for double points in $\mathbf{P}^n$, some of the points being in a fixed hyperplane.

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## 1. Introduction

Fix an integer $n \ge 2$ and a complete flag $H_0 \subsetneqq H_1 \subsetneqq \cdots \subsetneqq H_{n-1} \subsetneqq H_n := \mathbf{P}^n$ of linear subspaces. Thus $\dim(H_i) = i$ for all $i$. Consider the following two polynomial interpolation problems. Fix integers $d > 0$, $a_i \ge 0$, $0 \le i \le n$, such that $a_0 \in \{0,1\}$ and $m_i > 0$, $0 \le i \le n$. For any integral scheme $X$, any $P \in X_{reg}$ and any integer $m > 0$ let $\{mP, X\}$ denote the infinitesimal neighborhood of order $m-1$ of $P$ in $X$, i.e. the closed subscheme of $X$ with $(\mathcal{I}_{X,P})^m$ as its ideal sheaf. Thus $\{mP, X\}_{reg} = \{P\}$ and $\text{length}(\{mP, X\}) = \binom{m+\dim(X)-1}{\dim(X)}$. For any finite subset $A \subset X_{reg}$ set $\{mA, X\} := \cup_{P \in A}\{mP, X\}$. Fix a general $S_i \subset H_i$, $0 \le i \le n$, such that $\sharp(S_i) = a_i$ for all $i$. Set $Z := \cup_{i=0}^{n}\{m_iS_i, H_i\}$ and $W := \cup_{i=0}^{n}\{m_iS_i, H_n\}$. Notice that $\text{length}(Z) = \sum_{i=0}^{n} a_i \cdot \binom{m_i+i-1}{i}$ and $\text{length}(W) = \sum_{i=0}^{n} a_i \cdot \binom{m_i+n-1}{n}$. We will say that the first (resp. second) interpolation problem with respect to these data is nice if either $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$ or $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$ (resp. either $h^0(\mathbf{P}^n, \mathcal{I}_W(d)) = 0$ or $h^1(\mathbf{P}^n, \mathcal{I}_W(d)) =$

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0)). Notice that $h^0(H_i, \mathcal{O}_{H_i}(d)) = \binom{i+d}{i}$ and that for all integers $m > 0$ and $n \geq i > j \geq 0$ and all $P \in H_i$ we have $\{mP, H_i\}|H_j = \{mP, H_j\}$. Thus if $\mathrm{length}(Z) \leq \binom{n+d}{n}$ (resp. $\mathrm{length}(W) \leq \binom{n+d}{n}$) and there is an integer $k$ such that $0 \leq k \leq n-1$ and $\sum_{i=0}^{k} a_i \cdot \binom{m_i+i-1}{i} > \binom{k+d}{k}$ (resp. $\sum_{i=0}^{k} a_i \cdot \binom{m_i+k-1}{k} > \binom{k+d}{k}$) then the first (resp. second) interpolation problem is not nice. Hence we introduce the following conditions $\alpha$ and $\beta$ for the numerical data $n, d, a_i, m_i$. Obviously, we may also consider interpolation problems in which not all points on $H_i$ have the same prescribed multiplicity.

**Definition 1.** Fix numerical data $n, d, a_i, m_i$ as above. We will say that these numerical data have property $\alpha_k$ (resp. property $\beta_k$) for some integers $0 \leq k \leq n-1$ if $\sum_{i=0}^{k} a_i \cdot \binom{m_i+i-1}{i} \leq \binom{k+d}{k}$ (resp. $\sum_{i=0}^{k} a_i \cdot \binom{m_i+k-1}{k} \leq \binom{k+d}{k}$). We will say that they have property $\alpha$ (resp. property $\beta$) if they have property $\alpha_k$ (resp. $\beta_k$) for all $0 \leq k \leq n-1$.

Since we assume $d > 0$ and $a_0 \in \{0, 1\}$, properties $\alpha_0$ and $\beta_0$ are satisfied by all numerical data. The classical case $m_n = 2$ and $a_i = 0$ for all $i \neq 0$ is very important (see [1], [2], [3], [4], and [8]). Here we will assume $m_n = 2$ and $a_i = 0$ for all $i \leq n-2$, but we will consider different multiplicities for the points in the hyperplane $H := H_{n-1}$. We will write $2P$ or $2S$ insetad of $\{2P, \mathbf{P}^n\}$ or $\{2S, \mathbf{P}^n\}$. We will prove the following results.

**Theorem 1.** *Assume* $\mathrm{char}(\mathbb{K}) \neq 2$. *Fix integers* $n, a, b, c, d$ *such that* $n \geq 2$, $a \geq 0$, $b \geq 0$, $c > 0$, $d \geq 3$ *and* $a + nb \leq \binom{n+d-}{n-1}$. *If* $n \geq 3$ *and* $d = 3$ *assume* $a + nb \leq n$. *Fix general* $A \subset H$, $B \subset H$ *and* $S \subset \mathbf{P}^n$ *such that* $\sharp(A) = a$, $\sharp(B) = b$ *and* $\sharp(S) = c$. *Set* $Z := A \cup \{2B, H\} \cup 2S$. *Exclude the following cases:*

*(a)* $(n, d, a, b, c) = (2, 4, 0, 0, 5)$;
*(b)* $(n, d, a, b, c) = (3, 4, 0, 0, 9)$;
*(c)* $(n, d, a, b, c) = (4, 3, 0, 0, 7)$;
*(d)* $(n, d, a, b, c) = (4, 4, 0, 0, 14)$;
*(e)* $(n, d, c) = (2, 3, 2)$, *and either* $(a, b) = (0, 2)$ *or* $(a, b) = (4, 0)$;
*(f)* $(n, d, b) = (5, 4, 14)$; $b \leq 9$.
*Then either* $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$ *or* $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$.

**Theorem 2.** *Assume* $\mathrm{char}(\mathbb{K}) \neq 2$. *Fix integers* $n, a, b, c, d$ *such that* $n \geq 2$, $a \geq 0$, $b \geq 0$, $c > 0$, $d \geq 3$ *and* $a + nb \leq \binom{n+d-}{n-1}$. *If* $n \geq 3$ *and* $d = 3$ *assume* $a + nb \leq n$. *Fix general* $A \subset H$, $B \subset H$ *and* $S \subset \mathbf{P}^n$. *Set* $W := A \cup 2B \cup \cup 2S$. *Exclude the following cases:*

*(a)* $(n, d, a, b, c) = (2, 4, 0, 0, 5)$;
*(b)* $(n, d, a, b, c) = (3, 4, 0, 0, 9)$;

*(c)* $(n,d,a,b,c) = (4,3,0,0,7)$;
*(d)* $(n,d,a,b,c) = (4,4,0,0,14)$;
*(e)* $(n,d,c) = (2,3,2)$, *and either* $(a,b) = (0,2)$ *or* $(a,b) = (4,0)$;
*(f)* $(n,d,b) = (5,4,14)$; $b \leq 9$.
*Then either* $h^0(\mathbf{P}^n, \mathcal{I}_W(d)) = 0$ *or* $h^1(\mathbf{P}^n, \mathcal{I}_W(d)) = 0$.

**Theorem 3.** *Assume* $\mathrm{char}(\mathbb{K}) \neq 2$. *Fix integers* $n, d, a, b, c$ *such that* $n \geq 2$, $d \geq 5$, $a \geq 0$, $b \geq 0$, $a + nb \geq \binom{n+d-1}{n-1}$ *and* $(n+1)c \geq \binom{n+d-1}{n}$. *Exclude the following cases:*
*(a)* $(n,d) = (3,4)$;
*(b)* $(n,d) = (5,4)$;
*(c)* $(n,d) = (5,4)$;
*(d)* $(n,d,c) = (2,5,5)$;
*(e)* $(n,d,c) = (4,5,14)$.
*Fix general* $A \subset H$, $B \subset H$ *and* $S \subset \mathbf{P}^n$ *such that* $\sharp(A) = a$, $\sharp(B) = b$ *and* $\sharp(S) = c$. *Set* $Z := A \cup \{2B, H\} \cup 2S$. *Then* $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$

**Theorem 4.** *Assume* $\mathrm{char}(\mathbb{K}) \neq 2$. *Fix integers* $n, d, a, b, c$ *such that* $n \geq 2$, $d \geq 5$, $a \geq 0$, $b \geq 0$, $a + nb \geq \binom{n+d-1}{n-1}$ *and* $(n+1)c + b \geq \binom{n+d-1}{n}$. *Exclude the following cases:*
*(a)* $(n,d) = (3,4)$;
*(b)* $(n,d) = (5,4)$;
*(c)* $(n,d) = (5,4)$;
*(d)* $(n,d,c,b) = (2,5,5,0)$;
*(e)* $(n,d,c) = (4,5,14,0)$.
*Fix general* $A \subset H$, $B \subset H$ *and* $S \subset \mathbf{P}^n$ *such that* $\sharp(A) = a$, $\sharp(B) = b$ *and* $\sharp(S) = c$. *Set* $W := A \cup 2B \cup 2S$. *Then* $h^0(\mathbf{P}^n, \mathcal{I}_W(d)) = 0$.

**Remark 1.** Take the set-ups of Theorems 1 and 2. In the exceptional cases (a), (b), (c), (d) we have $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) = 1$ (resp. $h^0(\mathbf{P}^n, \mathcal{I}_W(d)) = 1$). In the exceptional cases (a), (c) and (d) we have $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 1$ (resp. $h^1(\mathbf{P}^n, \mathcal{I}_W(d)) = 1$). In the exceptional case (b) we have $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 2$ (resp. $h^1(\mathbf{P}^n, \mathcal{I}_W(d)) = 2$). In case (e) of Theorem 1 we have $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) = h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 1$. In the exceptional case (f) we have $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) \neq 0$, because $h^1(H, \mathcal{I}_{Z\cap H,H}(d)) \leq 0$ and $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) \neq 0$, because

$$h^0(\mathbf{P}^n, \mathcal{I}_{Z\setminus Z\cap H}(d-1)) \neq 0,$$

since $(n+1)b \leq \binom{n+d-1}{n}$.

## 2. The Proofs

For any closed subscheme $Z$ of any projective scheme $A$ and every effective Cartier divisor $D$ of $A$ let $\mathrm{Res}_D(Z)$ denote the residual scheme of of $Z$ with respect to $D$, i.e. the closed subscheme of $A$ with $\mathcal{I}_{Z,A} : \mathcal{I}_{D,A}$ as its ideal sheaf. For any effective Cartier divisor $D$ of $A$ such that $P \in D_{reg}$ we have $2\{P, A\} \cap D = 2\{P, D\}$ and $\mathrm{Res}_D(2\{P, A\}) = \{P\}$. We will often use the following elementary form of the so-called Horace Lemma.

**Lemma 1.** *Let $H \subset \mathbf{P}^n$ be a hyperplane and $Z \subset \mathbf{P}^n$ a closed subscheme. Then:*

*(a) $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) \le h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(Z)}}(d-1)) + h^0(H, \mathcal{I}_{Z\cap H}(d))$;*

*(b) $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) \le h^1(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(Z)}}(d-1)) + h^1(H, \mathcal{I}_{Z\cap H}(d))$.*

*Proof.* By the very definition of residual scheme with respect to $H$, there is the following exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{H(Z)}}(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Z\cap H}(d) \to 0 \qquad (1)$$

whose long cohomology exact sequence proves the lemma. □

The following result (called the differential Horace lemma) is a very particular case of [5], Lemma 2.3 (see in particular Figure 1 at p. 308).

**Lemma 2.** *Let $H \subset \mathbf{P}^n$ be hyperplane, $Z \subset \mathbf{P}^n$ a closed subscheme not containing $H$ and $s$ a positive integer. Let $U$ be the union of $Z$ and $s$ general double points of $\mathbf{P}^n$. Let $S$ be the union of $s$ general points of $H$. Let $E \subset H$ be the union of $s$ general double points of $H$ (not double points of $\mathbf{P}^n$, i.e. each of them has length $n$). To prove $h^1(\mathbf{P}^n, \mathcal{I}_U(d)) = 0$ (resp. $h^0(\mathbf{P}^n, \mathcal{I}_U(d)) = 0$) it is sufficient to prove*

$$h^1(H, \mathcal{I}_{(Z\cap H)\cup S}(d)) = h^1(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(Z)\cup E}}(d-1)) = 0$$

*(resp.*

$$h^0(H, \mathcal{I}_{(Z\cap H)\cup S}(d)) = h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(Z)\cup E}}(d-1)) = 0).$$

**Remark 2.** Assume $\mathrm{char}(\mathbb{K}) \neq 2$. Fix an integral projective variety, $L \in \mathrm{Pic}(X)$ and any linear subspace $W \subseteq H^0(X, L)$. We claim that $\dim(W \cap H^0(X, \mathcal{I}_v \otimes L)) = \max\{0, \dim(W)\}$ for a general $P \in X$ and a general tangent vector $v$ of $X$ with $v_{red} = P$. Indeed, as in [9] we reduce to the case in which $X$ is an integral curve. When $X$ is an integral curve the claim follows from [10] and the assumption $\mathrm{char}(\mathbb{K}) \neq 2$.

**Lemma 3.** *For all integers $n \ge 3$ and $d \ge 4$ we have*

$$\lfloor \binom{n+d-1}{n-1}/n \rfloor + n - 1 \le \lfloor \binom{n+d}{n}/(n+1) \rfloor . \qquad (2)$$

*Proof.* It is sufficient to prove that

$$\binom{n+d-1}{n-1}/n + n - 1 \le \binom{n+d}{n}/(n+1) - 1 . \qquad (3)$$

Assume that (3) fails, i.e. assume

$$(n+1)n(n-1) \ge (n+d-1)!(d-1)/((n-1)!d!) . \qquad (4)$$

The right hand side is an increasing function of $d$. Hence to find a contradiction it is sufficient to check that (4) fails when $d = 4$ for all $n \ge 3$. This is easy. □

**Lemma 4.** *For all integers $n \ge 3$, $d \ge 4$ and $0 \le b \le \lfloor \binom{n+d-1}{n-1}/n \rfloor$ we have*

$$\lfloor \binom{n+d-1}{n-1}/n \rfloor - b + n - 1 \le \lfloor (\binom{n+d}{n} - b)/(n+1) \rfloor . \qquad (5)$$

*Proof.* The case $b = 0$ is Lemma 3. For the general case just note that the difference between the right hand side of (5) and the left hand side of (5) is an increasing function of $b$. □

*Proof of Theorem 1.* We recall that when $a = b = 0$, then the result is true (see [1], [2], [3], [4], [8], [6]). We stress that in the checking of all cases we will always assume $a + b \neq 0$. $Z$ will always denote a scheme with numerical data $n, a, b, c$ or a "virtual" scheme with the same numerical data. Unless otherwise said, $Z$ will be a sufficiently general one, i.e. a general one with the costrains arising in each step of the proof. Set $e := \lfloor (\binom{n+d-1}{n-1} - a - nb)/n \rfloor$ and $f := \binom{n+d-1}{n-1} - a - nb - ne$. Hence $0 \le f \le n-1$. Increasing if necessary $c$, we may (and sometimes will) assume $(n+1)c \ge \binom{n+d-1}{n} - n$.

(a) Here we assume $n = 2$. First we will do the case $d = 3$. First assume $c \ge 3$. Any plane cubic with at least 3 non-collinear points is the union of 3 lines and it is uniquely determined by its singular points. This observation handle all cases with $c \ge 3$. Now assume $c = 2$ and set $D := \langle S \rangle$. The line $D$ is contained in the base locus of $|\mathcal{I}_Z(3)|$. The generality of $Z$ implies $\mathrm{Res}_D(Z) = (Z \backslash 2S) \cup S$. Using Lemma 2 with respect to $H$ and that $a + 2b \le d + 1 = 4$ by assumption, we get that the only exceptional cases are the ones described in case (e) of Theorem 1. Now assume $c = 1$. Since $h^1(\mathbf{P}^2, \mathcal{I}_{2P}(2)) = 0$ for any $P \in \mathbf{P}^2$, Lemma 2 with respect to $H$ gives $h^1(\mathbf{P}^2, \mathcal{I}_Z(3)) = 0$. Now assume $d = 4$. We also exclude case (a) of Theorem 1. Notice that $a + 2b \le d + 1 = 5$. If $a + 2b = 5$, then we apply Lemma 1 and reduce to the case with $d' := 3$, $c' = c$ and $a' = b' = 0$. Now assume $a + 2b \le 4$. First assume $a + 2b + 3c \le 15$. Here we need to check $h^1(\mathbf{P}^2, \mathcal{I}_Z(4)) = 0$. It is easy to check $h^1(\mathbf{P}^2, \mathcal{I}_{2S}(4)) = 0$. Let $W \subseteq h^0(H, \mathcal{O}_H(4))$ be the image of $H^0(\mathbf{P}^2, \mathcal{I}_{2S}(4))$ by the restriction map. Since $\mathrm{char}(\mathbb{K}) \ne 2$, we get that $A \cup \{2B, H\}$ gives $\max\{\dim(W), a + 2b\}$ independent conditions to $W$. Hence we easily get $h^1(\mathbf{P}^2, \mathcal{I}_Z(4)) = 0$. The case $a + 2b + 3c \ge 16$ is similar and left to the reader. Now assume $d \ge 5$. We use induction on $d$ specializing $\lfloor (d+1-a-2b)/2 \rfloor$ of the $c$ double points to double points with support on $H$. If $a + 2b \equiv d \pmod 2$, then we also use once Lemma 2.

(b) Here we assume $d = 3$ and $n \ge 3$. The assumption $a + nb \le n$, implies that every zero-dimensional subscheme of $\mathbf{P}^n$ isomorphic to $H \cap Z$ is projectively equivalent to a scheme contained in a hyperplane. Hence we may apply [7], Theorem 1.1.

(c) Here we assume $d = 4$ and $n \ge 3$. We fix $e + f$ general points and call them $P_1, \dots, P_e, Q_1, \dots, Q_f$ of $H$. We specialize $Z$ so that $e$ of of the double points of $2S$ are supported by the points $P_1, \dots, P_e$ and apply Lemma 1. We apply Lemma 2 to the points $Q_1, \dots, Q_f$ with respect to $f$ different connected components of $2S$. Of course, to do this construction we need to have $c \ge e + f$. This inequality is true by Lemma 3, because $f \le n - 1$. The scheme $\bigcup_{i=1}^{e} P_i \cup \bigcup_{i=1}^{f} \{2Q_i, H\}$ is the intersection with $H$ of the virtual residual scheme $A$ such that if $h^1(\mathbf{P}^n, \mathcal{I}_A(3)) = 0$, then we are done for these numerical data. If $f \ge 2$ we cannot apply part (b) to $A$. We first need to check that $\deg(A \cap H) \le \binom{n+2}{3}$, i.e. that $e + nf \le (n+2)(n+1)n/6$. Since $e + b \le \binom{n+3}{4}/n$ and $f \le n - 1$, it is sufficient to check the inequality

$$(n+3)(n+2)(n+1) + 24n(n-1) \le 4n(n+1)(n+2)\,. \tag{6}$$

This inequality is true for all $n \ge 3$. Hence $h^1(H, \mathcal{I}_{A \cap H}(3)) = 0$ (see [7]). The connected components of $A$ not intersecting $H$ are the union $\Delta$ of $c - e - f$ general double points. Call $\Psi$ the union of the $f$ double points of $\mathbf{P}^n$ contained the $f$ unreduced connected components of $A \cap H$. Since $f \le n - 1$ and any $n$ points of $\mathbf{P}^n$ are contained in a hyperplane, $\Delta \cup \Psi$ may be seen as a general union of $c - e$ double points. Hence $h^1(\mathbf{P}^n, \mathcal{I}_{\Delta \cup \Psi}(3)) = 0$ (see [4], [8]). Hence $h^1(\mathbf{P}^n, \mathcal{I}_{\Delta \cup (\Psi \cap H)}(3)) = 0$. Assume $h^1(\mathbf{P}^n, \mathcal{I}_A(3)) \ne 0$. Since $h^1(\mathbf{P}^n, \mathcal{I}_{\Delta \cup (\Psi \cap H)}(3)) = 0$, $h^1(H, \mathcal{I}_{A \cap H}(3)) = 0$ and the $e$ reduced connected components of $H \cap A$ are general in $H$, this implies $h^0(\mathbf{P}^n, \mathcal{I}_{\Delta \cup H}(3)) \ge h^0(\mathbf{P}^n, \mathcal{I}_{\Delta \cup (\Psi \cap H)}(3)) - e + 1$, i.e. $h^0(\mathbf{P}^n, \mathcal{I}_\Delta(2)) \le \binom{n+3}{3} - (n+1)(c - e - f) -$

$nf - e + 1$, where $\Delta$ is a general union of $c - e - f$ double points. Since the singular locus of a quadric is a linear subspace, we immediately get a numerical contradiction.

(c) Here we assume $n \geq 3$ and $d \geq 5$. We will use double induction on $n$ and $d$. In this case we just use Lemmas 1 and 2 exactly as in part (c). Here for the pair $(n, d-1)$ we may use the full inductive assumption, i.e. we do not need the last part of (c). □

*Proof of Theorem 2.* If $b = 0$, then use Theorem 1 Hence in the checking of all cases we may assume $b > 0$. $W$ will always denote a scheme with numerical data $n, a, b, c$ or a "virtual" scheme with the same numerical data. Unless otherwise said, $W$ will be a sufficiently general one, i.e. a general one with the costraines arising in each step of the proof. Set $e := \lfloor (\binom{n+d-1}{n-1} - a - nb)/n \rfloor$ and $f := \binom{n+d-1}{n-1} - a - nb - ne$.

(a) Here we assume $n = 2$. First we will do the case $d = 3$. First assume $c \geq 3$. Any plane cubic with at least 3 non-collinear points is the union of 3 lines and it is uniquely determined by its singular points. This observation handle all cases with $c \geq 3$. Now assume $c = 2$ and set $D := \langle \{S\} \rangle$. $D$ is contained in the base locus of $|\mathcal{I}_Z(3)|$. The generality of $Z$ implies $\mathrm{Res}_D(Z) = (Z \backslash 2S) \cup S$. Using Lemma 2 with respect to $H$ and that $a + 2b \leq d + 1 = 4$ by assumption, we get that the only exceptional cases are the ones described in case (e) of Theorem 2. Now assume $c = 1$. Since $h^1(\mathbf{P}^2, \mathcal{I}_{2P}(2)) = 0$ for any $P \in \mathbf{P}^2$, Lemma 2 with respect to $H$ gives $h^1(\mathbf{P}^2, \mathcal{I}_Z(3)) = 0$. Now assume $d = 4$. We also exclude case (a) of Theorem 2. Notice that $a + 2b \leq d + 1 = 5$. If $a + 2b = 5$, then we apply Lemma 1 and reduce to the case of Theorem 1 with $d' := 3$, $c' = c$, $a' = b$ and $b' = 0$. Now assume $a + 2b \leq 4$ and $b > 0$, i.e. either $b = 2$, $a = 0$ or $b = 1$ and $0 \leq a \leq 2$. First assume $a + 3b + 3c \leq 15$. Here we need to check $h^1(\mathbf{P}^2, \mathcal{I}_W(4)) = 0$. If $a + 2b = 3$, then we insert a further double point on $H$ and reduce to a statement for Theorem 1 with $d' = 3$, $a' = 2$, $b' = 0$ and $c' = c - 1$. If $a + 2b = 4$ we use Lemma 2 with respect to a point of $H$ and reduce to a case of Theorem 1 with $d' = 3$, $b' = 1$ and $a' = b$. If $(b, a) = (1, 0)$, then we specialize one of the $c$ double points outside $H$ to one with support on $H$ and use Lemma 2 with respect to another double point. We reduce to a case of Theorem 1 with $d' = 3$, $a' = 2$, $b' = 1$ and $c' = c - 2$. The proofs of the cases with $a + 3b + 3c \geq 16$ are easier. Now assume $d \geq 5$. We use induction on $d$ specializing $\lfloor (d + 1 - a - 2b)/2 \rfloor$ of the $c$ double points to double points with support on $H$. If $a + 2b \equiv d \pmod 2$, then we also use once Lemma 2. The only difference with respect to the case $n = 2$ of Theorem 1 is that now on $H$ for the integer $d - 1$ we have $f$ double points of $H$ and $b + e$ (instead of $e$) general points.

(b) Here we assume $d = 3$ and $n \geq 3$. The assumption $a + nb \leq n$, implies that every zero-dimensional subscheme of $\mathbf{P}^n$ isomorphic to $H \cap Z$ is projectively equivalent to a scheme contained in a hyperplane. Hence we may apply [7], Theorem 1.1.

(c) Here we assume $d = 4$ and $n \geq 3$. We fix $e + f$ general points $P_1, \dots, P_e, Q_1, \dots, Q_f$ of $H$. We specialize $Z$ so that $e$ of of the double points of $2S$ are supported by the points $P_1, \dots, P_e$ and apply Lemma 1. We apply Lemma 2 to the points $Q_1, \dots, Q_f$ with respect to $f$ different connected components of $2S$. Of course, to do this construction we need to have $c \geq e + f$. This inequality is true by Lemma 4, because $f \leq n - 1$. The scheme $B \cup \bigcup_{i=1}^{e} P_i \cup \bigcup_{i=1}^{f} \{2Q_i, H\}$ is the intersection with $H$ of the virtual residual scheme $A$ such that if $h^1(\mathbf{P}^n, \mathcal{I}_A(3)) = 0$, then Theorem 2 is true for these numerical data. If $f \geq 2$ we cannot apply part (b) to $A$. We first need to check that $\deg(A \cap H) \leq \binom{n+2}{3}$, i.e. that $e + nf \leq (n+2)(n+1)n/6$. Since $e + b \leq \binom{n+3}{4}/n$ and $f \leq n-1$, it is sufficient to use the inequality (6) proved in part (c) of the proof of Theorem 1. Hence $h^1(H, \mathcal{I}_{A\cap H}(3)) = 0$ (see [7]). The connected components of $A$ not intersecting $H$ are the union $\Delta$ of $c - e - f$ general double points. Call $\Psi$ the union of the $f$ double points of $\mathbf{P}^n$ contained the $f$ unreduced connected components of $A \cap H$. Since $f \leq n - 1$ and any $n$ points of $\mathbf{P}^n$ are contained in a hyperplane $\Delta \cup \Psi$ may be seen as a general union of $c - e$ double points. Hence $h^1(\mathbf{P}^n, \mathcal{I}_{\Delta\cup\Psi}(3)) = 0$ (see [4], [8]). Hence $h^1(\mathbf{P}^n, \mathcal{I}_{\Delta\cup(\Psi\cap H)}(3)) = 0$. Assume $h^1(\mathbf{P}^n, \mathcal{I}_A(3)) \neq 0$. Since $h^1(\mathbf{P}^n, \mathcal{I}_{\Delta\cup(\Psi\cap H)}(3)) = 0$, $h^1(H, \mathcal{I}_{A\cap H}(3)) = 0$ and the $b + e$ reduced connected components of $H \cap A$ are general in $A$, this implies $h^0(\mathbf{P}^n, \mathcal{I}_{\Delta\cup H}(3)) \geq h^0(\mathbf{P}^n, \mathcal{I}_{\Delta\cup(\Psi\cap H)}(3)) - e + 1$, i.e. $h^0(\mathbf{P}^n, \mathcal{I}_\Delta(2)) \leq \binom{n+3}{3} - (n+1)(c-e-f) - nf - e + 1$, where $\Delta$ is a general union of $c - e - f$ double points. Since the singular locus of a quadric is a linear subspace, we immediately get a numerical contradiction.

(d) Here we assume $n \geq 3$ and $d \geq 5$. Look at the corresponding step of the proof of Theorem 1. Now instead of the virtual scheme $Z'$ we have the virtual scheme $W'$ which differ from it only in $b$ connected components: $Z'$ has $\{2B, H\}$ as $b$ of its connected components, while $W'$ has $2B$ as $b$ of its connected components. The virtual residue of $Z'$ with respect to $H$ is of the form $E \cup \{2F, H\} \cup M$ with $M \cap H = \emptyset$, $E \cup F \subset H$, $\sharp(E) = e$, $\sharp(F) = f$, $E \cap F = \emptyset$, and $E \cup F$ general in $H$. The virtual residue of $W'$ with respect to $H$ is of the form $B \cup E \cup \{2F, H\} \cup M$ with $M \cap H = \emptyset$, $B \cup E \cup F \subset H$, $\sharp(B) = B$, $\sharp(E) = e$, $\sharp(F) = f$, $E \cap F = E \cap B = F \cap B = \emptyset$, and $E \cup F \cup B$ general in $H$. Of course, here for the same numerical data $n, d, a, b, c$ the integers $e, f$ may be different. □

*Proof of Theorem 3.* Since we excluded cases (a), (b) and (c), the Alexander-

Hirschowitz Theorem gives $h^0(H, \mathcal{I}_{Z\cap H}(d)) = 0$. Hence Lemma 1 shows that it is sufficient to prove $h^0(\mathbf{P}^n, \mathcal{I}_{2S}(d-1)) = 0$. Since we excluded cases (d) and (e), the last equation is true by the Alexander-Hirschowitz theorem. □

*Proof of Theorem 4.* Since we excluded cases (a), (b) and (c), the Alexander-Hirschowitz Theorem gives $h^0(H, \mathcal{I}_{W\cap H}(d)) = 0$. Since $\mathrm{Res}_H(Z) = b \cup 2S$, Lemma 1 shows that it is sufficient to prove $h^0(\mathbf{P}^n, \mathcal{I}_{B\cup 2S}(d-1)) = 0$. The Alexander-Hirschowitz theorem gives that either $h^0(\mathbf{P}^n, \mathcal{I}_{2S}(d-1)) = 0$ or $h^1(\mathbf{P}^n, \mathcal{I}_{B\cup 2S}(d-1)) \leq 1$ and that $h^1(\mathbf{P}^n, \mathcal{I}_{B\cup 2S}(d-1)) = 1$ if and only if $d = 5$ and either $n = 2$ or $n = 3$. Since $b + (n+1)c \geq \binom{n+d-1}{n}$, we have $h^0(\mathbf{P}^n, \mathcal{I}_{2S}(d-1)) \leq b+1$ and $h^0(\mathbf{P}^n, \mathcal{I}_{2S}(d-1)) = b+1$ only if $d = 5$, $b = 0$, $c = \binom{n+d-1}{n}/(n+1)$ and either $n = 2$ or $n = 3$. We excluded these cases. Hence it is sufficient to prove that $b$ general points of $H$ imposes $h^0(\mathbf{P}^n, \mathcal{I}_{2S}(d-1))$ condition to the linear system $|\mathcal{I}_{2S}(d-1)|$. This is true, because $h^0(\mathbf{P}^n, \mathcal{I}_{2S}(d-2)) = 0$ by a very weak form of the Alexander-Hirschowitz Theorem. □

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