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THE GEOMETRY OF QUANTUM PRINCIPAL BUNDLE COHERENT ALGEBRA SHEAVES

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Abstract: In this article, the main purpose is to continue the construction, using sheaf theory, of quantum principal bundle algebras sheaves started by Markus J. Pflaum in [47]. We generalize his construction to higher order differential calculi and extend this further to obtain quantum principal bundle coherent algebra sheaves (see definition below). The main results of our study include the construction of quantum connections, quantum curvatures, quantum Chern-Weil homomorphism and quantum characteristic classes for the quantum principal bundle coherent algebra sheaves.

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1. Introduction

In this paper we study the quantum (noncommutative) geometry of principal bundle coherent algebra sheaves (see definition below). Markus J. Pflaum was the first to introduce sheaf theoretic approach in noncommutative geometry to

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study the geometry of principal fiber bundles. In his article [47] he started with the principal fiber bundle (P, p, M, G), where M, P are topological spaces, $p : P \to M$ a continuous projection and G a topological group with $P \times G \longrightarrow P$ a free right action. He then used sheaf theory to transform (P, p, M, G) into the principal bundle algebra sheaf $(\mathcal{P}, \pi, \mathcal{M}, \mathcal{F})$, where $\mathcal{P}$ and $\mathcal{M}$ are algebra sheaves with $\mathcal{F}$ the algebra of continuous functions on the structure group G. Next, he replaced the algebra $\mathcal{F}$ by a Hopf algebra $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, S)$, to obtain a quantum principal bundle algebra sheaf $\mathcal{P} = (\mathcal{P}, \pi_p, \mathcal{M}, \mathcal{H}, (\gamma_\alpha)_{\alpha\in\mathfrak{U}})$ with a special coordinate system $(\gamma_\alpha)_{\alpha\in\mathfrak{U}}$.

In our study we extend the results obtained by Markus J. Pflaum in [47] to the study of higher order differential calculi, quantum connection, quantum curvature, quantum Chern-Weil homomorphism and quantum characteristic classes in the category of quantum principal bundle coherent algebra sheaves.

Finally, we give a description of the organization of the paper. In Section 2 we construct higher order quantum differential calculi on the structure quantum group $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, S)$. Section 3 introduces higher order quantum differential calculi on quantum principal bundle coherent algebra sheaves. Section 4 reviews quantum connection and constructs quantum curvature on quantum principal bundle coherent algebra sheaves. Section 5 states the main results and gives their proofs.

2. Higher Order Quantum Differential Calculi on Quantum Groups (Hopf Algebras)

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}$ the monoidal category of associative unital $\mathbb{K}$-algebras. In the sequel $\mathbb{K}$ will be specified to be mostly the field of complex numbers $\mathbb{C}$ with the real numbers $\mathbb{R}$ in few places.

Definition 1. Let (M, g) be an m-dimensional non-compact complete connected paracompact Riemannian manifold. A sheaf of associative unital $\mathbb{K}$-algebras on M is a triple $\mathcal{A}\colon = (\mathcal{A}, \rho, M)$ such that:

(1) $\mathcal{A}$ is a sheaf of rings on M, with multiplication: $m_{\mathcal{A}}\colon \mathcal{A}\otimes\mathcal{A} \longrightarrow \mathcal{A}$ and $\mathcal{A}$ is formally smooth (see [37]) if the $\mathcal{A}$-bimodul $\Omega^1(\mathcal{A}) := \ker\{m_{\mathcal{A}}\colon \mathcal{A}\otimes\mathcal{A} \longrightarrow \mathcal{A}\}$ is projective.

(2) $\rho\colon \mathcal{A} \longrightarrow M$ is a surjective morphism and for each $p \in M$, the stalk $\mathcal{A}_p\colon = \rho^{-1}(p)$ of $\mathcal{A}$ is an associative unital $\mathbb{K}$-algebra with the unit element $1_p \in \mathcal{A}_p$ so that the corresponding section: $M \longrightarrow \mathcal{A}\colon p \longmapsto 1_{(p)} = 1_p \in \mathcal{A}$ is

smooth, and

(3) The scalar multiplication in $\mathcal{A}$: $\bullet\colon \mathbb{K}\times\mathcal{A}\longrightarrow\mathcal{A}\colon (\lambda,a)\longmapsto\lambda\cdot a\in\mathcal{A}_p\subset\mathcal{A}$ is smooth, where $\mathbb{K}$ is endowed with the discrete topology. Denote the monoidal category of associative unital $\mathbb{K}$-algebra sheaves by $\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}$-$\mathcal{S}h_M$.

Definition 2. A (graded) algebra $A\in Ob(\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}})$ is called (graded) right (left) coherent, if the following equivalent conditions hold:

(i) every (homogeneous) finitely generated right-sided (left-sided) ideal in A is finitely presented, that is, A is (graded) coherent as a right (left) module over itself;

(ii) every finitely presented (graded) right (left) A-module is (graded) coherent. Note that all the finitely presented (graded) right (left) A-modules form an abelian category. We denote, this monoidal category of associative unital $\mathbb{K}$-coherent algebras by $\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}$-algebras and the associated monoidal category of associative unital $\mathbb{K}$- coherent algebra sheaves by $\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}$-$\mathcal{S}h_M$.

Throughout, this paper we will fix the m-dimensional non-compact complete paracompact connected Riemannian manifold (M,g) and take $Op(M)$ to be the category with objects the locally finite open cover $\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ of M and morphisms inclusion maps $\imath_{\alpha\beta}\colon U_\alpha\hookrightarrow U_\beta$ if and only if $U_\alpha\subset U_\beta$ and $\alpha\prec\beta$ for all $\alpha,\beta\in\mathfrak{A}$, where $\prec$ is an order relation satisfying the conditions:

(i) $\alpha\prec\alpha$, for all $\alpha\in\mathfrak{A}$,

(ii) if $\alpha\prec\beta$ and $\beta\prec\gamma$, then $\alpha\prec\gamma$,

(iii) for any $\alpha,\beta\in\mathfrak{A}$, there exists $\gamma\in\mathfrak{A}$ such that $\alpha\prec\gamma$ and $\beta\prec\gamma$.

It may happen that $\alpha\neq\beta$ but $\alpha\prec\beta$ and $\beta\prec\alpha$ simultaneously. The family $\{\cup_\alpha\}_{\alpha\in\mathfrak{A}}$ of open subsets of M indexed by the directed set $\mathfrak{A}$ will be called a directed system if, for any pair (α,β) with $\alpha\prec\beta$, there exists an inclusion morphism $\imath_{\alpha\beta}\colon U_\alpha\hookrightarrow U_\beta$ such that $\imath_{\alpha\alpha}\colon U_\alpha\hookrightarrow U_\alpha$, i.e. $\imath_{\alpha\alpha}=I_{U_\alpha}$ and $\imath_{\alpha\beta}\circ\imath_{\beta\gamma}=\imath_{\alpha\gamma}$, if $\alpha\prec\beta\prec\gamma$ and we have the inclusion composite morphism $\imath_{\alpha\gamma}\colon U_\alpha\hookrightarrow U_\beta\hookrightarrow U_\gamma$.

In our noncommutative setting the coherent algebra sheaves we use should be thought of in the sense of category of coherent algebra sheaves given by the notion of a bundle of localizations formulated in the following definition.

Definition 3. (see [52]) A bundle of localizations $((M,g),C,\rho)$, consists of the fixed m-dimensional non-compact complete paracompact connected Riemannian manifold (M,g) endowed with the category $Op(M)$ with objects the locally finite open cover $\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ of M and for each $U_\alpha\subset M$ a category $C(U_\alpha)$

of monoidal associative unital $\mathbb{K}$-coherent algebras such that for each inclusion morphism $\imath_{\alpha\beta}\colon U_\alpha \hookrightarrow U_\beta$ an exact right adjoint localization morphism $C(\imath_{\alpha\beta})\colon C(U_\beta) \longrightarrow C(U_\alpha)$. For any pair of inclusions $\imath_{\alpha\beta}\colon U_\alpha \hookrightarrow U_\beta$ and $\imath_{\beta\lambda}\colon U_\beta \hookrightarrow U_\lambda$, the function ρ assigns a functor isomorphism

$$\rho_{\imath_{\beta\lambda},\imath_{\alpha\beta}}\colon C(\imath_{\alpha\beta}) \circ C(\imath_{\beta\lambda}) \longrightarrow C(\imath_{\beta\lambda}\imath_{\alpha\beta}),$$

such that for any three inclusion morphisms $\imath_{\alpha\beta}\colon U_\alpha \hookrightarrow U_\beta$, $\imath_{\beta\lambda}\colon U_\beta \hookrightarrow U_\lambda$ and $\imath_{\lambda\gamma}\colon U_\lambda \hookrightarrow U_\gamma$ the composite map exists, the following diagram

$$\begin{array}{ccc} C(\imath_{\alpha\beta}) \circ C(\imath_{\beta\lambda}) \circ C(\imath_{\lambda\gamma}) & \xrightarrow{\rho_{\imath_{\beta\lambda},\imath_{\alpha\beta}}C(\imath_{\lambda\gamma})} & C(\imath_{\beta\lambda}\imath_{\alpha\beta}) \circ C(\imath_{\beta\gamma}\imath_{\alpha\beta}) \\ {\scriptstyle C(\imath_{\alpha\beta})\rho_{\imath_{\lambda\gamma},\imath_{\beta\lambda}}}\big\downarrow & & \big\downarrow{\scriptstyle \rho_{\imath_{\lambda\gamma},\imath_{\beta\lambda}\imath_{\alpha\beta}}} \\ C(\imath_{\alpha\beta}) \circ C(\imath_{\lambda\gamma}\imath_{\beta\lambda}) & \xrightarrow{\rho_{\imath_{\lambda\gamma}\imath_{\beta\lambda},\imath_{\alpha\beta}}} & C(\imath_{\lambda\gamma}\imath_{\beta\lambda}\imath_{\alpha\beta}) \end{array}$$

commutes and $C(id) = Id$, $\rho_{id,\imath_{\alpha\beta}} = \rho_{\imath_{\alpha\beta},id} = id$.

Definition 4. A presheaf $\mathcal{F}$ in a bundle of localizations $\mathcal{M} = ((M,g), C,\rho)$ is a function which assigns to an open set $U_\alpha \in \{U_\alpha\}_{\alpha\in\mathfrak{A}}$ an object $\mathcal{F}(U_\alpha)$ of the monoidal category $C(U_\alpha)$ of associative unital $\mathbb{K}$-coherent algebras and to any inclusion morphism $\imath_{\alpha\beta}\colon U_\alpha \hookrightarrow U_\beta$ a morphism $\mathcal{F}(\imath_{\alpha\beta})\colon\ \mathcal{F}(U_\alpha) \longrightarrow \mathcal{F}(U_\beta)$ satisfying the compatibilty conditions : For any pair of inclusion morphisms $\imath_{\alpha\beta}\colon U_\alpha \hookrightarrow U_\beta, \imath_{\beta\lambda}\colon U_\beta \hookrightarrow U_\lambda$, the following diagram is commutative

$$\begin{array}{ccc} C(\imath_{\alpha\beta}) \circ C(\imath_{\beta\lambda}) & \xrightarrow{C(\imath_{\alpha\beta})\mathcal{F}(\imath_{\beta\lambda}))} & C(\imath_{\alpha}\mathcal{F}(U_\lambda)) \\ {\scriptstyle \rho_{\imath_{\beta\lambda},\imath_{\alpha\beta}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{F}(\imath_{\alpha\beta})} \\ C(\imath_{\beta\lambda}\imath_{\alpha\beta}) & \xrightarrow{\mathcal{F}(\imath_{\beta\lambda}\imath_{\alpha\beta})} & \mathcal{F}(U_\alpha) \end{array}$$

Morphisms from presheaf $\mathcal{F}$ to presheaf $\mathcal{G}$ are functions η that assign to any open subset $U_\alpha \in \{U_\alpha\}_{\alpha\in\mathfrak{A}}$ a morphism $\eta(U_\alpha)\colon \mathcal{F}(U_\alpha \longrightarrow \mathcal{G}(U_\alpha)$ such that for any inclusion morphism $\imath_{\alpha\beta}\colon U_\alpha \hookrightarrow U_\beta$ we obtain

$$\mathcal{G}(\imath_{\alpha\beta}) \circ C(\imath_{\alpha\beta})(\eta(U_\beta)) = \eta(U_\alpha) \circ \mathcal{F}(\imath_{\alpha\beta}).$$

The composition of any two such morphisms η_1 and η_2 is defined by $\eta_2{\circ}\eta_1(U_\alpha) = \eta_2(U_\alpha) \circ \eta_1(U_\alpha)$. We denote the category of presheaves on $\mathcal{M}$ by $\mathcal{P}rsh_{\mathcal{M}}$ and call a presheaf $\mathcal{F}$ quasi-coherent, if the morphism $\mathcal{F}(\imath_{\alpha\beta})\colon \mathcal{F}(U_\alpha) \longrightarrow \mathcal{F}(U_\beta)$ is an isomorphism. Let $\mathcal{QC}oh\mathcal{M}$ be the subcategory of $\mathcal{P}rsh_{\mathcal{M}}$ generated by quasi-coherent presheaves. We easily obtain from this the Serre category of coherent algebra sheaves.

We begin this section by defining graded algebras and putting some notation in place for use throughout this study. We then construct higher order quantum differential calculi on the structure quantum group $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, S)$ (the unital Hopf $\mathbb{K}$-algebra) of the quantum principal bundle $\mathbb{K}$-coherent algebra sheaf $\mathcal{P} = (\mathcal{P}, \pi_{\mathcal{P}}, \mathcal{M}, \mathcal{H}, (\gamma_\alpha)_{\alpha\in\mathfrak{A}})$ over (M, g) with coordinate system $(\gamma_\alpha)_{\alpha\in\mathfrak{A}}$.

Definition 5. A graded unital $\mathbb{K}$-coherent algebra sheaf $\mathcal{A}^\bullet := \bigoplus_{n\geq 0} \mathcal{A}^n$ consists of $\mathcal{A}^n \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$ together with a graded tensor product $\hat{\otimes}\colon \mathcal{A}^\bullet \times \mathcal{A}^\bullet \longrightarrow \mathcal{A}^\bullet$ and a differential $d_{\mathcal{A}}\colon \mathcal{A}^\bullet \longrightarrow \mathcal{A}^\bullet$ such that:

(I) $\mathcal{A}^k \hat{\otimes} \mathcal{A}^l \subset \mathcal{A}^{k+l}$, for all $k, l \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$.

(II) $d_{\mathcal{A}}^2 = 0$ and $d_{\mathcal{A}}(\mathcal{A}^k) \subset \mathcal{A}^{k+l}$ for $k, l \in \mathbb{N}^0$.

(III) $d_{\mathcal{A}}(\omega \hat{\otimes} \eta) = d_{\mathcal{A}}\omega \hat{\otimes} \eta + (-1)^k \omega \hat{\otimes} d_{\mathcal{A}}\eta$, for all $\omega \in \mathcal{A}^k(U_\alpha)$ and $\eta \in \mathcal{A}^l(U_\alpha)$, where $(U_\alpha)_{\alpha\in\mathfrak{A}}$ is a locally finite open cover of (M, g) in the category $Op(M)$ with morphisms, inclusion maps $\imath_{\beta\alpha} : U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subset U_\alpha$ in $Op(M)$. Note that the graded $\mathbb{K}$-coherent algebra sheaf $\mathcal{A}^\bullet := \bigoplus_{n\geq 0} \mathcal{A}^n$ is said to be graded commutative if $s \hat{\otimes} t = (-1)^{kl} t \hat{\otimes} s$ for all $s \in \mathcal{A}^k(U_\alpha)$ and $t \in \mathcal{A}^l(U_\alpha)$.

Let $\mathcal{A}^\bullet = \bigoplus_{n\geq 0} \mathcal{A}^n$ and $\mathcal{B}^\bullet = \bigoplus_{m\geq 0} \mathcal{B}^m$ be two graded unital $\mathbb{K}$-coherent algebra sheaves. A $\mathbb{K}$-algebra sheaf morphism $\varphi : \mathcal{A}^\bullet \longrightarrow \mathcal{B}^{\bullet+j}$ of degree $j \in \mathbb{N}^0$ is a family of $\mathbb{K}$-algebra sheaf morphisms $\varphi : \mathcal{A}^k \longrightarrow \mathcal{B}^{k+j}, k \in \mathbb{N}^0$, such that for all $U_\alpha \supset U_\beta$ in $Op(M)$ the following diagrams:

$$\begin{array}{ccc} \mathcal{A}^k(U_\alpha) & \xrightarrow{\varphi_{U_\alpha}} & \mathcal{B}^{k+j}(U_\alpha) \\ {\scriptstyle \rho^{\mathcal{A}}_{U_\alpha U_\beta}}\big\downarrow & & \big\downarrow{\scriptstyle \rho^{\mathcal{B}}_{U_\alpha U_\beta}} \\ \mathcal{A}^k(U_\beta) & \xrightarrow{\varphi_{U_\beta}} & \mathcal{B}^{k+j}(U_\beta) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{A}^k(U_\alpha) & \xrightarrow{\varphi_{U_\alpha}} & \mathcal{B}^{k+j}(U_\alpha) \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ \mathcal{A}^{k+1}(U_\alpha) & \xrightarrow{\varphi_{U_\alpha}} & \mathcal{B}^{k+j+1}(U_\alpha) \end{array}$$

are commutative.

To construct general higher order quantum differential forms suitable for our purposes we introduce the following concepts following Max Karoubi [29].

Let $\mathcal{A} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$. A differential quasi-resolution of the $\mathbb{K}$-coherent algebra sheaf of a higher order differential calculus on $\mathcal{A}$ is a differential graded $\mathbb{K}$-coherent algebra sheaf $\Omega^\bullet(\mathcal{A}) := \bigoplus_{n\geq 0} \Omega^n(\mathcal{A})$, with product the wedge $\wedge$ and for all $n \in \mathbb{N}^0$, $U_\alpha \in (U_\alpha)_{\alpha\in\mathfrak{A}}, \Omega^n(\mathcal{A}(U_\alpha))$ is spanned by elements of the form $a_0 d_{\mathcal{A}} a_1 \wedge d_{\mathcal{A}} a_2 \cdots \wedge d_{\mathcal{A}} a_n$ for $a_j \in \mathcal{A}(U_\alpha)$ together with a differential $d_{\mathcal{A}}\colon \Omega^\bullet(\mathcal{A}) \longrightarrow \Omega^\bullet(\mathcal{A})$ such that:

(I) $\Omega^0(\mathcal{A}) = \mathcal{A}$.

(II) Leibniz Rule: $d_{\mathcal{A}}(\omega\wedge\eta) = d_{\mathcal{A}}\omega\wedge\eta+(-1)^k\omega\wedge d_{\mathcal{A}}\eta$ for all $\omega \in \Omega^k(\mathcal{A}(U_\alpha))$ and $\eta \in \Omega^l(\mathcal{A}(U_\alpha))$.

(III) $d_{\mathcal{A}}^2 = 0$ and $d_{\mathcal{A}}(\Omega^k(\mathcal{A})) \subset \Omega^{k+1}(\mathcal{A})$.

In particular, the homogeneous components $\Omega^k(\mathcal{A})$ are $\mathcal{A}$-bimodules. Further, suppose $\Omega^\bullet(\mathcal{A})$, $\Omega^\bullet(\mathcal{B})$ are higher order differential calculi on $\mathcal{A}, \mathcal{B} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$.

We call a $\mathbb{K}$-algebra sheaf morphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ differentiable if there exists a $\mathbb{K}$-algebra sheaf morphism $\Omega^\bullet(\varphi) : \Omega^\bullet(\mathcal{A}) \longrightarrow \Omega^\bullet(\mathcal{B})$ of differential graded $\mathbb{K}$-coherent algebra sheaves with $\Omega^0(\varphi) = \varphi$. That is,

$$\Omega^\bullet(\varphi) : \Omega^\bullet(\mathcal{A}) := \bigoplus_{n\geq 0} \Omega^n(\mathcal{A}) \longrightarrow \Omega^\bullet(\mathcal{B}) := \bigoplus_{n\geq 0} \Omega^n(\mathcal{B})$$

is a family of $\mathbb{K}$-algebra sheaf morphisms, $\Omega^n(\varphi) : \Omega^n(\mathcal{A}) \longrightarrow \Omega^n(\mathcal{B})$ such that the diagram:

$$\begin{array}{ccc} \Omega^n(\mathcal{A}) & \xrightarrow{\Omega^n(\varphi)} & \Omega^n(\mathcal{B}) \\ \downarrow d & & d\downarrow \\ \Omega^{n+1}(\mathcal{A}) & \xrightarrow{\Omega^{n+1}(\varphi)} & \Omega^{n+1+}(\mathcal{B}) \end{array}$$

is commutative.

Remark 1. If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is differentiable with respect to the differential calculi $\Omega^\bullet(\mathcal{A})$ and $\Omega^\bullet(\mathcal{B})$ then it is easily seen that the induced $\mathbb{K}$-algebra sheaf morphism $\Omega^\bullet(\varphi) : \Omega^\bullet(\mathcal{A}) \longrightarrow \Omega^\bullet(\mathcal{B})$ is unique, since $\Omega^\bullet(\mathcal{A})$ is generated by $\Omega^0(\mathcal{A}) = \mathcal{A}$ and $d_{\mathcal{A}}\mathcal{A}$.

Definition 6. Let

$$\mathcal{A} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}} - \mathcal{S}h_M), \quad \mathcal{M} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}} - \mathcal{S}h_M)$$

an $\mathcal{A}$-module and $\Omega^\bullet(\mathcal{M}) := \bigoplus_{n\geq 0} \Omega^n(\mathcal{M})$ a graded $\mathcal{A}$-module sheaf with product $\wedge$ not necessarily commutative together with a differential $d_{\mathcal{M}} : \Omega^\bullet(\mathcal{M}) \longrightarrow \Omega^{\bullet+1}(\mathcal{M})$ of degree 1 such that:

(I) $\Omega^0(\mathcal{M}) = \mathcal{M}$.

(II) Leibniz Rule: $d_{\mathcal{M}}(\omega \wedge \eta) = d_{\mathcal{M}}\omega \wedge \eta + (-1)^k\omega \wedge d_{\mathcal{M}}\eta$ for all $\omega \in \Omega^k(\mathcal{M}(U_\alpha))$ and $\eta \in \Omega^l(\mathcal{M}(U_\alpha))$, where $(U_\alpha)_{\alpha\in\mathfrak{A}}$ is a locally finite open cover of (M, g) in $Op(M)$ with morphisms, inclusion maps $\imath_{\beta\alpha} : U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subset U_\alpha$ in $Op(M)$.

(III) $d_{\mathcal{M}}^2\mathcal{M} = 0$.

In particular, the $\mathcal{A}$-modules $\Omega^n(\mathcal{M})$ are $\mathcal{M}$-bimodules. We call $\Omega^\bullet(\mathcal{M})$ a differential quasi-resolution of the $\mathbb{K}$-coherent algebra sheaf module $\mathcal{M}$.

An important example is the universal differential quasi-resolution.

Definition 7. Let $\mathcal{A} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$ with multiplication a $\mathbb{K}$-algebra sheaf morphism: $\mu : \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \longrightarrow \mathcal{A}$, set $\Omega^0_{univ}(\mathcal{A}) = \mathcal{A}$ and define,

$$\Omega^1_{univ}(\mathcal{A}) := \mathrm{Ker}\mu := \mathrm{Ker}\{\mu : \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \longrightarrow \mathcal{A}\}.$$

It is then immediate that $\Omega^1_{univ}(\mathcal{A})$ is an $\mathcal{A}$-bimodule. For any $n \in \mathbb{N}^0$, define $\Omega^n_{univ}(\mathcal{A})$ as the tensor product:

$$\Omega^n_{univ}(\mathcal{A}) := \Omega^1_{univ}(\mathcal{A}) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1_{univ}(\mathcal{A})$$

of n-copies of $\Omega^1_{univ}(\mathcal{A})$. It is clear that $\Omega^\bullet_{univ}(\mathcal{A}) := \bigoplus_{n\geq 0} \Omega^n_{univ}(\mathcal{A})$ is a graded $\mathbb{K}$-coherent algebra sheaf and an $\mathcal{A}$-bimodule. Next we need to define a differential

$$d_{univ} : \Omega^n_{univ}(\mathcal{A}) \longrightarrow \Omega^{n+1}_{univ}(\mathcal{A}).$$

First, for $n = 0$, define $d_{univ}(a) = d(a)$ for any $a \in \Omega^0_{univ}(\mathcal{A}(U_\alpha)) = \mathcal{A}(U_\alpha)$ by:

$$d_{univ}a := 1 \otimes a - a \otimes 1.$$

It is then easy to see that the Leibnitz rule $d_{univ}(ab) = d_{univ}(a)b - ad_{univ}(b)$, holds for all $a, b \in \mathcal{A}(U_\alpha)$.

For the other elements of $\Omega^\bullet_{univ}(\mathcal{A})$ the differential d_{univ} is defined as follows: Any element $\omega \in \Omega^1_{univ}(\mathcal{A}(U_\alpha))$ can be written as a finite sum $\omega = \sum b_j d_{univ} a_j$ with $a_j, b_j \in \mathcal{A}(U_\alpha)$ satisfying $\sum b_j a_j = 0$. Indeed by definition $\omega = \sum b_j \otimes a_j$, where $a_j, b_j \in \mathcal{A}(U_\alpha)$ and $\sum b_j a_j = 0$. Thus

$$\omega = \sum b_j \otimes a_j - (\sum b_j \otimes a_j) \otimes 1 = \sum_j b_j \otimes a_j - \sum (b_j a_j \otimes 1)$$
$$= \sum b_j (1 \otimes a_j - a_j \otimes 1) = \sum b_j d_{univ} a_j.$$

This expression is however, not unique.

Define $d_{univ}(\omega)$ for $\omega = \sum b_j da_j \in \Omega^1_{univ}(A(U_\alpha))$ by the formula

$$d_{univ}(\omega) = \sum d_{univ}(b_j) \otimes_{\mathcal{A}} d_{univ}(a_j) \in \Omega^2_{univ}(\mathcal{A}).$$

For general elements of $\Omega^\bullet_{univ}(\mathcal{A})$, define d_{univ} by the Leibnitz rule:

$$d_{univ}(\omega \wedge \eta) := d_{univ}(\omega) \wedge \eta + (-1)^k \omega \wedge d_{univ}\eta,$$

for all $\omega \in \Omega^k_{univ}(\mathcal{A}(U_\alpha))$, and $\eta \in \Omega^l_{univ}(\mathcal{A}(U_\alpha))$.

The following result shows why we call $\Omega^\bullet_{univ}(\mathcal{A})$ the universal quasi-resolution of $\mathcal{A} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$.

Theorem 1. *Let $\mathcal{A} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$ and $\Omega^\bullet(\mathcal{A}) = \bigoplus_{n\geq 0} \Omega^n(\mathcal{A})$ with differential $d_{univ} : \Omega^\bullet(\mathcal{A}) \longrightarrow \Omega^\bullet(\mathcal{A})$ be a differential quasi-resolution of $\mathcal{A}$. Then there exists a unique $\mathbb{K}$-algebra sheaf morphism of differential graded $\mathbb{K}$-coherent algebra sheaves $\Psi : \Omega^\bullet_{univ}(\mathcal{A}) \longrightarrow \Omega^\bullet(\mathcal{A})$ which is an identity on $\Omega^0_{univ}(\mathcal{A}) = \Omega^0(\mathcal{A}) = \mathcal{A}$.*

Proof. Let $d_{univ} : \Omega^\bullet_{univ}(\mathcal{A}) \longrightarrow \Omega^{\bullet+1}_{univ}(\mathcal{A})$ be the differential operator on $\Omega^\bullet_{univ}(\mathcal{A})$ and $d : \Omega^\bullet(\mathcal{A}) \longrightarrow \Omega^{\bullet+1}(\mathcal{A})$.

Uniqueness. We first show that Ψ is unique. The assumption $\Omega^0_{univ}(\mathcal{A}) = \Omega^0(\mathcal{A}) = \mathcal{A}$ implies that $\Psi(a) = a$, for every $a \in \mathcal{A}(U_\alpha)$.

Observe that the diagram:

$$\begin{array}{ccc} \Omega^\bullet_{univ}(\mathcal{A}) & \xrightarrow{\Psi} & \Omega^\bullet(\mathcal{A}) \\ \downarrow{\scriptstyle d_{univ}} & & {\scriptstyle d}\downarrow \\ \Omega^{\bullet+1}_{univ}(\mathcal{A}) & \xrightarrow{\Psi} & \Omega^{\bullet+1}(\mathcal{A}) \end{array}$$

is commutative, i.e. $d \circ \Psi = \Psi \circ d_{univ}$, since Ψ is a $\mathbb{K}$-algebra sheaf morphism of differential graded $\mathbb{K}$-coherent algebra sheaves. Thus

$$\Psi(\omega) = \sum b_{j_1} d_{univ} a_{j_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} b_{j_k} d_{univ} a_{j_k},$$

where $\omega = \sum b_{j_1} d_{univ} a_{j_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} b_{j_k} d_{univ} a_{j_k}$, for all $\omega \in \Omega^k_{univ}(\mathcal{A}(U_\alpha))$ with $\sum b_{j_i} d_{univ} a_{j_i} \in \Omega^1_{univ}(\mathcal{A}(U_\alpha))$.

Existence. To show existence, define

$$\Psi(\omega) := \sum b_{j_1} d_{univ} a_{j_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} b_{j_k} d_{univ} a_{j_k}.$$

It is then clear that Ψ is well-defined. □

Following Micho Durdevich [18, 19] and S.L. Woronowicz [57] we recall the following constructions for a unital Hopf $\mathbb{K}$-algebra $(\mathcal{H}, m, \eta, \Delta, \epsilon, S)$.

Let $ad_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ be the coadjoint action of $\mathcal{H}$ defined by

$$ad_{\mathcal{H}}(h) := \sum h_{(2)} S(h_{(1)}) h_{(3)},$$

for every $h \in \mathcal{H}$ and $(\Omega^1(\mathcal{H}), d_{\mathcal{H}})$ the first order quantum differential calculi on $\mathcal{H}$. Then for any $n \in \mathbb{N}^0$, with $n \geq 2$, define $T^n \underline{ns}(\mathcal{H})$ as the tensor product:

$$T^n \underline{ns}(\mathcal{H}) := \Omega^1_{inv}(\mathcal{H}) \otimes_{\mathcal{H}} \Omega^1_{inv}(\mathcal{H}) \otimes_{\mathcal{H}} \cdots \otimes_{\mathcal{H}} \Omega^1_{inv}(\mathcal{H}),$$

of n-copies of $\Omega^1(\mathcal{H})$. Further, let $T^\bullet \underline{ns}(\mathcal{H}) := \bigoplus_{n\geq 0} T^n \underline{ns}(\mathcal{H})$ be the graded $\mathbb{K}$- algebra over $\Omega^1(\mathcal{H})$ and $Sym^\bullet(\mathcal{H})$ the graded ideal in $T^n \underline{ns}(\mathcal{H})$ defined by

$$Sym^\bullet(\mathcal{H}) := \{\omega \in (\Omega^1(\mathcal{H}))^{\otimes^2_{\mathcal{H}}} : \omega = \sum_j d_{\mathcal{H}}(h_j) \otimes d_{\mathcal{H}}(k_j),$$
$$\sum h_j d_{\mathcal{H}}(k_j) = 0\,,\ \forall h_j, k_j \in \mathcal{H}\}.$$

Define the corresponding factor algebra by

$$\hat{T}^\bullet \underline{ns}(\mathcal{H}) := T^\bullet \underline{ns}(\mathcal{H}) \diagup Sym^\bullet(\mathcal{H}).$$

It is then shown (see Micho Durdevich [18, 19], S.L. Woronowicz [57]) that there exists a unique $\mathbb{K}$-linear map

$$\hat{d} : \hat{T}^\bullet \underline{ns}(\mathcal{H}) \longrightarrow \hat{T}^\bullet \underline{ns}(\mathcal{H})$$

extending the derivation $d_{\mathcal{H}} : \mathcal{H} \longrightarrow \Omega^1(\mathcal{H})$, such that:

(I) $\hat{d}^2 = 0$ and $\hat{d}(\hat{T}^k \underline{ns}(\mathcal{H})) \subset (\hat{T}^{k+1} \underline{ns}(\mathcal{H}))$, for the homogeneous components of $T^\bullet \underline{ns}(\mathcal{H})$.

(II) Leibnitz Rule: $\hat{d}(\omega \otimes \eta) = d(\omega) \otimes \eta + (-1)^k \omega \otimes \hat{d}\eta$, for each $\omega \in \hat{T}^k \underline{ns}(\mathcal{H})$ and $\eta \in \hat{T}^l \underline{ns}(\mathcal{H})$.

Definition 8. Let $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, S)$ be a unital Hopf $\mathbb{K}$-algebra and $(\Omega^1(\mathcal{H}), d_{\mathcal{H}})$ a first order quantum differential calculus on $\mathcal{H}$. We say that $(\Omega^1(\mathcal{H}), d_{\mathcal{H}})$ is left-covariant if

$$\sum h_j dk_j = 0 \Longrightarrow \sum \Delta(h_j)(1_{\mathcal{H}} \otimes d_{\mathcal{H}})\Delta(k_j) = 0$$

for all $h_j, k_j \in \mathcal{H}, 1 \leq j \leq n, n \in \mathbb{N}$.

Thus there exists a $\mathbb{K}$-linear morphism:

$$\Delta_{\Omega^1(\mathcal{H})} : \Omega^1(\mathcal{H}) \longrightarrow \mathcal{H} \otimes_{\mathbb{K}} \Omega^1(\mathcal{H})$$

such that the diagrams:

$$\begin{array}{ccc} \Omega^1(\mathcal{H} & \xrightarrow{{}_{left}\Delta_{\Omega^1(\mathcal{H})}} & \mathcal{H} \otimes_{\mathbb{K}} \Omega^1(\mathcal{H}) \\ {\scriptstyle {}_{left}\Delta_{\Omega^1(\mathcal{H})}} \downarrow & & \downarrow {\scriptstyle {}_{left}\Delta_{\Omega^1(\mathcal{H})} \otimes 1_{\Omega^1(\mathcal{H})}} \\ \mathcal{H} \otimes \Omega^1(\mathcal{H}) & \xrightarrow{1_{\mathcal{H}} \otimes {}_{left}\Delta_{\Omega^1(\mathcal{H})}} & \mathcal{H} \otimes \mathcal{H} \otimes \Omega^1(\mathcal{H}) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{d_{\mathcal{H}}} & \Omega^1(\mathcal{H}) \\ {\scriptstyle \Delta} \downarrow & & \downarrow {\scriptstyle {}_{left}\Delta_{\Omega^1(\mathcal{H})}} \\ \mathcal{H} \otimes \mathcal{H} & \xrightarrow{1_{\mathcal{H}} \otimes d_{\mathcal{H}}} & \mathcal{H} \otimes \Omega^1(\mathcal{H}) \end{array}$$

are commutative. Similarly, $(\Omega^1(\mathcal{H}), d_{\mathcal{H}})$ is right-covariant if

$$\sum h_j dk_j = 0 \Longrightarrow \sum \Delta(h_j)(d_{\mathcal{H}} \otimes 1_{\mathcal{H}})\Delta(k_j) = 0$$

for all $h_j, k_j \in \mathcal{H}, 1 \leq j \leq n, n \in \mathbb{N}$

and there exists a $\mathbb{K}$-linear morphism:

$$\Delta_{\Omega^1(\mathcal{H})} : \Omega^1(\mathcal{H}) \longrightarrow \Omega^1(\mathcal{H}) \otimes \mathcal{H}\,,$$

such that the diagrams:

$$\begin{array}{ccc} \Omega^1(\mathcal{H}) & \xrightarrow{\Delta_{\Omega^1(\mathcal{H})}} & \Omega^1(\mathcal{H}) \otimes \mathcal{H} \\ \big\downarrow \Delta_{\Omega^1(\mathcal{H})} & & 1_{\Omega^1(\mathcal{H})}\otimes\Delta \big\downarrow \\ \Omega^1(\mathcal{H}) \otimes \mathcal{H} & \xrightarrow{\Delta_{\Omega^1(\mathcal{H})}\otimes 1_{\mathcal{H}}} & \Omega^1(\mathcal{H}) \otimes \mathcal{H} \otimes \mathcal{H} \end{array}$$

and

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{d_{\mathcal{H}}} & \Omega^1(\mathcal{H}) \\ \Delta \big\downarrow & & \big\downarrow {}_{right}\Delta_{\Omega^1(\mathcal{H})} \\ \mathcal{H} \otimes \mathcal{H} & \xrightarrow[d_{\mathcal{H}}\otimes 1_{\mathcal{H}}]{} & \Omega^1(\mathcal{H}) \otimes \mathcal{H} \end{array}$$

are commutative.

It is then easily shown (see M. Durdevich [18], S.L. Woronowicz [57]) that the first order quantum differential calculus $(\Omega^1(\mathcal{H}), d_{\mathcal{H}})$ is a bicovariant first order quantum differential calculus if the following diagram

$$\begin{array}{ccc} \Omega^1(\mathcal{H}) & \xrightarrow{{}_{left}\Delta_{\Omega^1(\mathcal{H})}} & \mathcal{H} \otimes \Omega^1(\mathcal{H}) \\ {}_{right}\Delta_{\Omega^1(\mathcal{H})} \big\downarrow & & \big\downarrow 1_{\mathcal{H}}\otimes\Omega^1(\mathcal{H}) \\ \Omega^1(\mathcal{H}) \otimes \mathcal{H} & \xrightarrow[{}_{left}\Delta_{\Omega^1(\mathcal{H})}\otimes 1_{\mathcal{H}}]{} & H \otimes \Omega^1(\mathcal{H}) \otimes \mathcal{H} \end{array}$$

commutes.

Remark 2. If we consider $\Omega^1(\mathcal{H})$ as an $\mathcal{H}$-bimodule then we have a $\mathbb{K}$-linear morphism:

(1) ${}_{right}\Delta_{\Omega^1(\mathcal{H})} : \Omega^1(\mathcal{H}) \longrightarrow \Omega^1(\mathcal{H}) \otimes \mathcal{H}$, such that the pair $(\Omega^1(\mathcal{H}), {}_{right}\Delta_{\Omega^1(\mathcal{H})})$ is a right-covariant bimodule.

Similarly a $\mathbb{K}$-linear morphism:

(2) ${}_{left}\Delta_{\Omega^1(\mathcal{H})} : \Omega^1(\mathcal{H}) \longrightarrow \mathcal{H} \otimes \Omega^1(\mathcal{H})$
makes the pair $(\Omega^1(\mathcal{H}),_{left} \Delta_{\Omega^1(\mathcal{H})})$ into a left-covariant bimodule.

The morphisms (1) and (2) make $(\Omega^1(\mathcal{H}),_{right} \Delta_{\Omega^1(\mathcal{H})})$ into a bicovariant module so that the diagram

$$\begin{array}{ccc} \Omega^1(\mathcal{H}) & \xrightarrow{{}_{left}\Delta_{\Omega^1(\mathcal{H})}} & \mathcal{H}\otimes\Omega^1(\mathcal{H}) \\ {\scriptstyle {}_{right}\Delta_{\Omega^1(\mathcal{H})}}\downarrow & & \downarrow{\scriptstyle 1_{\mathcal{H}}\otimes {}_{right}\Delta_{\Omega^1(\mathcal{H})}} \\ \Omega^1(\mathcal{H})\otimes\mathcal{H} & \xrightarrow[{}_{left}\Delta_{\Omega^1(\mathcal{H})}\otimes 1_{\mathcal{H}}]{} & \mathcal{H}\otimes\Omega^1(\mathcal{H})\otimes\mathcal{H} \end{array}$$

is commutative.

Definition 9. Let $(\Omega^1(\mathcal{H}), d_{\mathcal{H}})$ be a left covariant first order quantum differential calculus and

$$ {}_{left}\Delta_{\Omega^1(\mathcal{H})} : \Omega^1(\mathcal{H}) \longrightarrow \mathcal{H}\otimes\Omega^1(\mathcal{H})$$

the corresponding left coaction of $\mathcal{H}$ on $\Omega^1(\mathcal{H})$.

Define the space $\Omega^1_{inv}(\mathcal{H})$ of left-invariant elements of $\Omega^1(\mathcal{H})$ by

$$\Omega^1_{inv}(\mathcal{H}) := \{\omega \in \Omega^1(\mathcal{H}) :_{left} \Delta_{\Omega^1(\mathcal{H})}(\omega) := 1_{\mathcal{H}}\otimes\omega\}.$$

Then the $\mathbb{K}$-linear surjective morphism:

$$\pi_{\Omega^1_{inv}(\mathcal{H})} : \mathcal{H} \longrightarrow \Omega^1_{inv}(\mathcal{H}) \subseteq \Omega^1(\mathcal{H})$$

defined by

$$\pi_{\Omega^1_{inv}(\mathcal{H})}(h) := \sum_{(h)} S(h_{(1)})dh_{(2)}$$

induces a right $\mathcal{H}$-module structure on $\Omega^1_{inv}(\mathcal{H})$ which we denote by

$$\diamond : \Omega^1_{inv}(\mathcal{H})\otimes\mathcal{H} \longrightarrow \Omega^1_{inv}(\mathcal{H})$$

and define,

$$\pi_{\Omega^1_{inv}(\mathcal{H})}(h)\diamond k := \pi_{\Omega^1_{inv}(\mathcal{H})}(hk - \epsilon(h)k),$$

for every $h, k \in \mathcal{H}$.

Now consider the tensor product of n-copies of $\Omega^1_{inv}(\mathcal{H})$:

$$T^n\underline{ns}_{inv}(\mathcal{H}) := \Omega^1_{inv}(\mathcal{H})\otimes_{\mathcal{H}}\cdots\otimes_{\mathcal{H}}\Omega^1_{inv}(\mathcal{H})$$

and denote the tensor algebra over $\Omega^1_{inv}(\mathcal{H})$ by

$$T^\bullet\underline{ns}_{inv}(\mathcal{H}) := \bigoplus_{n\geq 0} T^n\underline{ns}_{inv}(\mathcal{H}).$$

Let $\hat{T}^\bullet\underline{ns}_{inv}(\mathcal{H})$ denote the subalgebra of left-invariant elements of $T^\bullet\underline{ns}_{inv}(\mathcal{H})$ and by $\hat{T}^k\underline{ns}_{inv}(\mathcal{H})$ the space of left-invariant k-th order elements or k-homogeneous components of $\hat{T}^\bullet\underline{ns}_{inv}(\mathcal{H})$. Let $\pi_{inv} : \hat{T}^\bullet\underline{ns}(\mathcal{H}) \longrightarrow \hat{T}^\bullet\underline{ns}_{inv}(\mathcal{H})$ be the canonical projection morphism onto the left-invariant elements. We can then uniquely extend the $\mathcal{H}$-module structure:

$$\diamond : \Omega^1_{inv}(\mathcal{H}) \otimes \mathcal{H} \longrightarrow \Omega^1_{inv}(\mathcal{H})$$

to $\hat{T}\underline{ns}_{inv}(\mathcal{H})$ to obtain an $\mathcal{H}$-module structure:

$$\hat{\diamond} : \hat{T}\underline{ns}_{inv}(\mathcal{H}) \otimes \mathcal{H} \longrightarrow \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H})\,,$$

such that

$$\omega \hat{\diamond} h := \sum_{(h)} S(h_{(1)})\omega h_{(2)}$$

together with:

(i) $1\hat{\diamond}h = \epsilon(h)1$.

(ii) $(\omega\eta)\hat{\diamond}h = \sum_{(h)}(\omega\hat{\diamond}h_{(1)})(\eta\hat{\diamond}h_{(2)})$, for all $\omega, \eta \in \hat{T}\underline{ns}_{inv}(\mathcal{H})$.

Note that the algebra $\hat{T}\underline{ns}_{inv}(\mathcal{H}) \subseteq \hat{T}ns(\mathcal{H})$ is $\hat{d}_{inv}$-invariant and the following conditions are satisfied:

(1)

$$\hat{d}_{inv}(\omega\hat{\diamond}h) = \hat{d}_{inv}(\omega)\hat{\diamond}h - \sum_h \pi_{\Omega^1_{inv}(\mathcal{H})}(h_{(1)})(\omega\hat{\diamond}h_{(2)}) + (-1)^k \sum_h (\omega\hat{\diamond}h_{(1)})(\omega\hat{\diamond}h_{(2)}).$$

(2)

$$\hat{d}_{inv}\pi_{\Omega^1_{inv}(\mathcal{H})}(h) = -\sum \pi_{\Omega^1_{inv}(\mathcal{H})}(h_{(1)})\pi_{\Omega^1_{inv}(\mathcal{H})}(h_{(2)}),$$

for every $h \in \mathcal{H}$ and $\omega \in \hat{T}^k\underline{ns}_{inv}(\mathcal{H})$, where

$$\hat{d}_{inv} : \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H}) \longrightarrow \hat{T}^{\bullet+1}\underline{ns}_{inv}(\mathcal{H}).$$

Now suppose $(\Omega^1(\mathcal{H}), d_{\mathcal{H}})$ is bicovariant with

$${}_{right}\Delta_{\Omega^1(\mathcal{H})} : \Omega^1(\mathcal{H}) \longrightarrow \Omega^1(\mathcal{H}) \otimes \mathcal{H}$$

the right coaction of $\mathcal{H}$ on $\Omega^1(\mathcal{H})$. Then (see M. Durdevich [18, 19]) the comultiplication morphism:

$$\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$$

has a unique extension to the $\mathbb{K}$-algebra morphism:

$$\hat{\Delta} : \hat{T}^{\bullet}\underline{ns}(\mathcal{H}) \longrightarrow \hat{T}^{\bullet}\underline{ns}(\mathcal{H})\hat{\otimes}\hat{T}^{\bullet}\underline{ns}(\mathcal{H})$$

of graded-differential $\mathbb{K}$-algebras, where $\hat{\otimes}$ is the tensor product for graded $\mathbb{K}$-algebra, so that for all $\eta \in \Omega^1(\mathcal{H})$,

$$\hat{\Delta}(\eta) = {}_{left}\Delta_{\Omega^1(\mathcal{H})}(\eta) + {}_{right}\Delta_{\Omega^1(\mathcal{H})}(\eta).$$

Furthermore,

$$\hat{\Delta}(\hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H})) \subset \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H}) \,\hat{\otimes}\, \hat{T}^{\bullet}\underline{ns}(\mathcal{H}).$$

Finally, let

$$\hat{\bar{\omega}} : \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H}) \longrightarrow \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H})\hat{\otimes}\hat{T}^{\bullet}\underline{ns}(\mathcal{H})$$

be the corresponding restrictions of

$$\hat{\bar{\omega}} : \hat{T}^{\bullet}\underline{ns}(\mathcal{H}) \longrightarrow \hat{T}^{\bullet}\underline{ns}(\mathcal{H})\hat{\otimes}\hat{T}^{\bullet}\underline{ns}(\mathcal{H})$$

such that for all $\zeta \in \Omega^1_{inv}(\mathcal{H})$, we have $\hat{\bar{\omega}}(\zeta) = 1 \otimes \zeta + \bar{\omega}(\zeta)$.

The right coaction

$${}_{right}\Delta_{\Omega^1(\mathcal{H})} : \Omega^1(\mathcal{H}) \longrightarrow \Omega^1(\mathcal{H}) \otimes \mathcal{H}$$

of $\mathcal{H}$ on $\Omega^1(\mathcal{H})$ admits a natural extension to give the right coaction:

$${}_{right}\Delta_{\Omega^1(\mathcal{H})} : \hat{T}^{\bullet}\underline{ns}(\mathcal{H}) \longrightarrow \hat{T}^{\bullet}\underline{ns}(\mathcal{H}) \otimes \mathcal{H}.$$

Next, let $\bar{\omega} : \Omega^1_{inv}(\mathcal{H}) \longrightarrow \Omega^1_{inv}(\mathcal{H}) \otimes \mathcal{H}$ be the coadjoint action of $\mathcal{H}$ on $\Omega^1_{inv}(\mathcal{H})$. Then the space $\Omega^1_{inv}(\mathcal{H})$ is clearly right-invariant, that is

$${}_{right}\Delta_{\Omega^1(\mathcal{H})}(\Omega^1(\mathcal{H})) \subseteq \Omega^1(\mathcal{H}) \otimes \mathcal{H}$$

so that

$$\bar{\omega} = {}_{right}\Delta_{\Omega^1(\mathcal{H})} \restriction_{\Omega^1(\mathcal{H})},$$

i.e. the diagram:

$$\begin{array}{ccc} \Omega^1_{inv}(\mathcal{H}) & \xrightarrow{\bar{\omega}} & \Omega^1_{inv}(\mathcal{H}) \otimes \mathcal{H} \\ \pi_{\Omega^1_{inv}(\mathcal{H})}\Big\uparrow & & \Big\uparrow \pi_{\Omega^1_{inv}(\mathcal{H})} \otimes 1_{\mathcal{H}} \\ \mathcal{H} & \xrightarrow{ad_{\mathcal{H}}} & \mathcal{H} \otimes \mathcal{H} \end{array}$$

commutes:

$$\bar{\omega} \circ \pi_{\Omega^1_{inv}(\mathcal{H})} = (\pi_{\Omega^1_{inv}(\mathcal{H})} \otimes 1_{\mathcal{H}}) \circ ad_{\mathcal{H}}.$$

3. Higher Order Quantum Differential Calculi on Quantum Principal Bundle Coherent Algebra Sheaves

In this section we define the relevant higher order quantum differential calculus for quantum principal bundle coherent algebra sheaves over (M, g) with special coordinate systems.

Let $\mathcal{P} = (\mathcal{P}, \pi_{\mathcal{P}}, \mathcal{M}, \mathcal{H}, (\gamma_\alpha)_{\alpha \in \mathfrak{A}})$ be a quantum principal fiber bundle coherent algebra sheaf over (M, g), $(U_\alpha)_{\alpha \in \mathfrak{A}}$ a locally finite open cover of M in the category $Op(M)$ with morphisms, inclusion maps $\imath_{\beta\alpha} : U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subset U_\alpha$ in $Op(M)$ and $(\Omega^1(\mathcal{H}), d_{\mathcal{H}})$ a fixed bicovariant first order quantum differential calculus on the structure quantum group, $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, S)$, a unital Hopf $\mathbb{K}$-algebra. Define a graded $\mathbb{K}$-vector space $\Omega^\bullet_{vert}(\mathcal{P}) := \mathcal{P} \otimes \hat{T}^\bullet \underline{ns}_{inv}(\mathcal{H})$ with grading induced from $\hat{T}^\bullet \underline{ns}_{inv}(\mathcal{H})$. Let $\varphi_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{H}$ be the right coaction of $\mathcal{H}$ on $\mathcal{P}$ such that the diagrams:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\varphi_{\mathcal{P}}} & \mathcal{P} \otimes \mathcal{H} \\ \varphi_{\mathcal{P}} \downarrow & & \downarrow 1_{\mathcal{P}} \otimes \Delta \\ \mathcal{P} \otimes \mathcal{H} & \xrightarrow{\varphi_{\mathcal{P}} \otimes 1_{\mathcal{H}}} & \mathcal{P} \otimes \mathcal{H} \otimes \mathcal{H} \end{array}$$

and

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\varphi_{\mathcal{P}}} & \mathcal{P} \otimes \mathcal{H} \\ & 1_{\mathcal{P}} \searrow & \downarrow 1_{\mathcal{P}} \otimes \epsilon \\ & & \mathcal{P} \cong \mathcal{P} \otimes \mathbb{K} \end{array}$$

are commutative. Furthermore, the $\mathbb{K}$-linear morphism

$$\mathcal{P} \otimes \mathcal{P} \xrightarrow{\psi} \mathcal{P} \otimes \mathcal{H}$$

defined by $\psi(u \otimes v) := u\varphi(v)$, is surjective for every $u, v \in \mathcal{P}(U_\alpha)$ and $U_\alpha \in (U_\alpha)_{\alpha \in \mathfrak{A}}$. Also for each $h \in \mathcal{H}$, there exist $u, v \in \mathcal{P}(U_\alpha)$ such that

$$1 \otimes h = \sum_j u_j \varphi(v_j).$$

We can now define the structure of a differential graded $\mathbb{K}$-coherent algebra on $\Omega^\bullet_{vert}(\mathcal{P})$ by the following formulae:

$$(1) \qquad (u \otimes \omega)(v \otimes \eta) := \sum_j uv_j \otimes (\eta \hat{\diamond} h_j)\omega\,,$$

$$(2) \qquad \hat{d}_{vert}(u \otimes \omega) = u \otimes \hat{d}_{vert}\omega + \sum_j u \otimes \pi_{\mathcal{H}}(h_j)\omega\,,$$

where $\varphi_{\mathcal{P}}(u) := \sum_j u_j \otimes h_k$, for all $u, v \in \mathcal{P}(U_\alpha), \omega, \eta \in \hat{T}^\bullet \underline{ns}_{inv}(\mathcal{H})$ and

$$\hat{d}_{vert} : \Omega^\bullet_{vert}(\mathcal{P}) \longrightarrow \Omega^{\bullet+1}_{vert}(\mathcal{P}).$$

Thus as a differential graded $\mathbb{K}$-coherent algebra sheaf, $\Omega^\bullet_{vert}(\mathcal{P}) := \mathcal{P} \otimes \hat{T}^\bullet \underline{ns}_{inv}(\mathcal{H})$ is generated by $\mathcal{P} = \Omega^0_{vert}(\mathcal{P})$. It is easily seen that there exists a unique $\mathbb{K}$-algebra sheaf morphism:

$$\hat{\varphi}_{vert} : \Omega^\bullet_{vert}(\mathcal{P}) \longrightarrow \Omega^\bullet_{vert}(\mathcal{P}) \otimes \hat{T}^\bullet \underline{ns}_{inv}(\mathcal{H})$$

of differential graded $\mathbb{K}$-coherent algebra sheaves extending the coaction morphism

$$\varphi_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{H},$$

given by $\hat{\varphi}_{\mathcal{P}}(\omega \otimes \eta) := \sum_{i,j} u_j \otimes \eta_j h_i v_j$, where $\varphi_{\mathcal{P}}(u) = \sum_i u_i \otimes h_i$ and $\hat{\hat{\omega}}(\eta) = \sum_j \eta_j \otimes v_j$ for every $u \in \mathcal{P}(U_\alpha), \eta \in \hat{T}\underline{ns}_{inv}(\mathcal{H})$ with the morphism:

$$\hat{\hat{\omega}} : \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H}) \longrightarrow \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H}) \otimes \hat{T}^{\bullet}\underline{ns}(\mathcal{H})\,,$$

such that the diagram:

$$\begin{array}{ccc} \Omega^{\bullet}_{vert}(\mathcal{P}) & \xrightarrow{\hat{\varphi}_{vert}} & \Omega^{\bullet}_{vert}(\mathcal{P}) \otimes \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H}) \\ {\scriptstyle \hat{\varphi}_{vert}}\Big\downarrow & & \Big\downarrow {\scriptstyle 1_{\Omega^{\bullet}_{vert}(\mathcal{P})} \otimes \hat{\Delta}} \\ \Omega^{\bullet}_{vert}(\mathcal{P}) \otimes \hat{T}^{\bullet}\underline{ns}(\mathcal{H}) & \xrightarrow{\hat{\varphi}_{\mathcal{P}} \otimes 1_{\hat{T}\underline{ns}(\mathcal{H})}} & \Omega^{\bullet}_{vert}(\mathcal{P}) \otimes \hat{T}^{\bullet}\underline{ns}(\mathcal{H}) \otimes \hat{T}^{\bullet}\underline{ns}(\mathcal{H}) \end{array}$$

is commutative, i.e.

$$(\hat{\varphi}_{\mathcal{P}} \otimes 1_{\hat{T}^{\bullet}\underline{ns}(\mathcal{H})}) \circ \hat{\varphi}_{vert} = (1_{\Omega^{\bullet}_{vert}(\mathcal{P})} \otimes \hat{\Delta}) \circ \hat{\varphi}_{vert}$$

(see M. Durdevich [20, 21]).

For each $n \in \mathbb{N}^0$, let $proj_n : \hat{T}^{\bullet}\underline{ns}(\mathcal{H}) \longrightarrow \hat{T}^{n}\underline{ns}(\mathcal{H})$ be the natural surjective projection on the n-th homogeneous component and $\pi_{\Omega^1_{inv}(\mathcal{H})} : \mathcal{H} \longrightarrow \Omega^1_{inv}(\mathcal{H})$ defined by

$$\pi_{\Omega^1_{inv}(\mathcal{H})}(h) := \sum_{(h)} S(h_{(1)}) dh_{(2)},$$

with $S : \mathcal{H} \longrightarrow \mathcal{H}$ an invertible antipode. Define coadjoint $\mathbb{K}$-algebra coaction of $\mathcal{H}$ on itself: $ad_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ by

$$ad_{\mathcal{H}}(h) = \sum_{(h)} h_{(2)} \otimes S(h_{(1)}) h_{(3)}.$$

Now suppose that $\Omega^{\bullet}(\mathcal{P})$ is endowed with a differential graded $\mathbb{K}$-coherent algebra sheaf structure. We can then represent the elements of $\Omega^{\bullet}_{vert}(\mathcal{P})$ as the vertical higher order quantum differential forms on the quantum principal bundle coherent algebra sheaves $\mathcal{P} = (\mathcal{P}, \pi_{\mathcal{P}}, \mathcal{M}, \mathcal{H}, (\gamma_\alpha)_{\alpha \in \mathfrak{A}})$ with coordinate system $(\gamma_\alpha)_{\alpha \in \mathfrak{A}}$ over (M, g).

Theorem 2. *Let $\mathcal{P} = (\mathcal{P}, \pi_{\mathcal{P}}, \mathcal{M}, \mathcal{H}, (\gamma_\alpha)_{\alpha \in \mathfrak{A}})$ be a quantum principal fiber bundle coherent algebra sheaf over (M, g) and $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$ a locally finite open cover of M in the category $Op(M)$ with morphisms inclusion maps $\imath_{\beta\alpha} : U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subset U_\alpha$ in $Op(M)$. Then there exists a unique $\mathbb{K}$-algebra sheaf morphism:*

$$\hat{\varphi}_{vert} : \Omega^{\bullet}_{vert}(\mathcal{P}) \longrightarrow \Omega^{\bullet}_{vert}(\mathcal{P}) \hat{\otimes} \hat{T}^{\bullet}\underline{ns}(\mathcal{H})$$

of differential graded $\mathbb{K}$-coherent algebra sheaves extending the $\mathbb{K}$-algebra sheaf *coaction morphism:* $\varphi_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{H}$ *such that the following diagram*

$$\begin{array}{ccc} \Omega^{\bullet}_{vert}(\mathcal{P}) & \xrightarrow{\hat{\varphi}_{vert}} & \Omega^{\bullet}_{vert}(\mathcal{P})\hat{\otimes}\hat{T}^{\bullet}\underline{ns}(\mathcal{H}) \\ {\scriptstyle \hat{\varphi}_{vert}}\downarrow & & \downarrow{\scriptstyle 1_{\Omega^{\bullet}_{vert}(\mathcal{P})}\otimes\hat{\Delta}} \\ \Omega^{\bullet}_{vert}(\mathcal{P})\hat{\otimes}\hat{T}^{\bullet}\underline{ns}(\mathcal{H}) & \xrightarrow{\hat{\varphi}_{vert}\otimes 1_{\hat{T}^{\bullet}\underline{ns}(\mathcal{H})}} & \Omega^{\bullet}_{vert}(\mathcal{P})\hat{\otimes}\hat{T}^{\bullet}\underline{ns}(\mathcal{H})\hat{\otimes}\hat{T}^{\bullet}\underline{ns}(\mathcal{H}) \end{array}$$

commutes.

Proof. Define the product on $\Omega^{\bullet}_{vert}(\mathcal{P}) := \mathcal{P} \otimes \hat{T}^{\bullet}\underline{ns}(\mathcal{H})$ by

$$(u \otimes \omega)(v \otimes \eta) = \sum_j uv_j \otimes (\eta \hat{\diamond} h_j)\omega$$

and the differential operator $\hat{d}_{vert} : \Omega^{\bullet}_{vert}(\mathcal{P}) \longrightarrow \Omega^{\bullet+1}_{vert}(\mathcal{P})$ by

$$\hat{d}_{vert}(u \otimes \omega) := u \otimes \hat{d}\omega + \sum_j u \otimes \pi_{\Omega^1_{inv}(\mathcal{H})}(h_j)\omega.$$

Now use the properties of the $\mathbb{K}$-algebra sheaf coaction morphism $\varphi_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{H}$ defined by $\varphi_{\mathcal{P}}(u) := \sum_j u_j \otimes h_j$ for every $u_j \in \mathcal{P}(U_\alpha), h_j \in \mathcal{H}$ and $U_\alpha \in (U_\alpha)_{\alpha\in\mathfrak{A}}$ and the $\mathcal{H}$-module structure

$$\hat{\diamond} : \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H}) \otimes \mathcal{H} \longrightarrow \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H})$$

given by $(\omega \wedge \eta)\hat{\diamond} h := \sum_{(h)}(\omega\hat{\diamond} h_{(1)})(\eta\hat{\diamond} h_{(2)})$, for every $\omega, \eta \in \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H})$ and $h \in \mathcal{H}$ to show the associativity of the product:

$$\begin{aligned}(\omega \otimes \zeta)(u \otimes \nu)(v \otimes \eta) &:= \sum_j (\omega u_j \otimes (\zeta\hat{\diamond} h_j)\nu)(v \otimes \eta) \\ &= \sum_{j,k} \omega u_j v_k \otimes ((\zeta\hat{\diamond} h_j)\nu)\hat{\diamond} l_k)\eta = (\omega \otimes \zeta)\sum_k \omega v_k \otimes (\nu\hat{\diamond} l_k)\eta \\ &= (\omega \otimes \zeta)((u \otimes \nu)(v \otimes \eta)),\end{aligned}$$

where $\varphi_{\mathcal{P}}(v) = \sum_k v_k \otimes l_k$, for every $w, u, v \in \mathcal{P}(U_\alpha)$, $\zeta, \nu, \eta \in \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H})$, $h_j, l_k \in \mathcal{H}$. It is easily seen that $\Omega^{\bullet}_{vert}(\mathcal{P})$ is a unital associative $\mathbb{K}$-coherent algebra sheaf with unit $1 \otimes 1$.

We can show by computation that the operator $\hat{d}_{vert}$ actually defines a differential operator compatible with the product structure defined on $\Omega^{\bullet}_{vert}(\mathcal{P})$.

$$\begin{aligned}\hat{d}_{vert}[(u \otimes \zeta)(v \otimes \eta)] &= \sum_j \hat{d}_{vert}(uv_j \otimes (\zeta\hat{\diamond} h_j)\eta) \\ &= \sum_j uv_j \otimes \hat{d}((\zeta\hat{\diamond} h_j)\eta) + \sum_{j,k} u_k v_j \otimes \pi_{\Omega^1_{inv}(\mathcal{H})}(l_k h_{j(1)})(\zeta\hat{\diamond} h_{j(2)})\end{aligned}$$

$$= \sum_j uv_j \otimes (\hat{d}(\eta)\hat{\circ}h_j)\eta - \sum_j uv_j \otimes \pi_{\Omega^1_{inv}(\mathcal{H})}(h_{j(1)})(\zeta\hat{\circ}h_{j(2)})\eta$$

$$+ (-1)^q \sum uv_j \otimes (\zeta\hat{\circ}h_{j(1)})\pi_{\Omega^1_{inv}(\mathcal{H})}(h_{j(2)})\eta + (\zeta\hat{\circ}h_j)\hat{d}\eta)$$

$$+ \sum_{j,k} u_k v_j \otimes \pi_{\Omega^1_{inv}(\mathcal{H})}(h_{j(1)})(\zeta\hat{\circ}h_{j(2)})\eta + \sum_j uv_j \otimes \pi_{\Omega^1_{inv}(\mathcal{H})}(h_{j(1)})(\zeta\hat{\circ}h_{j(2)})\eta$$

$$= (u \otimes \hat{d}\zeta)(v \otimes \eta) + (-1)^q (u \otimes \zeta) \sum_j v_j \otimes \pi_{\Omega^1_{inv}(\mathcal{H})}(h_j)\eta$$

$$= [\hat{d}_{vert}(u \otimes \zeta)](v \otimes \eta) + (-1)^q (u \otimes \zeta)\hat{d}_{vert}(v \otimes \eta),$$

for every

$$u, v \in \Omega^\bullet_{vert}(U_\alpha), h_j, l_j \in \mathcal{H}, \zeta \in \hat{T}^q \underline{ns}_{inv}(\mathcal{H})$$

and $\eta \in \hat{T}^\bullet \underline{ns}_{inv}(\mathcal{H})$. Next we show that $(\hat{d}_{vert})^2 = 0$. For this we have that:

$$\hat{d}^2_{vert}(u \otimes \zeta) = \hat{d}_{vert}(u \otimes \hat{d}_{vert}\zeta + \sum_j u_j \otimes \pi_{\Omega^1_{inv}(\mathcal{H})}(h_j)\zeta)$$

$$= \sum_j u_j \otimes \pi_{\Omega^1_{inv}(\mathcal{H})}(h_j)\hat{d}_{inv}\zeta$$

$$+ \sum_j u_j \otimes \pi_{\Omega^1_{inv}(\mathcal{H})}(h_{j(1)})\pi_{\Omega^1(\mathcal{H})}(h_{j(2)})\zeta + \sum_j u_j \otimes \hat{d}_{vert}(\pi_{\Omega^1(\mathcal{H})}(h_j)\zeta) = 0.$$

Now we show that $\Omega^\bullet_{vert}(\mathcal{P})$, as a differential graded $\mathbb{K}$-coherent algebra sheaf is generated by $\Omega^0_{vert}(\mathcal{P}) = \mathcal{P}$. First we observe that elements of the form $u\hat{d}_{vert}(h)$ linearly generate $\Omega^1_{vert}(\mathcal{P})$, since for each $h \in \mathcal{H}$, there exist elements $u_j, v_j \in \mathcal{P}(U_\alpha)$, for every $U_\alpha \in \{U_\alpha\}_{\alpha\in\mathfrak{A}}$ so that $1 \otimes h = \sum_j u_j \varphi_{\mathcal{P}}(v_j)$. This last result implies that the $\mathbb{K}$-algebra sheaf morphism

$$\hat{\varphi}_{vert} : \Omega^\bullet_{vert}(\mathcal{P}) \longrightarrow \Omega^\bullet_{vert}(\mathcal{P})\hat{\otimes}\hat{T}^\bullet \underline{ns}(\mathcal{H})$$

if it exists is unique. Define $\hat{\varphi}_{vert}$ by $\hat{\varphi}_{vert}(u \otimes \zeta) := \sum_{j,k}\sum u_j \otimes \zeta_k \otimes h_j v_k$ where $\varphi_{\mathcal{P}}(u) = \sum_j u_j \otimes h_j$ and

$$\hat{\hat{\omega}} : \hat{T}^\bullet \underline{ns}_{inv}(\mathcal{H}) \longrightarrow \hat{T}^\bullet \underline{ns}_{inv}(\mathcal{H})\hat{\otimes}\hat{T}^\bullet \underline{ns}_{inv}(\mathcal{H})$$

is given by $\hat{\hat{\omega}}(\zeta) = \sum_k \zeta_k \otimes v_k$. Then it is immediate that the $\mathbb{K}$-algebra sheaf morphism

$$\hat{\varphi}_{vert} : \Omega^\bullet_{vert}(\mathcal{P}) \longrightarrow \Omega^\bullet_{vert}(\mathcal{P})\hat{\otimes}\hat{T}^\bullet \underline{ns}(\mathcal{H})\,,$$

so defined is a differential graded $\mathbb{K}$-algebra sheaf morphism. □

The $\mathbb{K}$-algebra sheaf morphism $\varphi_{vert} : \Omega^\bullet_{vert}(\mathcal{P}) \longrightarrow \Omega^\bullet_{vert}(\mathcal{P}) \otimes \mathcal{H}$ is clearly an extension of the $\mathbb{K}$-algebra sheaf coaction morphism $\varphi_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{H}$ such that the following diagram,

$$\begin{array}{ccc}
\Omega^{\bullet}_{vert}(\mathcal{P}) & \underrightarrow{\hat{\varphi}_{vert}} & \Omega^{\bullet}_{vert}(\mathcal{P}) \otimes \hat{T}^{\bullet}\underline{ns}(\mathcal{H}) \\
{}^{\wedge}\varphi_{vert} \searrow & & \swarrow 1_{\Omega^{\bullet}_{vert}(\mathcal{P})} \otimes proj_0 \\
& \Omega^{\bullet}_{vert}(\mathcal{P}) \otimes \mathcal{H} &
\end{array}$$

is commutative.

Further, we also have the following commutative diagrams:

$$(1) \qquad \begin{array}{ccc}
\Omega^{\bullet}_{vert}(\mathcal{P}) & \xrightarrow{{}^{\wedge}\varphi_{vert}} & \Omega^{\bullet}_{vert}(\mathcal{P}) \otimes \mathcal{H} \\
{}^{\wedge}\varphi_{vert}\Big\downarrow & & \Big\downarrow 1_{\Omega^{\bullet}_{vert}(\mathcal{P})} \otimes \Delta \\
\Omega^{\bullet}_{vert}(\mathcal{P}) \otimes \mathcal{H} & \xrightarrow{{}^{\wedge}\varphi_{vert} \otimes 1_{\mathcal{H}}} & \Omega^{\bullet}_{vert}(\mathcal{P}) \otimes \mathcal{H} \otimes \mathcal{H},
\end{array}$$

$$(2) \qquad \begin{array}{ccc}
\Omega^{\bullet}_{vert}(\mathcal{P}) & \underrightarrow{{}^{\wedge}\varphi_{vert}} & \Omega^{\bullet}_{vert}(\mathcal{P}) \otimes \mathcal{H} \\
1_{\Omega^{\bullet}_{vert}(\mathcal{P})} \searrow & & \swarrow 1_{\Omega^{\bullet}_{vert}(\mathcal{P})} \otimes \epsilon \\
& \Omega^{\bullet}_{vert}(\mathcal{P}) \cong \Omega^{\bullet}_{vert}(\mathcal{P}) \otimes \mathbb{K} &
\end{array}$$

and

$$(3) \qquad \begin{array}{ccc}
(3)\Omega^{\bullet}_{vert}(\mathcal{P}) & \xrightarrow{\hat{d}_{vert}} & \Omega^{\bullet+1}_{vert}(\mathcal{P}) \\
{}^{\wedge}\varphi_{vert}\Big\downarrow & & \Big\downarrow {}^{\wedge}\varphi_{vert} \\
\Omega^{\bullet}_{vert}(\mathcal{P}) \otimes \mathcal{H} & \xrightarrow{\hat{d}_{vert} \otimes 1_{\mathcal{H}}} & \Omega^{\bullet+1}_{vert}(\mathcal{P}) \otimes \mathcal{H},
\end{array}$$

showing the compatibility of φ_{vert} with the comultiplication $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$, the counit $\epsilon : \mathcal{H} \longrightarrow \mathbb{K}$ and the $\mathbb{K}$-algebra sheaf morphism $\hat{d}_{vert} : \Omega^{\bullet}_{vert}(\mathcal{P}) \longrightarrow \Omega^{\bullet+1}_{vert}(\mathcal{P})$.

Let a higher order quantum differential calculus on the quantum principal bundle coherent algebra sheaf $\mathcal{P} = (\mathcal{P}, \pi_{\mathcal{P}}, \mathcal{M}, \mathcal{H}, (\gamma_\alpha)_{\alpha \in \mathfrak{A}})$ be the differential graded $\mathbb{K}$-coherent algebra sheaf $\Omega^{\bullet}(\mathcal{P})$ defined in section 3 such that:

<u>HOQDC</u> [I] $\Omega^{\bullet}(\mathcal{P})$ is generated as a differential graded $\mathbb{K}$-coherent algebra sheaf by

$$\Omega^0(\mathcal{P}) = \mathcal{P}.$$

<u>HOQDC</u> [II] The $\mathbb{K}$-algebra sheaf coaction $\varphi_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{H}$ extends to a morphism

$$\hat{\varphi}_{\mathcal{P}} : \Omega^{\bullet}(\mathcal{P}) \longrightarrow \Omega^{\bullet}(\mathcal{P}) \hat{\otimes} \hat{T}^{\bullet}\underline{ns}(\mathcal{H})$$

of differential graded $\mathbb{K}$-coherent algebra sheaves such that the diagram:

$$\begin{array}{ccc} \Omega^\bullet(\mathcal{P}) & \xrightarrow{\hat{\varphi}_\mathcal{P}} & \Omega^\bullet(\mathcal{P})\hat{\otimes}\hat{T}^\bullet \underline{ns}(\mathcal{H}) \\ {\scriptstyle \hat{\varphi}_\mathcal{P}}\downarrow & & \downarrow{\scriptstyle 1_{\Omega^\bullet(\mathcal{P})}\otimes\hat{\Delta}} \\ \Omega^\bullet(\mathcal{P})\hat{\otimes}\hat{T}^\bullet \underline{ns}(\mathcal{H}) & \xrightarrow{\hat{\varphi}_\mathcal{P}\otimes 1_{\hat{T}^\bullet \underline{ns}(\mathcal{H})}} & \Omega^\bullet(\mathcal{P})\hat{\otimes}\hat{T}^\bullet \underline{ns}(\mathcal{H})\hat{\otimes}\hat{T}^\bullet \underline{ns}(\mathcal{H}) \end{array}$$

commutes. The $\mathbb{K}$-algebra sheaf coaction morphism ${}^\wedge\varphi_\mathcal{P} : \Omega^\bullet(\mathcal{P}) \longrightarrow \Omega^\bullet(\mathcal{P}) \otimes\mathcal{H}$ is defined by the commutativity of the diagram:

$$\begin{array}{ccc} \Omega^\bullet(\mathcal{P}) & \underrightarrow{\hat{\varphi}_\mathcal{P}} & \Omega^\bullet(\mathcal{P})\hat{\otimes}\hat{T}^\bullet \underline{ns}(\mathcal{H}) \\ {}^\wedge\varphi_\mathcal{P} \searrow & & \swarrow 1_{\Omega^\bullet(\mathcal{P})} \otimes proj_0 \\ & \Omega^\bullet(\mathcal{P}) \otimes \mathcal{H}. & \end{array}$$

We then have the following commutative diagrams:

$$\text{(I)} \qquad \begin{array}{ccc} \Omega^\bullet(\mathcal{P}) & \xrightarrow{{}^\wedge\varphi_\mathcal{P}} & \Omega^\bullet(\mathcal{P}) \otimes \mathcal{H} \\ {\scriptstyle {}^\wedge\varphi_\mathcal{P}}\downarrow & & \downarrow{\scriptstyle 1_{\Omega^\bullet(\mathcal{P})}\otimes\Delta} \\ \Omega^\bullet(\mathcal{P}) \otimes \mathcal{H} & \xrightarrow{{}^\wedge\varphi_\mathcal{P}\otimes 1_H} & \Omega^\bullet(\mathcal{P}) \otimes \mathcal{H} \otimes \mathcal{H}, \end{array}$$

$$\text{(II)} \qquad \begin{array}{ccc} \Omega^\bullet(\mathcal{P}) & \underrightarrow{{}^\wedge\varphi_\mathcal{P}} & \Omega^\bullet(\mathcal{P}) \otimes \mathcal{H} \\ 1_{\Omega^\bullet(\mathcal{P})} \searrow & & \swarrow 1_{\Omega^\bullet(\mathcal{P})} \otimes \epsilon \\ & \Omega^\bullet(\mathcal{P}) \cong \Omega^\bullet(\mathcal{P}) \otimes \mathbb{K} & \end{array}$$

and

$$\text{(III)} \qquad \begin{array}{ccc} \Omega^\bullet(\mathcal{P}) & \xrightarrow{d_\mathcal{P}} & \Omega^{\bullet+1}(\mathcal{P}) \\ {\scriptstyle {}^\wedge\varphi_\mathcal{P}}\downarrow & & \downarrow{\scriptstyle {}^\wedge\varphi_\mathcal{P}} \\ \Omega^\bullet(\mathcal{P}) \otimes \mathcal{H} & \xrightarrow{d_\mathcal{P}\otimes 1_\mathcal{H}} & \Omega^{\bullet+1}(\mathcal{P}) \otimes \mathcal{H}. \end{array}$$

We are now in a position to introduce the concept of basic or horizontal higher order quantum differential forms $\Omega^\bullet_{hor}(\mathcal{P})$.

Definition 10. Let ${}^\wedge\varphi_\mathcal{P} \colon \Omega^\bullet(\mathcal{P}) \longrightarrow \Omega^\bullet(\mathcal{P}) \otimes \mathcal{H}$ be the $\mathbb{K}$-algebra sheaf morphism extending the $\mathbb{K}$-algebra coaction morphism $\varphi_\mathcal{P} : \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{H}$. We call the elements of the graded $\mathbb{K}$-coherent subalgebra sheaf $\Omega^\bullet_{hor}(\mathcal{P}) := {}^\wedge\varphi_\mathcal{P}^{-1}(\Omega^\bullet(\mathcal{P}) \otimes \mathcal{H})$ of the higher order quantum differential forms $\Omega^\bullet(\mathcal{P})$ basic or horizontal quantum differential forms with $\Omega^0_{hor}(\mathcal{P}) = \mathcal{P}$.

Theorem 3. *Let* ${}^\wedge\varphi_\mathcal{P} \colon \Omega^\bullet(\mathcal{P}) \longrightarrow \Omega^\bullet(\mathcal{P}) \otimes \mathcal{H}$ *be the extension of the* $\mathbb{K}$*-algebra sheaf coaction morphism* $\varphi_\mathcal{P} : \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{H}$. *Then the differen-*

tial graded $\mathbb{K}$-coherent subalgebra sheaf, $\Omega^\bullet_{hor}(\mathcal{P}) := \hat{\varphi}_{\mathcal{P}}^{-1}(\Omega^\bullet(\mathcal{P}) \otimes \mathcal{H})$ of the higher order quantum differential forms $\Omega^\bullet(\mathcal{P})$ on the total quantum space $\mathcal{P} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}Sh_M)$ is ${}^\wedge\varphi_{\mathcal{P}}$-invariant. That is, ${}^\wedge\varphi_{\mathcal{P}}(\Omega^\bullet_{hor}(\mathcal{P})) \subseteq \Omega^\bullet_{hor}(\mathcal{P}) \otimes \mathcal{H}$.

Proof. Let $\omega \in \Omega^\bullet_{hor}(\mathcal{P})(U_\alpha) = \Omega^\bullet_{hor}(\mathcal{P}(U_\alpha))$. Then the commutativity of the diagram:

$$\begin{array}{ccc} \Omega^\bullet_{hor}(\mathcal{P}(U_\alpha)) & \xrightarrow{{}^\wedge\varphi_{\mathcal{P}}\restriction_{U_\alpha}} & \Omega^\bullet_{hor}(\mathcal{P}(U_\alpha)) \otimes \mathcal{H} \\ {\scriptstyle {}^\wedge\varphi_{\mathcal{P}}\restriction_{U_\alpha}}\downarrow & & \downarrow{\scriptstyle 1_{\Omega^0_{hor}(\mathcal{P}U_\alpha)}\otimes\hat{\Delta}} \\ \Omega^\bullet_{hor}(\mathcal{P}(U_\alpha)) \otimes \mathcal{H} & \xrightarrow{{}^\wedge\varphi_{\mathcal{P}}\otimes 1_{\mathcal{H}}} & \Omega^\bullet_{hor}(\mathcal{P}(U_\alpha)) \otimes \mathcal{H} \otimes \mathcal{H} \end{array}$$

implies by computation the claim. Indeed,

$$\begin{aligned} ({}^\wedge\varphi_{\mathcal{P}} \restriction_{U_\alpha} \otimes 1_H) \circ^\wedge \varphi_{\mathcal{P}} \restriction_{U_\alpha} (\omega) &= (1_{\Omega^\bullet(\mathcal{P}(U_\alpha)} \otimes \hat{\Delta}) \circ^\wedge \varphi_{\mathcal{P}} \restriction_{U_\alpha} (\omega) \\ &= (1_{\Omega^\bullet(\mathcal{P}(U_\alpha))} \otimes \Delta) \circ^\wedge \varphi_{\mathcal{P}} \restriction_{U_\alpha} (\omega). \end{aligned}$$

But $(1_{\Omega^\bullet(\mathcal{P}(U_\alpha))} \otimes \Delta) \circ^\wedge \varphi_{\mathcal{P}} \restriction_{U_\alpha} (\omega) \in \Omega^\bullet(\mathcal{P}(U_\alpha)) \otimes \mathcal{H} \otimes \mathcal{H}$. Thus

$${}^\wedge\varphi_{\mathcal{P}} U_\alpha(\omega) \in \Omega^\bullet(\mathcal{P}(U_\alpha)) \otimes \mathcal{H}, \text{ for all } U_\alpha \in (U_\alpha)_{\alpha\in\mathfrak{A}}.$$ □

Theorem 4. *Let $\mathcal{P} = (\mathcal{P}, \pi_{\mathcal{P}}, \mathcal{M}, \mathcal{H}, (\gamma_\alpha)_{\alpha\in\mathfrak{A}})$ be a quantum principal bundle coherent algebra sheaf over (M, g) with coordinate system $(\gamma_\alpha)_{\alpha\in\mathfrak{A}}, (U_\alpha)_{\alpha\in\mathfrak{A}}$ a locally finite open cover of M in $Op(M)$ together with inclusion maps $\imath_{\beta\alpha} : U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subset U_\alpha$ in $Op(M)$ and $(\Omega^\bullet(\mathcal{P}), d)$ a higher order quantum differential calculus on the total quantum space $\mathcal{P} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$. Further, suppose $\omega \in \Omega^n(\mathcal{P}(U_\alpha))$ is a homogeneous element of $\Omega^\bullet(\mathcal{P}(U_\alpha))$ such that for $0 \le k \le n$,*

$$(1_{\Omega^n(P(U_\alpha))} \otimes proj_l) \circ \hat{\varphi}_{\mathcal{P}(U_\alpha)}(\omega) = 0,$$

for all $l > k$, i.e.the composite morphism

$$\Omega^n(\mathcal{P}(U_\alpha)) \overset{\hat{\varphi}_{\mathcal{P}(U_\alpha)}}{\longrightarrow} \Omega^n(\mathcal{P}(U_\alpha))\hat{\otimes}\hat{T}^n\underline{ns}(\mathcal{H}) \overset{1_{\Omega^n(P(U_\alpha))}\otimes Proj_l}{\longrightarrow} \Omega^n(\mathcal{P}(U_\alpha))\hat{\otimes}\hat{T}^{n-l}\underline{ns}(\mathcal{H}),$$

vanishes for $l > k$. Then there exists basic or horizontal quantum differential forms:

$$\eta_1, \cdots, \eta_m \in \Omega^{n-k}_{hor}(\mathcal{P}(U_\alpha))$$

and elements $\zeta_1, \cdots, \zeta_m \in \hat{T}^n\underline{ns}_{inv}(\mathcal{H})$ such that

$$(1_{\Omega^n(\mathcal{P}(U_\alpha))} \otimes proj_k) \circ \hat{\varphi}_{\mathcal{P}(U_\alpha)}(\omega) = \sum_{j=1}^{m} ({}^\wedge\varphi_{\mathcal{P}(U_\alpha)}(\eta_j)\zeta_j)\,. \tag{3.1}$$

Proof. If the composite morphism $(1_{\Omega^n(\mathcal{P}(U_\alpha))} \otimes proj_k) \circ \hat{\varphi}_{\mathcal{P}(U_\alpha)}$ evaluated at $\omega \in \Omega^n(\mathcal{P}(U_\alpha))$ is non-trivial, i.e.

$$(1_{\Omega^n(\mathcal{P}(U_\alpha))} \otimes proj_k) \circ \hat{\varphi}_{\mathcal{P}(U_\alpha)}(\omega) \neq O,$$

then for some linearly independent elements $\zeta_i \in T^\bullet \underline{ns}_{inv}(\mathcal{H}), h_{ij} \in \mathcal{H}$ and $\chi_{ij} \in \Omega^{n-k}(\mathcal{P}(U_\alpha))$, we have

$$((1_{\Omega^n(\mathcal{P}(U_\alpha))} \otimes proj_k)\hat{\varphi}_{\mathcal{P}(U_\alpha)}(\omega) = \sum_{i,j} \chi_{i,j} \otimes h_{ij}\zeta_i \,. \tag{3.2}$$

Using the equations:

(i) $\hat{\bar{\omega}}(\zeta_i) = 1 \otimes \zeta_i + \bar{\omega}(\zeta_i)$,

(ii) $((\hat{\varphi}_{\mathcal{P}(U_\alpha)} \otimes 1) \circ \hat{\varphi}_{\mathcal{P}(U_\alpha)}(\omega)) = ((1 \otimes \hat{\Delta})) \circ \hat{\varphi}_{\mathcal{P}(U_\alpha)}(\omega)$,

(iii) $(1_{\Omega^n(P(U_\alpha)} \otimes proj_k)\hat{\varphi}_{\mathcal{P}(U_\alpha)}(\omega) = \sum_{i,j} \chi_{i,j} \otimes h_{ij}\zeta_i$ and the fact that $0 \leq k \leq n$ we obtain

$$\sum_{i,j} \hat{\varphi}_{\mathcal{P}}(\chi_{i,j}) \otimes h_{ij}\zeta = ((\hat{\varphi} \otimes proj_k) \circ \hat{\varphi}_{\mathcal{P}})(\omega) = (1 \otimes 1 \otimes proj_k) \sum_{i,j} \chi_{ij} \otimes \hat{\Delta}(h_{ij}\zeta_i)$$
$$= \sum_{i,j} \chi_{ij} \otimes h_{ij(1)} \otimes h_{ij(2)}\zeta_i \,. \tag{3.3}$$

Applying $1 \otimes 1 \otimes \pi_{inv}$ to both sides of the equation (3.3) we get

$$\sum \hat{\varphi}_{\mathcal{P}(U_\alpha)}(\eta_i) \otimes \zeta_i = \sum_{i,j} \chi_{ij} \otimes h_{ij} \otimes \zeta_i,$$

where $\eta_i = \sum_j \chi_{ij}\epsilon(h_{ij})$. That is,

$$\hat{\varphi}_{\mathcal{P}(U_\alpha)}(\eta_i) = \sum_j \chi_{ij} \otimes h_{ij}, \tag{3.4}$$

which says $\eta_i \in \Omega^{n-k}(\mathcal{P}(U_\alpha))$, for every $U_\alpha \in (U_\alpha)_{\alpha\in\mathfrak{A}}$. Thus (3.2) and (3.4) give the claim (3.1), where

$$\bar{\omega} : \Omega^1_{inv}(\mathcal{H}) \longrightarrow \Omega^1_{inv}(\mathcal{H}) \otimes \mathcal{H}, \hat{\bar{\omega}} : \hat{T}^\bullet \underline{ns}_{inv}(\mathcal{H}) \longrightarrow \hat{T}^\bullet \underline{ns}_{inv}(\mathcal{H}) \hat{\otimes} \hat{T}^\bullet \underline{ns}_{inv}(\mathcal{H}),$$
$$\hat{\Delta} \ : \hat{T}^\bullet \underline{ns}(\mathcal{H}) \longrightarrow \hat{T}^\bullet \underline{ns}(\mathcal{H}) \hat{\otimes} \hat{T}^\bullet \underline{ns}(\mathcal{H}), \pi_{inv} : \hat{T}^\bullet \underline{ns}(\mathcal{H}) \longrightarrow \hat{T}^\bullet \underline{ns}_{inv}(\mathcal{H})$$

and

$$proj_k : \hat{T}^\bullet \underline{ns}(\mathcal{H}) \longrightarrow \hat{T}^{n-k} \underline{ns}(\mathcal{H}), \text{ for } 0 \leq k \leq n. \qquad \square$$

Theorem 5. *Let $\mathcal{P} = (\mathcal{P}, \pi_{\mathcal{P}}, \mathcal{M}, \mathcal{H}, (\gamma_\alpha)_{\alpha\in\mathfrak{A}})$ be a quantum principal bundle coherent algebra sheaf over $(M, g), \{U_\alpha\}_{\alpha\in\mathfrak{A}}$ a locally finite open cover of M in the category $Op(M)$ with morphisms inclusion maps $\imath_{\beta\alpha} : U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subset U_\alpha$ in $Op(M)$ and $(\Omega^\bullet(\mathcal{P}), d)$ a higher order quantum differential calculus on the total quantum space $\mathcal{P} \in Ob(\mathcal{C}ohAssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$. Then there exists a* *unique surjective differential graded $\mathbb{K}$-algebra sheaf morphism*

$$\pi_{vert}\colon \Omega^\bullet(\mathcal{P}) \longrightarrow \Omega^\bullet_{vert}(\mathcal{P})$$

which reduces to the identity on $\mathcal{P} := \Omega^0(\mathcal{P})$. Furthermore, the following diagrams:

(I)
$$\begin{array}{ccc} \Omega^\bullet(\mathcal{P}) & \xrightarrow{\wedge\varphi_{\mathcal{P}}} & \Omega^\bullet(\mathcal{P}) \otimes \mathcal{H} \\ {\scriptstyle \pi_{vert}}\downarrow & & \downarrow{\scriptstyle \pi_{vert}\otimes 1_{\mathcal{H}}} \\ \Omega^\bullet_{vert}(\mathcal{P}) & \xrightarrow{\wedge\varphi_{\mathcal{P},vert}} & \Omega^\bullet_{vert}(\mathcal{P}) \otimes \mathcal{H} \end{array}$$

and

(II)
$$\begin{array}{ccc} \Omega^\bullet(\mathcal{P}) & \underrightarrow{\wedge\varphi_{\mathcal{P},vert}} & \Omega^\bullet(\mathcal{P}) \otimes \mathcal{H} \\ \pi_{vert} \searrow & & \swarrow 1_{\Omega^\bullet(\mathcal{P})} \otimes \epsilon \\ & \Omega^\bullet_{vert}(\mathcal{P}) \cong \Omega^\bullet_{vert}(\mathcal{P}) \otimes \mathbb{K} & \end{array}$$

are commutative.

Proof. For every $U_\alpha \in \{U_\alpha\}_{\alpha\in\mathfrak{A}}$ define a $\mathbb{K}$-linear grade-preserving morphism

$$\pi_{vert} : \Omega^\bullet(\mathcal{P}) \longrightarrow \Omega^\bullet_{vert}(\mathcal{P})$$

implicitly by the equation:

$$\pi_{vert}(\omega) = (1 \otimes \pi_{inv,proj_k} \hat{\varphi}_{\mathcal{P}}(\omega))$$

for every $\omega \in \Omega^k(\mathcal{P}(U_\alpha))$. It is then immediate that π_{vert} reduces to the identity on $\mathcal{P} = \Omega^0(\mathcal{P})$. We need now to show that π_{vert} is a differential $\mathbb{K}$-algebra sheaf morphism. To do this, choose

$$\omega \in \Omega^k(\mathcal{P}(U_\alpha)), \eta \in \Omega^l(\mathcal{P}(U_\alpha)), u_j, v_j \in \mathcal{P}(U_\alpha) = \Omega^0(\mathcal{P}(U_\alpha))$$

with $\zeta_i \in \hat{T}^k \underline{ns}_{inv}(\mathcal{H})$ and $\xi_j \in \hat{T}^l \underline{ns}_{inv}(\mathcal{H})$ such Theorem 4 above gives

$$(1 \otimes proj_k)\hat{\varphi}_{\mathcal{P}}(\omega) = \sum_j \varphi_{\mathcal{P}(u)} \otimes \zeta_i$$

and

$$(1 \otimes proj_l)\hat{\varphi}_{\mathcal{P}}(\eta) = \sum_i \varphi_{\mathcal{P}}\,,$$

where $\varphi_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{H}$ is the $\mathbb{K}$-algebra sheaf coaction morphism given by

$$\varphi_{\mathcal{P}} = \sum u_j \otimes h_j,$$

for every $h \in \mathcal{H}$. We then have $\pi_{vert}(\omega) = \sum_i u_i \otimes \zeta_i$ and $\pi_{vert}(\eta) = \sum_j v_j \otimes \xi_j$. By directly computing we get:

$$\pi_{vert}(\omega \wedge \eta)$$
$$= (1 \otimes \pi_{inv} proj_{k+l})\hat{\varphi}_{\mathcal{P}}(\omega \wedge \eta) = \sum_{i,j}(1 \otimes \pi_{inv})\{\varphi_{\mathcal{P}}(u_i)\zeta_i)(\varphi_{\mathcal{P}})(v_j)\xi_j)\}$$
$$= \sum_{i,j}(1 \otimes \pi_{inv})\sum_{r,s}(u_{ir}v_{js} \otimes h_{ir}k_{js(1)}(\zeta_i \circ k_{js(2)}\xi_j)$$
$$= \sum_{i,j,s} u_i v_{js} \otimes (\zeta_i \circ k_{js})\xi_j = (\sum_i u_i \otimes \zeta_i) = \pi_{ert}(\omega) \wedge \pi_{vert}(\eta),$$

where $\varphi_{\mathcal{P}}(u_i) = \sum_r u_{ir} \otimes h_{ir}$ and $\varphi_{\mathcal{P}}(v_j) \sum_s v_{js} \otimes k_{js}$, with $h, k \in \mathcal{H}$. Furthermore, we have:

$$\pi_{vert} d(\omega) = (1 \otimes \pi_{inv} proj_{k+1}) d\hat{\varphi}_{\mathcal{P}}(\omega) = (1 \otimes \pi_{inv} d)(\sum_i (u_i) \otimes \zeta_i)$$
$$= (1 \otimes \pi_{inv} d)(\sum_{ir} u_{ir} \otimes h_{ir} d\zeta_i + u_{ir} \otimes d(h_{ir})\zeta_i) = \sum_i u_i \otimes d\zeta_i$$
$$+ \sum_{i,r} u_{ir} \otimes \pi(h_{ir})\zeta_i = d_{vert}(u_i \otimes \zeta_i) = d_{vert}\pi_{vert}(\omega).$$

Hence π_{vert} is a differential $\mathbb{K}$-algebra sheaf morphism. It is clear that the differential graded $\mathbb{K}$-coherent algebra sheaf $\Omega^\bullet(\mathcal{P})$ is generated by $\mathcal{P}$.

Commuativity of the diagram (I) is clear since $(\pi_{vert} \otimes 1_{\mathcal{H}}) \circ \hat{\varphi}_{\mathcal{P}}$ and $(\hat{\varphi}_{\mathcal{P},vert} \circ \pi_{vert})$ are differential $\mathbb{K}$-algebra sheaf morphisms agreeing with $\varphi_{\mathcal{P}} \colon \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{H}$ on $\mathcal{P}$. Commutativity of diagram (II) follows from that of (I) and the definitions of ${}^\wedge\varphi_{\mathcal{P}} \colon \Omega^\bullet(\mathcal{P}) \longrightarrow \Omega^\bullet(\mathcal{P})\hat{\otimes}\mathcal{H}$. □

Proposition 1. *The sequence*

$$0 \longrightarrow \Omega^1_{hor}(\mathcal{P}) \hookrightarrow \Omega^1(\mathcal{P}) \xrightarrow{\pi_{vert}} \Omega^1_{vert}(\mathcal{P}) \longrightarrow 0$$

of natural $\mathbb{K}$-algebra sheaf morphisms of $\mathcal{P}$-bimodules is exact.

Proof. Let $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$ a locally finite open cover of (M, g) in $Op(M)$ together with inclusion maps $\imath_{\beta\alpha} : U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subset U_\alpha$ in $Op(M)$ for every $U_\alpha \in \{U_\alpha\}_{\alpha \in \mathfrak{A}}$. It is easily seen that $\Omega^1_{hor}(\mathcal{P}(U_\alpha)) \subseteq \mathrm{Ker}(\pi_{vert} \restriction_{U_\alpha}) \cap \Omega^1(\mathcal{P}(U_\alpha))$ and $\pi_{vert}(\Omega^1(\mathcal{P}(U_\alpha)) = \Omega^1_{vert}(\mathcal{P}(U_\alpha))$. Hence for every $\omega \in \Omega^1_{vert}(\mathcal{P}(U_\alpha))$ we get :

$$(1 \otimes proj_1)\hat{\varphi}_{\mathcal{P}} \restriction_{U_\alpha})(\omega) = \sum_i \varphi_{\mathcal{P}} \restriction_{U_\alpha})(u_i)\zeta_i$$

for some $u_i \in \mathcal{P}(U_\alpha)$ and $\zeta_i \in T^{\hat{\bullet}}\underline{ns}_{inv}(\mathcal{H})$. This in turn gives

$$\pi_{vert}(\omega) = \sum_i u_i \otimes \zeta_i \,.$$

Thus if $\omega \in \mathrm{Ker}\pi_{vert}\restriction_{U_\alpha}$, then
$$(1 \otimes proj_1)\hat{\varphi}\restriction_{U_\alpha}(\omega) = 0$$
implies $\omega \in \Omega^1_{hor}(\mathcal{P}(U_\alpha))$. □

Remark 3. Note that if $(\Omega^\bullet(\mathcal{P}), d_{\mathcal{P}})$ is the higher order quantum differential calculus on the total quantum space $\mathcal{P} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$, then from the exact sequence
$$0 \longrightarrow \mathcal{M} \xrightarrow{\ \pi\ } \mathcal{P}$$
we deduce that $(\Omega^\bullet(\mathcal{M}), d_{\mathcal{M}})$, the higher order quantum differential calculus on the base quantum space $\mathcal{M} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$ with $\Omega^\bullet(\mathcal{M}) \subseteq \Omega^\bullet(\mathcal{P})$ consisting of all right-invariant basic or horizontal quantum differential forms:
$$\Omega^\bullet(\mathcal{M}) := \{\omega \in \Omega(\mathcal{P}) : \hat{\varphi}(\omega) := \omega \otimes 1\}.$$
The differential graded $\mathbb{K}$-coherent algebra sheaf $\Omega^\bullet(\mathcal{M})$ is in general not generated by its subalgebra $\Omega^0(\mathcal{M}) = \mathcal{M}$.

4. Quantum Connections and Quantum Curvatures

We define the quantum connection on $\mathcal{P}$ (see [48]) as the splitting of the sequence
$$0 \longrightarrow \Omega^\bullet_{hor}(\mathcal{P}) \xrightarrow{\imath_{\Omega^\bullet_{hor}(\mathcal{P})}} \Omega^\bullet(\mathcal{P}) \xrightarrow{\pi_{vert}} \Omega^\bullet_{vert}(\mathcal{P}) := \Omega^\bullet(\mathcal{P}) \big/ \Omega^\bullet_{hor}(\mathcal{P}) \longrightarrow 0$$
where $\Omega^\bullet(\mathcal{P})$ is a differential graded $\mathbb{K}$-coherent algebra sheaf on $\mathcal{P}$, $\Omega^\bullet_{hor}(\mathcal{P})$ and $\Omega^\bullet_{vert}(\mathcal{P})$ are the horizontal and vertical components of $\Omega^\bullet(\mathcal{P})$ respectively.

Let $(\Omega^\bullet(\mathcal{H}), d_{\mathcal{H}})$ be the higher order quantum differential calculus on the structure quantum group $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, S)$ based on the universal envelope $\hat{T}^\bullet \underline{ns}(\mathcal{H})$ of the given first order quantum differential bicovariant calculus $\Gamma := \Omega^1(\mathcal{H})$, with differential $d_{\mathcal{H}} : \mathcal{H} \longrightarrow \Omega^1(\mathcal{H})$. Notable references for this section are the papers of M. Durdevich [19, 20], E.M. Einstein-Matthews [23] and S.L. Woronowicz [57]. Suppose that the differential graded $\mathbb{K}$-coherent algebra sheaf $(\Omega^\bullet(\mathcal{P}), d_{\mathcal{P}})$, is the quantum differential calculus on the total quantum space
$$\mathcal{P} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}} - \mathcal{S}h_M)$$
with $\Omega^0(\mathcal{P}) = \mathcal{P}$.

The right coaction $\varphi_{\mathcal{P}} \colon \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{H}$, then uniquely extends to a differential graded $\mathbb{K}$-algebra sheaf morphism
$$\hat{\varphi}_{\mathcal{P}} : \Omega^\bullet(\mathcal{P}) \longrightarrow \Omega^\bullet(\mathcal{P}) \hat{\otimes} \hat{T}^\bullet \underline{ns}(\mathcal{H}),$$

with

$$^{\wedge}\varphi_{\mathcal{P}} \colon \Omega^{\bullet}(\mathcal{P}) \longrightarrow \Omega^{\bullet}(\mathcal{P}) \otimes \mathcal{H}$$

given by the commutativity of the following diagram:

$$\begin{array}{ccc} \Omega^{\bullet}(\mathcal{P}) & \xrightarrow{\hat{\varphi}_{\mathcal{P}}} & \Omega^{\bullet}(\mathcal{P})\hat{\otimes}\hat{T}^{\bullet}\underline{ns}(\mathcal{H}) \\ {}^{\wedge}\varphi_{\mathcal{P}} \searrow & & \downarrow 1_{\Omega^{\bullet}(\mathcal{P})} \otimes proj_0 \\ & \Omega^{\bullet}(\mathcal{P}) \otimes \mathcal{H} & \end{array}$$

i.e.

$$^{\wedge}\varphi_{\mathcal{P}} := (1_{\Omega^{\bullet}(\mathcal{P})} \otimes proj_0)\hat{\varphi}_{\mathcal{P}}\,,$$

where $proj_0 : \hat{T}^{\bullet}\underline{ns}(\mathcal{H}) \longrightarrow \mathcal{H}$ is the projection $\mathbb{K}$-algebra morphism.

Let $\hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H})$ be the $\mathbb{K}$-subalgebra of left invariant elements of $\hat{T}^{\bullet}\underline{ns}(\mathcal{H})$ and

$$\pi_{inv} : \hat{T}^{\bullet}\underline{ns}(\mathcal{H}) \longrightarrow \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H})$$

the natural $\mathbb{K}$-algebra projection morphism.

We denote the adjoint coaction of $\mathcal{H}$ on $\Omega^1_{inv}(\mathcal{H})$ by

$$\bar{\omega} \colon \Omega^1_{inv}(\mathcal{H}) \longrightarrow \Omega^1_{inv}(\mathcal{H}) \otimes \mathcal{H}\,,$$

such that the diagram:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{ad_{\mathcal{H}}} & \mathcal{H} \otimes \mathcal{H} \\ {\scriptstyle \pi_{\Omega^1_{inv}(\mathcal{H})}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\Omega^1_{inv}(\mathcal{H})}\otimes 1_H} \\ \Omega^1_{inv}(\mathcal{H}) & \xrightarrow{\bar{\omega}} & \Omega^1_{inv}(\mathcal{H}) \otimes \mathcal{H} \end{array}$$

is commutative with the natural extension $\mathbb{K}$-algebra morphism:

$$\hat{\bar{\omega}} : \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H}) \longrightarrow \hat{T}^{\bullet}\underline{ns}_{inv}(\mathcal{H})\hat{\otimes}\hat{T}^{\bullet}\underline{ns}(\mathcal{H})$$

given by

$$\hat{\bar{\omega}}(\eta) = 1 \otimes \eta + \bar{\omega}(\eta),$$

for every $\eta \in \Omega^1_{inv}(\mathcal{H})$.

Definition 11. Define $\Theta(\mathcal{P}) := \Theta(\bar{\omega}, \mathcal{P})$ to be the space of all $\mathbb{K}$-linear morphisms:

$$\kappa : \Omega^1_{inv}(\mathcal{H}) \longrightarrow \Omega^{\bullet}(\mathcal{P})$$

such that the diagram

$$\begin{array}{ccc} \Omega^1_{inv}(\mathcal{H}) & \xrightarrow{\kappa} & \Omega^\bullet(\mathcal{P}) \\ \bar{\omega}\big\downarrow & & \big\downarrow{\scriptstyle \wedge\varphi_{\mathcal{P}}} \\ \Omega^1_{inv}(\mathcal{H})\otimes\mathcal{H} & \xrightarrow{\kappa\otimes 1_H} & \Omega^1(\mathcal{P})\otimes\mathcal{H} \end{array}$$

commutes.

We can now see that $\Theta(\mathcal{P}) := \Theta(\bar{\omega},\mathcal{P})$ is graded with $\Theta^\bullet(\mathcal{P}) := \Theta^\bullet(\bar{\omega},\mathcal{P}) := \bigoplus_{n\geq 0}\Theta^n(\bar{\omega},\mathcal{P})$, where $\Theta^n(\mathcal{P}) := \Theta^n(\bar{\omega},\mathcal{P})$ consists of all $\mathbb{K}$-linear maps $\kappa : \Omega^1_{inv}(\mathcal{H}) \longrightarrow \Omega^n(\mathcal{P})$. Clearly $\Theta^\bullet(\mathcal{P})$ is closed with respect to compositions with the differential operator $d_{\mathcal{P}} : \Omega^\bullet(\mathcal{P}) \longrightarrow \Omega^{\bullet+1}(\mathcal{P})$ and is a module over the subalgebra of the left-invariant differential forms.

We call the elements $\zeta \in \Theta^n(\mathcal{P})$ pseudotensorial quantum n-forms on the total quantum space $\mathcal{P} \in (\mathcal{C}ohAssoc\mathcal{A}lg_{\mathbb{K}}\text{-}Sh_M)$ with values in $\Omega^1_{inv}(\mathcal{H})$.

Definition 12. Let $\mathcal{P} = (\mathcal{P},\pi_{\mathcal{P}},\mathcal{M},\mathcal{H},(\gamma_\alpha)_{\alpha\in\mathfrak{A}})$ be a quantum principal bundle coherent algebra sheaf over $(M,g),\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ a locally finite open cover of M in $Op(M)$ together with inclusion maps $\imath_{\beta\alpha} : U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subset U_\alpha$ in $Op(M)$ and $\Theta^\bullet(\mathcal{P}) = \bigoplus_{n\geq 0}\Theta^n(\mathcal{P})$. We call an element $\omega \in \Theta^1(\mathcal{P})$ a connection on $\mathcal{P}$ if for each $\nu \in \Omega^1_{inv}(\mathcal{H})$-left invariant parts of differential forms on $(\mathcal{H},m,\eta,\Delta,\epsilon,S)$, we have $\pi_{vert}\omega(\nu) = 1\otimes\nu$, given by the composition:

$$\Omega^1_{inv}(\mathcal{H}) \xrightarrow{\omega} \Omega^1(\mathcal{P}) \xrightarrow{\pi_{vert}} \Omega^1_{vert}(\mathcal{P}) := \mathcal{P}\otimes\Omega^1_{inv}(\mathcal{H}),$$

for every $\nu \in \Omega^1_{inv}(\mathcal{H}), \omega(\nu) \in \Omega^1_{vert}(\mathcal{P}(U_\alpha))$, where

$$\pi_{vert} : \Omega^\bullet(\mathcal{P}) \to \Omega^\bullet_{vert}(\mathcal{P}) := \mathcal{P}\otimes\Omega^\bullet_{inv}(\mathcal{H}),$$

is the unique differential graded $\mathbb{K}$-algebra sheaf morphism reducing to $\Omega^0(\mathcal{P}) = \mathcal{P}$ and $\pi_{vert}(\Omega^\bullet_{vert}(\mathcal{P})\cap\Omega^\bullet(\mathcal{P})) = \Omega^\bullet_{vert}(\mathcal{P})$

We refer to [1] for the conditions for existence of a connection on a quantum principal bundle coherent algebra sheaf.

Theorem 6. *Let $\mathcal{P} = (\mathcal{P},\pi_{\mathcal{P}},\mathcal{M},\mathcal{H},(\gamma_\alpha)_{\alpha\in\mathcal{A}})$ be a quantum principal bundle coherent algebra sheaf over $(M,g),\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ a locally finite open cover of M in $Op(M)$ together with inclusion maps $\imath_{\beta\alpha} : U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subset U_\alpha$ in $Op(M)$. Then $\mathcal{P} = (\mathcal{P},\pi_{\mathcal{P}},\mathcal{M},\mathcal{H},(\gamma_\alpha)_{\alpha\in\mathfrak{A}})$ admits at least a connection if the following condition is satisfied:*

(ATC): Atiyah Condition: There is an Atiyah exact sequence of $\mathcal{P}$-coherent module sheaves

$$0 \longrightarrow \Omega^\bullet_{hor}(\mathcal{P}) \xrightarrow{\imath_{\Omega^\bullet_{hor}(\mathcal{P})}} \Omega^\bullet(\mathcal{P}) \xrightarrow{\pi_{vert}} \Omega^\bullet_{vert}(\mathcal{P}) := \Omega^\bullet(\mathcal{P})\big/\Omega^\bullet_{hor}(\mathcal{P}) \longrightarrow 0$$

which splits, i.e. there exists $\mathcal{P}$*-coherent module sheaf morphism:* $q_\omega : \Omega^\bullet_{vert}(\mathcal{P}) \longrightarrow \Omega^\bullet(\mathcal{P})$ *such that*

$$\pi_{vert} \circ q_\omega = 1_{\Omega^\bullet_{vert}(\mathcal{P})}.$$

In this case a connection is the existence of a $\mathcal{P}$*-coherent module sheaf* $\Omega^\bullet_{vert}(\mathcal{P})$ *such that*

$$\Omega^\bullet(\mathcal{P}) = \Omega^\bullet_{hor}(\mathcal{P}) \oplus \Omega^\bullet_{vert}(\mathcal{P}).$$

Proof. To show that the Atiyah sequence (ATC) is exact, it suffices to prove that for $\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ a locally finite open cover of M that trivializes $\mathcal{P} = (\mathcal{P}, \pi_{\mathcal{P}}, \mathcal{M}, \mathcal{H}, (\gamma_\alpha)_{\alpha\in\mathfrak{A}})$ so that the the sheaf morphism

$$\gamma_\alpha : \mathcal{M}\restriction_{U_\alpha} \otimes \mathcal{H} \longrightarrow \mathcal{P}\restriction_{U_\alpha}$$

for each $\alpha \in \mathfrak{A}$ define sheaf morphisms

$$\gamma_{\beta,\alpha} : \mathcal{M}\restriction_{U_\beta} \otimes \mathcal{H} \longrightarrow \mathcal{M}\restriction_{U_\alpha} \otimes \mathcal{H}$$

for all $\alpha, \beta \in \mathfrak{A}$ by

$$(\gamma_{\beta,\alpha})_{U_\delta} = (\gamma_\beta)^{-1}_{U_\delta} \circ (\gamma_\alpha)_{U_\delta},$$

for $U_\delta \in \{U_\alpha\}_{\alpha\in\mathfrak{A}}$ satisfying:

(1) $(\gamma_\alpha)_{U_\delta}(s \otimes 1) = \rho_U(s)$, for $s \in M(U_\delta), U_\delta \subset U_\alpha$, and

(2) $((\gamma_{\beta,\alpha})_{U_\delta} \otimes 1_{\mathcal{H}}) \circ (1 \otimes \Delta) = (1 \otimes \Delta) \circ (\gamma_{\beta,\alpha})_{U_\delta}$ for $U_\delta \subset U_{\alpha\beta} = U_\alpha \cap U_\beta$, where $\Delta\colon \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ is the comultiplication and

$$m\colon \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$$

the multiplication of the unital Hopf $\mathbb{K}$-algebra $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, S)$ and

$$\gamma_\alpha : \mathcal{M}\restriction_{U_\alpha} \otimes \mathcal{H} \longrightarrow \mathcal{P}\restriction_{U_\alpha}$$

is the $\mathcal{H}$-trivial quantum principal bundle coherent algebra sheaf over U_α for each $\alpha \in \mathfrak{A}$ and every graded quantum differential form $\nu \in \Omega^\bullet(\mathcal{P}\restriction_{U_\alpha})$ arises from a quantum differential form $s \in \Omega^\bullet(\mathcal{M}\restriction_{U_\alpha} \otimes \mathcal{H})$. Now since

$$s = \sigma \diamond 1,$$

where $\sigma\colon \mathcal{M}\restriction_{U_\alpha} \longrightarrow \mathcal{P}\restriction_{U_\alpha}$ extends to higher order differential forms defines a quantum differential graded form on $\mathcal{M}\restriction_{U_\alpha} \otimes \mathcal{H}$, then the push-forward

$$(\gamma_\alpha)_*\colon \Omega^\bullet(\mathcal{M}\restriction_{U_\alpha} \otimes \mathcal{H}) \longrightarrow \Omega^\bullet(\mathcal{P}\restriction_{U_\alpha} \otimes \mathcal{H})$$

gives the quantum differential graded form ν on $\mathcal{P}\restriction_{U_\alpha} \otimes \mathcal{H}$.

It is then easily seen that

$$\Omega^\bullet_{hor}(\mathcal{P}(U_\alpha)) \subset \mathrm{Ker}(\pi_{vert\restriction_{U_\alpha}}) \cap \Omega^\bullet(\mathcal{P}(U_\alpha))$$

and

$$\pi_{vert}\Omega^{\bullet}(\mathcal{P}(U_\alpha)) = \Omega^{\bullet}_{vert}(\mathcal{P}(U_\alpha)).$$

Thus for every $\omega \in \Omega^{\bullet}(\mathcal{P}(U_\alpha))$ we obtain the equation:

$$(1_{\Omega^{\bullet}(\mathcal{P})}\hat{\otimes} proj_k)\hat{\varphi}_{\mathcal{P}} \restriction_{U_\alpha} (\omega) = \sum_j (^\wedge\varphi^k{}_{\mathcal{P}} \restriction_{U_\alpha} (\omega_j)\zeta_j),$$

for every $\zeta_j \in \hat{T}^{\bullet - k}\underline{ns}_{inv}(\mathcal{H})$. Consequently,

$$\pi_{vert}(\omega) = \sum_j (^\wedge\varphi^k_{\mathcal{P}}) \restriction_{U_\alpha} (\omega_j) \;\hat{\otimes}\; \zeta_j.$$

Next if $\omega \in \mathrm{Ker}\pi_{vert} \restriction_{U_\alpha}$, then

$$(1_{\Omega^{\bullet}(\mathcal{P})}\hat{\otimes} proj_k)\hat{\varphi}_{\mathcal{P}} \restriction_{U_\alpha} (\omega) = 0$$

which implies that $\omega \in \Omega^{\bullet}_{hor}(\mathcal{P}(U_\alpha))$. Thus the Atiyah sequence is exact.

To show the existence of a connection we define a $\mathcal{P}$-module sheaf morphism:

$$q_\omega \colon \Omega^{\bullet}_{vert}(\mathcal{P}) \longrightarrow \Omega^{\bullet}(\mathcal{P})$$

by

$$q_\omega(u \otimes \nu) = u\omega(\nu),$$

for every $u \in \Omega^{\bullet}_{vert}(\mathcal{P}), \nu \in \hat{T}^{\bullet - k}\underline{ns}_{inv}(\mathcal{H})$ such that the diagram:

$$\begin{array}{ccc}
\Omega^{\bullet}_{vert}(\mathcal{P})\hat{\otimes}\hat{T}^{\bullet - k}\underline{ns}_{inv}(\mathcal{H}) & \xrightarrow{q_\omega \hat{\otimes} 1_{\hat{T}^{\bullet - k}\underline{ns}_{inv}(\mathcal{H})}} & \Omega^{\bullet}_{vert}(\mathcal{P})\hat{\otimes}\hat{T}^{\bullet - k}\underline{ns}_{inv}(\mathcal{H}) \\
{\scriptstyle ^\wedge q^k_{\mathcal{P}}\restriction_{\Omega^{\bullet}_{vert}(\mathcal{P}(U_\alpha))}}\Big\uparrow & & \Big\uparrow{\scriptstyle ^\wedge\varphi_{\mathcal{P}}} \\
\Omega^{\bullet}_{vert}(\mathcal{P}) & \xrightarrow{q_\omega} & \Omega^{\bullet}(\mathcal{P})
\end{array}$$

is commutative. Then one easily shows that

$$\pi_{vert} \restriction_{U_\alpha} \circ\, q_\omega \restriction_{U_\alpha} = 1_{\Omega^{\bullet}_{vert}(\mathcal{P}(U_\alpha))}$$

for every $U_\alpha \in \{U_\alpha\}_{\alpha \in \mathcal{A}}$, that

$$\pi_{vert} \circ q_\omega = 1_{\Omega^{\bullet}_{vert}(P)}.$$

So the Atiyah exact sequence splits and we get the decomposition:

$$\Omega^{\bullet}(\mathcal{P}) = \Omega^{\bullet}_{hor}(\mathcal{P}) \oplus \Omega^{\bullet}_{vert}(\mathcal{P}).$$

This completes the proof. □

In the sequel, we will denote the space of all connections, which are both regular and multiplicative, on the quantum total space of our principal bundle coherent algebra sheaf by $\mathcal{C}onn(\mathcal{P})$.

Definition 13. Let $\omega \in \mathcal{C}onn(\mathcal{P})$ be any connection on the quantum total space $\mathcal{P}$. A morphism

$$Curv_\omega : \Omega^1_{inv}(\mathcal{H}) \longrightarrow \Omega^\bullet(\mathcal{P})$$

given by

$$Curv_\omega = d_{\mathcal{P}}\omega - < \omega, \omega >$$

is a pseudotensorial 2-form called the quantum curvature of the connection ω.

5. Main Results

Let $\mathcal{P} = (\mathcal{P}, \pi_{\mathcal{P}}, \mathcal{M}, \mathcal{H}, (\gamma_\alpha)_{\alpha\in\mathfrak{A}})$ be a principal bundle coherent algebra sheaf over (M, g) with coordinate system $(\gamma_\alpha)_{\alpha\in\mathfrak{A}}$ and structure quantum group, the unital Hopf $\mathbb{K}$-algebra, $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, S)$. Let

$$\mathcal{A}, \mathcal{B} \in Ob(\mathcal{C}ohAssocAlg_{\mathbb{K}} - \mathcal{S}h_M).$$

We denote by $Z(\mathcal{A})$ the center of $\mathcal{A}$ and $Z(\mathcal{B})$ that of $\mathcal{B}$. The morphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is called a center preserving $\mathbb{K}$-algebra sheaf morphism if it maps the center $Z(\mathcal{A})$ of $\mathcal{A}$ into the center $Z(\mathcal{B})$ of $\mathcal{B}$ i.e. $\varphi(Z(\mathcal{A})) \subset Z(\mathcal{B})$.

Definition 14. Let $\mathcal{A}$, $\mathcal{M} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$ with $\mathcal{M}$ an $\mathcal{A}$-bimodule sheaf. We say that $\mathcal{M}$ is a central bimodule sheaf if for every $z \in Z(\mathcal{A}(U_\alpha))$ and $m \in \mathcal{M}(U_\alpha)$ we have the condition:

$$mz = zm\,,$$

where $(U_\alpha)_{\alpha\in\mathfrak{A}}$ is a locally finite open cover of M in $Op(M)$ together with inclusion maps $\imath_{\beta\alpha} : U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subset U_\alpha$ in $Op(M), \alpha, \beta \in \mathfrak{A}$.

We now let $[Z(\mathcal{A}), \mathcal{M}]$, be the sub-bimodule sheaf of $\mathcal{M}$ generated by $zm - mz$ for every $z \in Z(\mathcal{A}(U_\alpha))$ and $m \in \mathcal{M}(U_\alpha)$. Define the quotient bimodule sheaf $\mathcal{M}_Z$ by

$$\mathcal{M}_Z := \mathcal{M} \diagup [Z(\mathcal{A}), \mathcal{M}]$$

with canonical projection algebra sheaf morphism $\pi_Z : \mathcal{M} \longrightarrow \mathcal{M}_Z$. It is then immediate that every algebra sheaf morphism of central bimodule sheaves $\mathcal{M}, \mathcal{N} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$, i.e. $\varphi : \mathcal{M} \longrightarrow \mathcal{N}$ vanishes on $[Z(\mathcal{M}), \mathcal{M}]$. Thus there exists a unique algebra sheaf morphism

$$\varphi_Z : \mathcal{M}_Z \longrightarrow \mathcal{N}$$

such that the diagram:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\varphi} & \mathcal{N} \\ \pi_Z \searrow & & \nearrow \varphi_Z \\ & \mathcal{M}_Z & \end{array}$$

commutes, i.e.

$$\varphi = \varphi_Z \circ \pi_Z.$$

Thus an $(Z(\mathcal{A}), Z(\mathcal{A}))$-bimodule sheaf $\mathcal{M}$ is the underlying bimodule sheaf of $Z(\mathcal{A})$-bimodule sheaf. Define the sub-bimodule of $\Omega^1_{inv}(\mathcal{A})$ by

$$[Z(\mathcal{A}(U_\alpha)), \Omega^1_{inv}(\mathcal{A}(U_\alpha))] := \{z\omega - \omega z : z \in Z(\mathcal{A}(U_\alpha)), \omega \in \Omega^1_{inv}(\mathcal{A}(U_\alpha))\}.$$

Then, the quotient

$$\Omega^1_Z(\mathcal{A}) := \Omega^1_{univ}(\mathcal{A}) \diagup [Z(\mathcal{A}), \Omega^1_{univ}(\mathcal{A})]$$

is by definition a central bimodule sheaf.

Let $\pi_Z : \Omega^1_{univ}(\mathcal{A}) \longrightarrow \Omega^1_Z(\mathcal{A})$ be the canonical projection sheaf morphism and the derivation $d_Z \colon \mathcal{A} \longrightarrow \Omega^1_Z(\mathcal{A})$ given by the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{d_{univ}} & \Omega^1_{univ}(\mathcal{A}) \\ d_Z \searrow & & \downarrow \pi_Z \\ & \Omega^1_Z(\mathcal{A}), & \end{array}$$

i.e. $d_Z = \pi_Z \circ d_{univ}$. Thus $(\Omega^1_Z(\mathcal{A}), d_Z)$ is a first order quantum differential calculus on $\mathcal{A}$. If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is such that $\varphi(Z(\mathcal{A})) \subset Z(\mathcal{B})$, then there is a unique $\mathcal{A}$-bimodule sheaf morphism:

$$\varphi_* := \Omega^1_Z(\varphi) : \Omega^1_Z(\mathcal{A}) \longrightarrow \Omega^1_Z(\mathcal{B})\,,$$

such that the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \\ d_Z \downarrow & & \downarrow d_Z \\ \Omega^1_Z(\mathcal{A}) & \xrightarrow{\Omega^1_Z(\varphi)} & \Omega^1_Z(\mathcal{B}) \end{array}$$

is commutative.

Now define the $(\mathcal{A}, \mathcal{A})$-bimodule for $n \geq 0$ by $\Omega^0_{univ}(\mathcal{A}) = \mathcal{A}$ and

$$\Omega^n_{univ}(\mathcal{A}) = \Omega^1_{univ}(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_{univ}(\mathcal{A}) \cdots \otimes_{\mathcal{A}} \Omega^1_{univ}(\mathcal{A}),$$

n factors for $n \geq 1$. Write the canonical graded coherent algebra sheaf as:

$$\Omega^\bullet_{univ}(\mathcal{A}) = \bigoplus_{n \geq 0} \Omega^n_{univ}(\mathcal{A}),$$

which is the tensor coherent algebra sheaf $T_{\mathcal{A}}(\Omega^1_{univ}(\mathcal{A}))$ over $\mathcal{A}$ of the $(\mathcal{A},\mathcal{A})$-bimodule $\Omega^1_{univ}(\mathcal{A})$. The derivation

$$d_{univ}\colon \mathcal{A} \longrightarrow \Omega^1_{univ}(\mathcal{A})$$

has a unique extension

$$d_{univ}\colon \Omega^{\bullet}_{univ}(\mathcal{A}) \longrightarrow \Omega^{\bullet+1}_{univ}(\mathcal{A})$$

with $d^2_{univ} = 0$.

Let $\mathcal{J}_Z$ be the two-sided ideal generated by

$$[Z(\mathcal{A}), \Omega^1_{univ}(\mathcal{A})] \ \text{ and } \ d_{univ}([Z(\mathcal{A}), \Omega^1_{univ}(\mathcal{A})]).$$

The space $\mathcal{J}_Z$ is a graded two-sided coherent ideal sheaf which is closed such that

$$\mathcal{J}_Z \cap \Omega^1_{univ}(\mathcal{A}) = [Z(\mathcal{A}), \Omega^1_{inv}(\mathcal{A})].$$

This shows that the quotient $\Omega^{\bullet}_Z(\mathcal{A})$ is a differential graded coherent algebra sheaf which coincides in degree 1 with $\Omega^1_Z(\mathcal{A})$ and its differential the image of d_{univ} extends the differential

$$d_Z\colon \mathcal{A} \longrightarrow \Omega^1_Z(\mathcal{A})$$

which we again denote by

$$d_Z\colon \Omega^{\bullet}_Z(\mathcal{A}) \longrightarrow \Omega^{\bullet+1}_Z(\mathcal{A}).$$

Note that $\Omega^{\bullet}_Z(\mathcal{A})$ as a differential graded coherent algebra sheaf is the quotient of tensor coherent algebra sheaf $T_{\mathcal{A}}(\Omega^1_Z(\mathcal{A}))$ over $\mathcal{A}$ of the central bimodule $\Omega^1_Z(\mathcal{A})$. Thus the $(\mathcal{A},\mathcal{A})$-bimodule sheaves $\Omega^n_Z(\mathcal{A}), n \geq 0$ with $\Omega^0_Z(\mathcal{A}) = \mathcal{A}$, are central bimodule sheaves and $\Omega^{\bullet}_Z(\mathcal{A}) := \bigoplus_{n\geq 0}\Omega^n_Z(\mathcal{A})$. Therefore, the differential graded coherent algebra sheaf $(\Omega^{\bullet}_Z(\mathcal{A}), d_Z)$ is a higher order quantum differential calculus on $\mathcal{A}$.

Definition 15. The quantum de Rham cohomology $H^{\bullet}_{de\,R}Z(\mathcal{A})$ is the cohomology of the bimodule of quantum differential forms $\Omega^{\bullet}_Z(\mathcal{A})$, i.e.

$$H^{\bullet}_{de\,R}Z(\mathcal{A}) := H^{\bullet}(\Omega^{\bullet}_Z(\mathcal{A})) := \mathrm{Ker} d_Z \diagup Imd_Z.$$

Definition 16. Let $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \epsilon, S)$ be a unital Hopf $\mathbb{K}$-algebra. We define its quantum Weil algebra as the differential graded algebra $(ChW^{\bullet}(\mathcal{H}), d_{ChW})$ freely generated by elements h of degree $1, Curv_h$ of degree 2 and linear on $h \in \mathcal{H}$. The unique differential operator:

$$d_{ChW}\colon ChW^{\bullet}(\mathcal{H}) \longrightarrow ChW^{\bullet+1}(\mathcal{H})$$

is defined on generators by:

$$d_{ChW}(h) = Curv_h - \sum_{(h)} h_{(0)} \otimes h_{(1)}$$

and

$$d_{ChW}(Curv_h) = \sum_{(h)} Curv_{h(0)\otimes h(1)} - \sum_{(h)} h_{(0)} \otimes h_{(1)}.$$

Let $(\Omega^\bullet(\mathcal{H}), d)$ be a differential graded $\mathbb{K}$-algebra and $\psi : \mathcal{H} \longrightarrow \Omega^1(\mathcal{H})$ a $\mathbb{K}$-linear morphism. Define the curvature:

$$Curv_\psi : \mathcal{H} \longrightarrow \Omega^2(\mathcal{P})$$

of ψ by

$$Curv_\psi(h) := d_{\mathcal{P}}\psi(h) + \sum_{(h)} \psi(h_{(0)}) \otimes \psi(h_{(1)}).$$

Then there exists a unique $\mathbb{K}$-algebra morphism:

$$\kappa(\psi) : ChW^\bullet(\mathcal{H}) \longrightarrow \Omega^\bullet(\mathcal{H})$$

such that $\kappa(\psi)(h) = h$ and

$$\kappa(\psi)(Curv_h) = Curv_\psi(h).$$

Let $I(\mathcal{H})$ be the ideal of $ChW^\bullet(\mathcal{H})$ generated by the curvature $Curv_h$. The powers of $I(\mathcal{H})$ and the induced truncations denoted by:

$$I^{[n]}(\mathcal{H}) := (I(\mathcal{H}))^{(n+1)}$$

and

$$ChW^n(\mathcal{H}) := ChW^\bullet(\mathcal{H}) \diagup (I(\mathcal{H}))^{(n+1)}.$$

Note that

$$ChW^0(\mathcal{H}) := T^\bullet(\mathcal{H})$$

is the differential graded $\mathbb{K}$-algebra of $\mathcal{H}$.

We can also introduce the complex dual to even higher traces,

$$ChW^n(\mathcal{H})_\wedge := ChW^n(\mathcal{H}) \diagup [ChW^{(n-1)}(\mathcal{H}), ChW^n(\mathcal{H})]$$

by factoring the graded commutators. Dual to higher traces we have the graded ideal

$$I^{[n]}(\mathcal{H})_\wedge := I^{[n]}(\mathcal{H}) \diagup [I(\mathcal{H}), I^{[n-1]}(\mathcal{H})].$$

We now describe the quantum Weil differential graded $\mathbb{K}$-algebra in the form useful for our purposes. Let $\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, S)$ be a unital Hopf $\mathbb{K}$-algebra and $\mathcal{P} = (\mathcal{P}, \pi_{\mathcal{P}}, \mathcal{M}, \mathcal{H}, (\gamma_\alpha)_{\alpha\in\mathfrak{A}})$ a quantum principal bundle coherent algebra sheaf over (M, g) with coordinate system $(\gamma_\alpha)_{\alpha\in\mathfrak{A}}$. Further, suppose $(\Omega^\bullet(\mathcal{P}), d_{\mathcal{P}})$ is the quantum differential calculus on the total quantum space $\mathcal{P} \in Ob(\mathcal{C}oh\mathcal{A}ssoc\mathcal{A}lg_{\mathbb{K}}\text{-}\mathcal{S}h_M)$. For every $k \in \mathbb{N}^0$, let $I^k(\mathcal{H}) \subseteq T^k\underline{ns}_{inv}(\mathcal{H})$ be the subspace of $T^k\underline{ns}_{inv}(\mathcal{H})$ consisting of $\bar{\omega}^k$-invariant elements under the coaction:

$$\bar{\omega}^k \colon T^k \underline{ns}_{inv}(\mathcal{H}) \longrightarrow T^k \underline{ns}_{inv}(\mathcal{H}) \otimes \mathcal{H}.$$

Now set

$$I^\bullet(\mathcal{H}) := \bigoplus_{n \geq 0} I^k(\mathcal{H}).$$

Then $I^\bullet(\mathcal{H})$ is the unital graded $\mathbb{K}$-subalgebra of $T^\bullet \underline{ns}_{inv}(\mathcal{H})$ consisting of elements invariant under the coadjoint action

$$\bar{\omega}^\bullet \colon T^\bullet \underline{ns}_{inv}(\mathcal{H}) \longrightarrow T^\bullet \underline{ns}_{inv}(\mathcal{H}) \otimes \mathcal{H}.$$

Let $\omega \in \mathcal{C}onn(\mathcal{P})$. Then there exists a unique horizontal valued unital $\mathbb{K}$-algebra morphism:

$$Curv^\bullet_\omega \colon T^\bullet \underline{ns}_{inv}(\mathcal{H}) \longrightarrow \Omega^\bullet(\mathcal{P})$$

which multiplies degrees by 2 and is an expression of $\mathbb{K}$-algebra morphism:

$$Curv^\bullet_\omega \colon \Omega^1_{inv}(\mathcal{H}) \longrightarrow \Omega^\bullet(\mathcal{P}).$$

Then the diagram

$$\begin{array}{ccc} T^\bullet \underline{ns}_{inv}(\mathcal{H}) & \xrightarrow{Curv^\bullet_\omega} & \Omega^\bullet(\mathcal{P}) \\ {\scriptstyle \bar{\omega}^\bullet}\downarrow & & \downarrow{\scriptstyle \wedge\varphi_{\mathcal{P}}} \\ T^\bullet \underline{ns}_{inv}(\mathcal{H}) \otimes \mathcal{H} & \xrightarrow{Curv^\bullet_\omega \otimes 1_{\mathcal{H}}} & \Omega^\bullet(\mathcal{P}) \otimes \mathcal{H} \end{array}$$

is commutative. We now consider only the restriction of $Curv^\bullet_\omega$ to the unital $\mathbb{K}$-subalgebra $I^\bullet(\mathcal{H})$ of $T^\bullet \underline{ns}_{inv}(\mathcal{H})$ making the diagram

$$\begin{array}{ccc} I^\bullet(\mathcal{H}) & \xrightarrow{Curv_\omega \restriction_{I^\bullet(\mathcal{H})}} & \Omega^\bullet(\mathcal{P}) \\ {\scriptstyle \bar{\omega}^\bullet \restriction_{I^\bullet(\mathcal{H})}}\downarrow & & \downarrow{\scriptstyle \wedge\varphi_{\mathcal{P}}} \\ I^\bullet(\mathcal{H}) \otimes \mathcal{H} & \xrightarrow{Curv_\omega \restriction_{I^\bullet(\mathcal{H})} \otimes 1_{\mathcal{H}}} & \Omega^\bullet(\mathcal{P}) \otimes \mathcal{H} \end{array}$$

commutative.

Theorem 7. *Let $\mathcal{P} = (\mathcal{P}, \pi_{\mathcal{P}}, \mathcal{M}, \mathcal{H}, (\gamma_\alpha)_{\alpha \in \mathfrak{A}})$ be a quantum principal bundle coherent algebra sheaf over (M, g) with coordinate system $(\gamma_\alpha)_{\alpha \in \mathfrak{A}}$, $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$ a locally finite open cover of M in $Op(M)$ together with inclusion maps $\imath_{\beta\alpha} :$ $U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subset U_\alpha$ in $Op(M)$. Then with the notation above, if $\nu \in I^k(\mathcal{H})$, the form $Curv^k_\omega(\nu)$ belongs to $\Omega^{2k}_Z(\mathcal{M})$, the differential graded $\mathbb{K}$-coherent subalgebra of $\Omega^{2k}(\mathcal{M})$. Furthermore, if $\omega \in \mathcal{C}onn(\mathcal{P})$, then the $2k$-form $Curv^k_\omega(\nu)$ is $d_{\mathcal{M}}$-closed.*

Proof. To prove the first statement, since $\omega \in \mathcal{C}onn(\mathcal{P})$ we have the Leibniz rule:

(1) $d_\omega(\zeta \wedge \eta) = d_\omega(\zeta) \wedge \eta + (-1)^k \zeta \wedge d_\omega \eta$ for every $\zeta \in \Omega^k_{hor}(\mathcal{P})$ and $\eta \in \Omega^l_{hor}(\mathcal{P})$, where $d_\omega : \Omega^\bullet_{hor}(\mathcal{P}) \longrightarrow \Omega^{\bullet+1}_{hor}(\mathcal{P})$ is given by

$$d_\varphi = d_{\mathcal{P}}\varphi - (-1)^k \sum \varphi \wedge \omega \pi_{\Omega^1_{inv}(\mathcal{H})}(h_j)$$

for every $h_j \in \mathcal{H}$. This implies that

$$\hat{\varphi}_{\mathcal{P}} Curv^k_\omega(\nu) = \sum_j Curv^k_\omega(\nu_j) \otimes h_j$$

for every $\nu \in T^k \underline{ns}_{inv}(\mathcal{H})$ with

$$\sum_j \nu_j \otimes h_j = \bar{\omega}^k(\nu),$$

Finally, the proof of the first statement follows from the assumption that $\nu \in T^k \underline{ns}_{inv}(\mathcal{H})$ and the definition of $\Omega^\bullet_Z(\mathcal{M})$.

The second statement is a consequence of the Leibniz Rule (1), the condition

(2) $(d_{\mathcal{P}} - d_\omega)(\Omega^\bullet(\mathcal{M})) = \{0\}$, for every $\omega \in \mathcal{C}onn(\mathcal{P})$ and the quantum Bianchi Identity:

(3) $d_\omega Curv_\omega = 0$, for every $\mathcal{C}onn(\mathcal{P})$. □

Theorem 8. *For every* $\omega \in \mathcal{C}onn(\mathcal{P})$,

$$Curv_\omega = d_\omega \omega.$$

Proof. This is a consequence of the definitions of

$$\lambda_\omega : \Omega^\bullet(\mathcal{P}) \longrightarrow \Omega^\bullet_{vert,hor}(\mathcal{P})$$

and

$$Curv_\omega : \Omega^1_{inv}(\mathcal{H}) \longrightarrow \Omega^\bullet(\mathcal{P}).$$

Thus for any $\nu \in \Omega^1_{inv}(\mathcal{H})$ we have

$$Curv_\omega(\nu) \otimes 1 = \lambda^1_\omega d_{\mathcal{P}} \omega(\nu) - 1 \otimes \delta(\nu).$$

Therefore,

$$d_\omega \omega = h_\omega d_{\mathcal{P}} = Curv_\omega,$$

where

$$\delta : \Omega^1_{inv}(\mathcal{H}) \longrightarrow \Omega^1_{inv}(\mathcal{H}) \otimes \Omega^1_{inv}(\mathcal{H})$$

is the embedding differential. □

In the following theorem we show that the cohomology class of the $2k$-form $Curv^k_\omega(\nu)$ in $\Omega^{2k}_Z(\mathcal{M})$ is independent of the choice of the connection $\omega \in$

$\mathcal{C}onn(\mathcal{P})$.

Theorem 9. *Let* $\mathcal{P} = (\mathcal{P}, \pi_{\mathcal{P}}, \mathcal{M}, \mathcal{H}, (\gamma_\alpha)_{\alpha\in\mathfrak{A}})$ *be a quantum principal bundle coherent algebra sheaf over* (M, g) *with coordinate system* $(\gamma_\alpha)_{\alpha\in\mathfrak{A}}$, $\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ *a locally finite open cover of* M *in* $Op(M)$ *together with inclusion maps* $\imath_{\beta\alpha} : U_\beta \hookrightarrow U_\alpha$ *if* $U_\beta \subset U_\alpha$ *in* $Op(M)$. *Let* $\omega_1, \omega_2 \in \mathcal{C}onn(\mathcal{P})$ *and set*

(1) $\omega_t = \omega_1 + t(\omega_2 - \omega_1)$, *for each* $t \in [0, 1]$, *such that* ω_t *is a line segment in* $\mathcal{C}onn(\mathcal{P})$, *determined by* ω_1 *and* ω_2. *Then:*

(2) $\frac{d}{dt}(Curv_{\omega t}) = d_{\omega_t}(\eta)$, *where* $\eta = \omega_2 - \omega_1$, *and*

(3) $\quad d_{\omega_t} : \Omega^{\bullet}_{hor}(\mathcal{P}) \longrightarrow \Omega^{\bullet+1}_{hor}(\mathcal{P})$.

Proof. It is clear from the relations:

(4) $q_\omega \restriction_{\zeta^\bullet(\mathcal{P})} = 0$,

(5) $d_{\omega_t}\varphi = d_{\mathcal{P}}\varphi - (-1)^k[\varphi, \omega_t]$, for every $\omega_t \in \mathcal{C}onn(\mathcal{P})$ and $\varphi \in \zeta^k(P(U_\alpha))$ and the Leibniz Rule:

(6) $d_{\omega_t}(\varphi \wedge \psi) = d_{\omega_t}(\varphi) \wedge \psi + (-1)^k \varphi \wedge d_{\omega_t}\psi$ for all $\psi \in \zeta^l(\mathcal{P}(U_\alpha))$ that:

$$\begin{aligned}\frac{d}{dt}Curv_{\omega_t} &= \frac{d}{dt}\{d_{\mathcal{P}}\omega_1 + t d_{\mathcal{P}}\eta - < \omega_1 + t\eta, \omega_1 + t\eta >\} \\ &= d_{\mathcal{P}}\eta - < \eta, \omega_1 > - < \omega_1, \eta > -2t < \eta, \eta > \\ &= d_{\mathcal{P}}\eta - < \omega_1 + t\eta, \eta > - < \eta, \omega_1 + t\eta > = d_{\omega_t}(\eta). \quad \square\end{aligned}$$

Theorem 10. *Let* $\mathcal{P} = (\mathcal{P}, \pi_{\mathcal{P}}, \mathcal{M}, \mathcal{H}, (\gamma_\alpha)_{\alpha\in\mathfrak{A}})$ *be a quantum principal bundle coherent algebra sheaf over* (M, g) *with coordinate system* $(\gamma_\alpha)_{\alpha\in\mathfrak{A}}$ *and* $\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ *a locally finite open cover of* M *in* $Op(M)$ *together with inclusion maps* $\imath_{\beta\alpha} : U_\beta \hookrightarrow U_\alpha$ *if* $U_\beta \subset U_\alpha$ *in* $Op(M)$. *Furthermore, suppose* $H^{\bullet}_{de\,R}Z(\mathcal{M})$ *is the quantum de Rham cohomology of the classes associated to the graded* $\mathcal{M}$*-bimodule sheaf of quantum differential forms* $\Omega^{\bullet}_{Z}(\mathcal{M})$. *Then:*

(1) The cohomology class $[Curv^{\bullet}_{\omega}(\nu)] \in H^{2\bullet}_{de\,R}Z(\mathcal{M})$ *is independent of the choice of connection* $\omega \in \mathcal{C}onn(\mathcal{P})$, *for each* $\nu \in I^{\bullet}(\mathcal{H})$.

(2) The quantum Weil Homomorphism

$$ChW : I^{\bullet}(\mathcal{H}) \longrightarrow H^{\bullet}_{de\,R}Z(\mathcal{M})$$

given by

$$ChW(\nu) = [Curv^{\bullet}_{\omega}(\nu)]$$

is a unital $\mathbb{K}$*-algebra morphism.*

Proof. The proof of this theorem follows from Theorem 7 and the following consideration. Pick an element $\nu \in T^k\underline{ns}_{inv}(\mathcal{H})$ represented by $\nu = \sum_j z_j \nu_1^j \otimes$

$\cdots\otimes\nu_s^j$, where $z_j\in\mathbb{K}=\mathbb{C}$ and $\nu_s^j\in\Omega^1_{inv}(\mathcal{H})$. Then Theorem 3 and the Leibniz rule:

(1) $d_\omega(\zeta\wedge\eta)=d_\omega(\zeta)\wedge\eta+(-1)^k\zeta d_\omega(\eta)$ for the differential operator:

$$d_\omega\colon\Omega^\bullet_{hor}(\mathcal{P})\longrightarrow\Omega^{\bullet+1}_{hor}(\mathcal{P})$$

and $\eta\in\Omega^l_{hor}(\mathcal{P}(U_\alpha))$ together with

(2) $Curv^\bullet_\omega: T^\bullet\underline{ns}_{inv}(\mathcal{H})\longrightarrow\Omega^\bullet(\mathcal{P})$ and the quantum Bianchi Identity

(3) $d_\omega Curv_\omega=0$, for $\omega\in\mathcal{C}onn(\mathcal{P})$ imply that we can extend the results of the Theorem 3 to get

(4) $\frac{d}{dt}Curv^\bullet_{\omega_t}(\nu)=\sum_j z_j\{d_{\omega_t}\zeta(\nu_1^j)\cdots Curv_{\omega_t}(\nu_s^j)+\cdots+Curv_{\omega_t}(\nu_1^j)\cdots d_{\omega_t}\zeta(\nu_s^j)\}=\sum_j z_j d_{\omega_t}\{\zeta(\nu_1^j)\cdots Curv_{\omega_t}(\nu_s^j)+\cdots+Curv_{\omega_t}(\nu_j^1)\cdots\zeta(\nu_s^j)\}$.

The tensorial properties of ζ and $Curv_{\omega_t}$ imply that for $\nu\in I^k(\mathcal{H})$ we have that the $2k$-form

(5) $\Theta_t(\nu):=\sum_j z_j\{\zeta(\nu_1^j)\cdots Curv_{\omega_t}(\nu_s^j)+\cdots+Curv_{\omega_t}(\nu_1^j)\cdots\zeta(\nu_s^j)\}$ belong to $\Omega^{2k}_Z(\mathcal{M})$. Thus

(6) $\frac{d}{dt}Curv^\bullet_{\omega_t}(\nu)=d_{\mathcal{P}}\Theta_t(\nu)$. This follows from the fact that

(7) $(d_{\mathcal{P}}-d_{\omega_t})(\Omega^\bullet_Z(\mathcal{M}))=\{0\}$. Now integrate (6) from 0 to 1 to get

$$Curv_{\omega_t}(\nu)=Curv_{\omega_1}(\nu)+d(\int_0^1\Theta_t(\nu)dt).$$

Finally, consider the left-invariant canonical braid operator

$$\beta:\Omega^1_{inv}(\mathcal{H})\otimes\Omega^1_{inv}(\mathcal{H})\longrightarrow\Omega^1_{inv}(\mathcal{H})\otimes\Omega^1_{inv}(\mathcal{H}),$$

given by

$$\beta(\nu\otimes\xi)=\sum_j\xi_j\otimes(\nu_j\diamond h_j),$$

where the $\mathbb{K}$-algebra morphism:

$$\bar{\omega}\colon\Omega^1_{inv}(\mathcal{H})\longrightarrow\Omega^1_{inv}(\mathcal{H})\otimes\mathcal{H},$$

is defined by

$$\bar{\omega}(\nu)=\sum_j\nu_j\otimes h_j,$$

for all $h_j\in\mathcal{H}$.

Define a graded $\mathbb{K}$-algebra

$$\wedge^\bullet(\mathcal{H}):=T^\bullet\underline{ns}_{inv}(\mathcal{H})\diagup G(\mathcal{H}),$$

where $G(\mathcal{H})$ is the graded ideal of $T^\bullet\underline{ns}_{inv}(\mathcal{H})$ generated by the image $Im(I-\beta)\subseteq\Omega^1_{inv}(\mathcal{H})\otimes\Omega^1_{inv}(\mathcal{H})$.

The coadjont action

$$\bar{\omega}^{\bullet}\colon T^{\bullet}\underline{ns}_{inv}(\mathcal{H}) \longrightarrow T^{\bullet}\underline{ns}_{inv}(\mathcal{H}) \otimes \mathcal{H}$$

naturally induces the coaction:

$$\bar{\omega}_{\wedge^{\bullet}(\mathcal{H})} : \wedge^{\bullet}(\mathcal{H}) \longrightarrow \wedge^{\bullet}(\mathcal{H}) \otimes \mathcal{H}.$$

Let $I_{\wedge^{\bullet}(\mathcal{H})} \subseteq \wedge^{\bullet}(\mathcal{H})$ be a $\mathbb{K}$-subalgebra of $\wedge^{\bullet}(\mathcal{H})$ consisting of the $\bar{\omega}_{\wedge^{\bullet}(\mathcal{H})}$-invariant elements. Thus

$$I^{\bullet}_{\wedge^{\bullet}(\mathcal{H})} = I^{\bullet}(\mathcal{H}) \diagup I^{\bullet}_{\wedge^{\bullet}(\mathcal{H})} \cap G^{\bullet}(\mathcal{H}).$$ □

Theorem 11. *Let $\mathcal{P} = (\mathcal{P}, \pi_{\mathcal{P}}, \mathcal{M}, \mathcal{H}(\gamma_\alpha)_{\alpha\in\mathfrak{A}})$ be a quantum principal fiber bundle coherent algebra sheaf over (M, g) with coordinate system $(\gamma_\alpha)_{\alpha\in\mathfrak{A}}$ and $\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ a locally finite open cover of M in $Op(M)$ together with inclusion maps $\imath_{\beta\alpha} : U_\beta \hookrightarrow U_\alpha$ if $U_\beta \subset U_\alpha$ in $Op(M)$. Then with the notation above, if $\omega \in \mathcal{C}onn(\mathcal{P})$, we have*

$$(Curv^{\bullet}_{\omega} \circ \beta)(\nu) = Curv^{\bullet}_{\omega}(\nu),$$

for each $\nu \in \Omega^1_{inv}(\mathcal{H}) \otimes \Omega^1_{inv}(\mathcal{H})$, i.e. the following diagram:

$$\begin{array}{ccc} T^{\bullet}\underline{ns}_{inv}(\mathcal{H}) & \xrightarrow{Curv^{\bullet}_{\omega}} & \Omega^{\bullet}(\mathcal{P}) \\ \beta\uparrow & & \nearrow Curv^{\bullet}_{\omega} \\ & T^{\bullet}\underline{ns}_{inv}(\mathcal{H}) & \end{array}$$

is commutative, i.e. $Curv^{\bullet}_{\omega} \circ \beta = Curv^{\bullet}_{\omega}$.

Proof. From the use of the equations of commutativity:

(1) $Curv_\omega(\nu)\eta = \sum_j \eta_j Curv_\omega(\nu \diamond h_j)$, and

(2) $\eta Curv_\omega(\nu) = \sum_j Curv_\omega(\nu \diamond S^{-1}(h_j))\eta_j$,

for every $\eta \in \Omega^{\bullet}_{hor}(\mathcal{P}(U_\alpha))$ and $\nu \in \Omega^1_{inv}(\mathcal{H})$ we get the relation:

$$\mu_\Omega(Curv_\omega \otimes \eta) = \mu_\Omega(\eta \otimes Curv_\omega) \circ \beta,$$

for every $\eta \in \zeta^{\bullet}(\mathcal{P}(U_\alpha))$. This proves the assertion. □

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