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POSITIVE SOLUTIONS OF DELAYED DISCRETE EQUATIONS

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Abstract: The main goal of this paper is to give a new criterion for the existence of positive solutions for delayed discrete equations

$$\Delta u(n+k) = f(n, u(n), u(n+1), \ldots, u(n+k)).$$

Sufficient conditions with respect to f are formulated in order to guarantee the existence of a positive solution for $n \to \infty$. The upper estimate for it is given as well. We show that the result presented generalizes the previous results in this direction. As an example, the result obtained is applied to a linear difference equation with delay.

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1. Introduction

The phenomenon of existence of a positive solution of differential or difference equations often arises when we discuss mathematical models describing various processes. The existence of positive solutions is very often substantial for a concrete model considered. This provides a motivation to study the conditions guaranteeing the existence of positive solutions of differential and difference equations and the properties of such solutions.

Let us note that investigations in this field can be found e.g. in [1]–[12].

In this paper, conditions guaranteeing the existence of a positive solution are given for a class of nonlinear delayed discrete equations. We indicate sufficient conditions for the existence of a positive solution. Moreover, we also give an upper estimate for it. With respect to the results in the existing literature, we remark that, mostly, only concrete classes of linear and nonlinear discrete equations were considered. We will show that the given sufficient conditions are sharp in a sense. This is illustrated by a simple linear equation with a suitable right-hand side and with a coefficient satisfying the indicated inequalities.

2. Preliminary

For given integers s, q, $s<q$, we set $\mathbf{Z}_s^q := \{s, s+1, \dots, q\}$. The case $s=-\infty$ or $q=\infty$ is admitted, too. We will consider the scalar delayed discrete equation

$$\Delta u(n+k) = f(n, u(n), u(n+1), \dots, u(n+k)), \tag{1}$$

where $f(n, u_0, u_1, \dots, u_k)$ is defined on $\mathbf{Z}_a^\infty \times \mathbf{R}^{n+1}$, $a \in \mathbf{N}$ with values in $\mathbf{R}$, $a \in \mathbf{N}$ and $k \in \mathbf{N}$ are fixed, $\mathbf{N} := \{1, 2, \dots\}$. In this paper, we are interested in the existence of a positive solution of equation (1) for $n \to \infty$.

Together with (1) we consider an initial problem. It is defined as follows: for a given $s \in \mathbf{Z}_a^\infty$ we are looking for the solution of (1) satisfying $k+1$ initial conditions

$$u(a+s+m) = u^{s+m} \in \mathbf{R},\ m = 0, 1, \dots, k \tag{2}$$

with prescribed constants u^{s+m}.

Let us recall that the solution of the initial problem (1), (2) is defined as an infinite sequence of numbers

$$\{u(a+s) = u^s, u(a+s+1) = u^{s+1}, \dots, u(a+s+k) = u^{s+k},$$
$$u(a+s+k+1), u(a+s+k+2), \dots$$

such that for any $n \in N(a+s)$ the equality (1) holds. The existence and uniqueness of the solution of the initial problem (1), (2) is obvious for every $n \in \mathbf{Z}_{a+s}^{\infty}$. Moreover, if the function $f(n, u_0, u_1, \ldots, u_k)$ is continuous with respect to the variables $u_0, u_1, \ldots, u_k$, then the initial problem (1), (2) depends continuously on the initial data. For every $n \in \mathbf{Z}_a^{\infty}$, we define a set

$$\omega(n) := \{u \in \mathbf{R} : b(n) < u < c(n)\}, \tag{3}$$

where b and c are real functions defined on $\mathbf{Z}_a^{\infty}$ satisfying $b(n) < c(n)$.

The following theorem, which will be used below, is a slight modification of Theorem 1 in [3] and can be proved in a similar way. Therefore we omit its proof.

Theorem 1. *Let $f(n, u_0, u_1, \ldots, u_k) : \mathbf{Z}_a^{\infty} \times \mathbf{R}^{n+1} \to \mathbf{R}$ be continuous with respect to the variables $u_0, u_1, \ldots, u_k$. If, moreover,*

$$f(n, u_0, u_1, \ldots, u_{k-1}, b(n+k)) - b(n+k+1) + b(n+k) < 0 \tag{4}$$

and

$$f(n, u_0, u_1, \ldots, u_{k-1}, c(n+k)) - c(n+k+1) + c(n+k) > 0 \tag{5}$$

hold for every $n \in \mathbf{Z}_a^{\infty}$ and every

$$u_0 \in \omega(n), u_1 \in \omega(n+1), \ldots, u_{k-1} \in \omega(n+k-1),$$

then there exists an initial problem

$$u^*(a+m) = u_m^* \in \mathbf{R},\ m = 0, 1, \ldots, k$$

with

$$u_0^* \in \omega(a), u_1^* \in \omega(a+1), \ldots, u_n^* \in \omega(a+k)$$

such that the corresponding solution $u = u^*(n)$ of (1) satisfies

$$b(n) < u^*(n) < c(n)$$

for every $n \in \mathbf{Z}_a^{\infty}$.

3. Auxiliary Lemmas

We formulate auxiliary results which will be used later. Their proofs are omitted since they can be performed easily using binomial and Taylor formulas. Let us recall that the symbol O used below means the Landau order symbol.

Lemma 1. *Let $\sigma \in \mathbf{R}$ and $d \in \mathbf{R}$ be fixed. Then the asymptotic decom-*

position

$$\left(1+\frac{d}{n}\right)^{\sigma}=1+\frac{\sigma d}{n}+\frac{\sigma(\sigma-1)d^2}{2n^2}+O\left(\frac{1}{n^3}\right) \tag{6}$$

holds for $n\to\infty$.

Lemma 2. *Let* $\sigma\in\mathbf{R}$ *be fixed. Then the asymptotic decomposition*

$$[\ln(n-\sigma)]^{1/2} = (\ln n)^{1/2}\left[1-\frac{\sigma}{2n\ln n}-\frac{\sigma^2}{4n^2\ln n}-\frac{\sigma^2}{8(n\ln n)^2}+O\left(\frac{1}{n^3}\right)\right] \tag{7}$$

holds for $n\to\infty$.

4. Existence of a Positive Solution

In this part we give sufficient conditions guaranteeing the existence of a positive solution of equation (1).

Theorem 2. *Let* $a\in\mathbf{N}$ *and* $k\in\mathbf{N}$ *be fixed. Let* $f(n,u_0,u_1,\dots,u_k):\mathbf{Z}_a^\infty\times\mathbf{R}^{n+1}\to\mathbf{R}$ *be continuous with respect to the variables* $u_0,u_1,\dots,u_k$. *If, moreover, there exists a constant* $\theta\in[0,1)$ *such that*

$$-\left(\frac{k}{k+1}\right)^{n+k}\cdot\sqrt{n\ln n}\cdot\left(\frac{1}{k+1}+\frac{k}{8n^2}+\frac{\theta k}{8n^2(\ln n)^2}\right) < f\left(n,u_0,u_1,\dots,u_{k-1},\sqrt{(n+k)\ln(n+k)}\cdot\left(\frac{k}{k+1}\right)^{n+k}\right) \tag{8}$$

and

$$f\left(n,u_0,u_1,\dots,u_{k-1},0\right)<0 \tag{9}$$

for every $n\in N(a)$ *and every* $u_0\in\omega(n), u_1\in\omega(n+1),\dots,u_{k-1}\in\omega(n+k-1)$ *with*

$$b(n):=0,\quad c(n):=\sqrt{n\ln n}\cdot\left(\frac{k}{k+1}\right)^{n},$$

then there exists a positive integer $a_1\geq a$ *and a solution* $u=u(n)$, $n\in\mathbf{Z}_{a_1}^\infty$ *of equation* (1) *such that*

$$u(n)>0 \tag{10}$$

holds for every $n\in\mathbf{Z}_{a_1}^\infty$.

Proof. In the proof, Theorem 1 is used. In our case we have (see (3))

$$\omega(n) = \left\{ u \in \mathbf{R} : 0 < u < \sqrt{n \ln n} \cdot \left(\frac{k}{k+1} \right)^n \right\}$$

for every $n \in \mathbf{Z}_a^\infty$. Let us verify that inequality (4) holds. It is easy to verify that (due to inequality (9))

$$f(n, u_0, u_1, \dots, u_{k-1}, b(n+k)) - b(n+k+1) + b(n+k) = f(n, u_0, u_1, \dots, u_{k-1}, 0) < 0$$

for every $n \in \mathbf{Z}_a^\infty$ and every

$$u_0 \in \omega(n), u_1 \in \omega(n+1), \dots, u_{k-1} \in \omega(n+k-1).$$

More difficult is the verification of inequality (5). For sufficiently large $n \in \mathbf{Z}_a^\infty$ and for every

$$u_0 \in \omega(n), u_1 \in \omega(n+1), \dots, u_{k-1} \in \omega(n+k-1),$$

we obtain

$$\begin{aligned} &f(n, u_0, u_1, \dots, u_{k-1}, c(n+k)) - c(n+k+1) + c(n+k) \\ &= f\left(n, u_0, u_1, \dots, u_{k-1}, \sqrt{(n+k)\ln(n+k)} \cdot \left(\frac{k}{k+1} \right)^{n+k} \right) \\ &\quad - \sqrt{(n+k+1)\ln(n+k+1)} \cdot \left(\frac{k}{k+1} \right)^{n+k+1} \\ &\quad + \sqrt{(n+k)\ln(n+k)} \cdot \left(\frac{k}{k+1} \right)^{n+k}. \end{aligned}$$

Further, the inequality (8) yields

$$\begin{aligned} &f(n, u_0, u_1, \dots, u_{k-1}, c(n+k)) - c(n+k+1) + c(n+k) \\ &> -\left(\frac{k}{k+1} \right)^{n+k} \cdot \sqrt{n \ln n} \cdot \left(\frac{1}{k+1} + \frac{k}{8n^2} + \frac{\theta k}{8n^2 (\ln n)^2} \right) \\ &\quad - \sqrt{(n+k+1)\ln(n+k+1)} \cdot \left(\frac{k}{k+1} \right)^{n+k+1} \\ &\quad + \sqrt{(n+k)\ln(n+k)} \cdot \left(\frac{k}{k+1} \right)^{n+k} \\ &= \left(\frac{k}{k+1} \right)^{n+k} \cdot \sqrt{n \ln n} \cdot \left[-\left(\frac{1}{k+1} + \frac{k}{8n^2} + \frac{\theta k}{8n^2} \right) \right. \end{aligned}$$

$$-\sqrt{\left(1+\frac{k+1}{n}\right)\frac{\ln(n+k+1)}{\ln n}}\cdot\left(\frac{k}{k+1}\right)+\sqrt{\left(1+\frac{k}{n}\right)\frac{\ln(n+k)}{\ln n}}\,\Bigg].$$

Now we develop asymptotic decompositions of functions in the last expression. Using formula (6) with $\sigma = 1/2$, $d = k+1$ and with $\sigma = 1/2$, $d = k$ we get

$$\sqrt{1+\frac{k+1}{n}} = 1+\frac{k+1}{2n}-\frac{(k+1)^2}{8n^2}+O\left(\frac{1}{n^3}\right)$$

and

$$\sqrt{1+\frac{k}{n}} = 1+\frac{k}{2n}-\frac{k^2}{8n^2}+O\left(\frac{1}{n^3}\right).$$

Moreover, using formula (7) with $\sigma = -k-1$ and with $\sigma = -k$ we have

$$\sqrt{\frac{\ln(n+k+1)}{\ln n}} = 1+\frac{k+1}{2n\ln n}-\frac{(k+1)^2}{4n^2\ln n}-\frac{(k+1)^2}{8(n\ln n)^2}+O\left(\frac{1}{n^3}\right)$$

and

$$\sqrt{\frac{\ln(n+k)}{\ln n}} = 1+\frac{k}{2n\ln n}-\frac{k^2}{4n^2\ln n}-\frac{k^2}{8(n\ln n)^2}+O\left(\frac{1}{n^3}\right).$$

Therefore

$$\begin{aligned}
&f(n,u_0,u_1,\dots,u_{k-1},c(n+k))-c(n+k+1)+c(n+k)\\
&\quad>\left(\frac{k}{k+1}\right)^{n+k}\cdot\sqrt{n\ln n}\cdot\Bigg[-\left(\frac{1}{k+1}+\frac{k}{8n^2}+\frac{\theta k}{8n^2(\ln n)^2}\right)\\
&\qquad-\left(1+\frac{k+1}{2n}-\frac{(k+1)^2}{8n^2}+O\left(\frac{1}{n^3}\right)\right)\\
&\qquad\times\left(1+\frac{k+1}{2n\ln n}-\frac{(k+1)^2}{4n^2\ln n}-\frac{(k+1)^2}{8(n\ln n)^2}+O\left(\frac{1}{n^3}\right)\right)\cdot\left(\frac{k}{k+1}\right)\\
&\qquad+\left(1+\frac{k}{2n}-\frac{k^2}{8n^2}+O\left(\frac{1}{n^3}\right)\right)\\
&\qquad\cdot\left(1+\frac{k}{2n\ln n}-\frac{k^2}{4n^2\ln n}-\frac{k^2}{8(n\ln n)^2}+O\left(\frac{1}{n^3}\right)\right)\Bigg].
\end{aligned}$$

It is easy to verify that the expression in square brackets equals

$$\begin{aligned}
&-\left(\frac{1}{k+1}+\frac{k}{8n^2}+\frac{\theta k}{8n^2(\ln n)^2}\right)\\
&\quad-\left(1+\frac{k+1}{2n}+\frac{k+1}{2n\ln n}-\frac{(k+1)^2}{8n^2}-\frac{(k+1)^2}{8(n\ln n)^2}+O\left(\frac{1}{n^3}\right)\right)\cdot\left(\frac{k}{k+1}\right)\\
&+1+\frac{k}{2n}+\frac{k}{2n\ln n}-\frac{k^2}{8n^2}-\frac{k^2}{8(n\ln n)^2}+O\left(\frac{1}{n^3}\right)=\frac{k\cdot(1-\theta)}{8n^2\ln^2 n}+O\left(\frac{1}{n^3}\right)
\end{aligned}$$

and, consequently,

$$f(n,u_0,u_1,\dots,u_{k-1},c(n+k))-c(n+k+1)+c(n+k)$$
$$> \left(\frac{k}{k+1}\right)^{n+k}\cdot\sqrt{n\ln n}\cdot\left[\frac{k\cdot(1-\theta)}{8n^2\ln^2 n}+O\left(\frac{1}{n^3}\right)\right].$$

We conclude that there exists an integer $a_1 \ge a$ such that the inequality

$$\frac{k(1-\theta)}{8n^2}+O\left(\frac{1}{n^3}\right)>0$$

holds for every $n\in \mathbf{Z}_{a_1}^{\infty}$. Consequently,

$$f(n,u_0,u_1,\dots,u_{k-1},c(n+k))-c(n+k+1)+c(n+k)>0,$$

i.e. the inequality (5) holds for every $n\in \mathbf{Z}_{a_1}^{\infty}$. This means that assumptions of Theorem 1 are true with $a:=a_1$. Then there exists an initial problem

$$u^*(a_1+m)=u^*_m\in\mathbf{R},\ \ m=0,1,\dots,k$$

with

$$u^*_0\in\omega(a_1),u^*_1\in\omega(a_1+1),\dots,u^*_k\in\omega(a_1+k)$$

such that the corresponding solution $u=u^*(n)$ of equation (1) satisfies $u^*(n)>0$ for every $n\in\mathbf{Z}_{a_1}^{\infty}$, i.e. (10) holds. The theorem is proved. □

Taking into account the form of the set $\omega(n)$ for every $n\in\mathbf{Z}_a^{\infty}$ given by inequality (4), it is easy to improve the assertion of Theorem 2.

Theorem 3. (Estimate of a Positive Solution) *Let all the assumptions of Theorem 2 be true. Then there exists a positive integer $a_1\ge a$ and a solution $u=u(n)$, $n\in\mathbf{Z}_{a_1}^{\infty}$ of equation* (1) *such that*

$$0<u(n)<\sqrt{n\ln n}\cdot\left(\frac{k}{k+1}\right)^{n} \tag{11}$$

holds for every $n\in\mathbf{Z}_{a_1}^{\infty}$.

Remark 4. Let us note that the assumptions $n\in\mathbf{N}$ in Theorems 2, 3 cannot be weakened to $n\in\mathbf{N}\cup\{0\}$. Indeed, if $n=0$, it is easy to see that the proof of Theorem 2 as well as the formula (11) lose any sense. This means that results presented can only be applied in the case of (substantially) delayed discrete equations.

5. An Application

Let us consider the delayed scalar linear discrete equation

$$\Delta u(n+k) = -p(n)u(n) \tag{12}$$

with fixed $k \in \mathbf{N}$ and variable $n \in \mathbf{Z}_a^\infty$. We assume the function $p\colon \mathbf{Z}_a^\infty \to \mathbf{R}$ to be positive. Let us apply Theorems 2, 3 to the case of equation (12). The following result is a consequence of the theorems mentioned in the case of

$$f(n, u(n), u(n+1), \dots, u(n+k)) := -p(n)u(n).$$

Theorem 5. *Let $a \in \mathbf{N}$ and $n \in \mathbf{N}$ be fixed. Let there exist a constant $\theta \in [0,1)$ such that the function $p\colon \mathbf{Z}_a^\infty \to \mathbf{R}$ satisfies the inequalities*

$$0 < p(n) \le \left(\frac{k}{k+1}\right)^k \cdot \left(\frac{1}{k+1} + \frac{k}{8n^2} + \frac{\theta k}{8(n \ln n)^2}\right) \tag{13}$$

for every $n \in \mathbf{Z}_a^\infty$. Then there exists a positive integer $a_1 \ge a$ and a solution $u = u(n)$, $n \in \mathbf{Z}_{a_1}^\infty$ of equation (12) such that the inequalities

$$0 < u(n) < \sqrt{n \ln n} \cdot \left(\frac{k}{k+1}\right)^n$$

hold for every $n \in \mathbf{Z}_{a_1}^\infty$.

6. Concluding Remarks

Problems concerning the asymptotic behaviour of solutions of discrete equations are widely investigated. Let us remark that the result given by Theorem 4 improves the previous one given in [10, p. 192]. Our results are sharp in a sense. Indeed, it is known (see e.g. [10, p. 179]) that, if $p(n) = p = \text{const}$ and an opposite inequality with respect to (13) holds, namely the inequality

$$p > \left(\frac{k}{k+1}\right)^k \cdot \frac{1}{k+1},$$

then all the solutions of (12) oscillate. This inequality is a necessary and sufficient condition for the oscillation of all the solutions of the discrete equation (12) with a constant coefficient.

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