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## ON CHAOTIC MAPS IN BI-INFINITE SYMBOL SPACE

Inese Bula[1 §], Irita Rumbeniece[2]

[1,2]Department of Mathematics
University of Latvia
19, Rainis Blvd., Riga, LV-1586, LATVIA
[1]e-mail: ibula@lanet.lv
[2]e-mail: navita@one.lv

**Abstract:** This paper analyzes mappings in bi-infinite symbol space and proves that the maps are chaotic. It formulates sufficient conditions for non-chaotic mapping in a symbol space.

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### 1. Introduction

Dynamical systems orginally arose in the study of systems of differential equations used to model physical phenomena. The technique of characterizing the orbit structure of a dynamical system via infinite sequences of "symbols" (in our case 0's and 1's) is known as *symbolic dynamics*. The technique is not new and appears to have orginally been applied by Hadamard (see [4]) in the study of geodesics on surfaces of negative curvature. The first exposition of symbolic dynamics as an independent subject was given by Morse and Hedlund (see [10]). They showed that in many circumstances such a finite description of the dynamics is possible. Applications of this idea to differerntial equations can be found in N. Levinson (see [7]), where Smale got inspiration for his construction of the horseshoe map (see [12], [13]).

Other ideas in symbolic dynamics come from data storage and transmission. D. Lind and B. Marcus in 1995 have published the first general textbook on symbolic dynamics (see [8]) and its applications to coding. This book and B.P.

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§Correspondence author

Kitchens (see [6]) give a good account of the history of symbolic dynamics and its applications.

In this article we consider two symbols 0 and 1 bi-infinite sequence space and investigate some different mappings in this symbol space. Well known is shift map, which is a chaotic map in symbolic space. But there exist other chaotic maps in symbol space as well. We consider some continuous mappings and show that they are chaotic. In our fantasy it is possible to make topological conjugacy or semi-conjugacy between chaotic map in symbol space and real valued functions. If it is possible, then the last functions are chaotic too. At the moment we cannot make this conjugacy.

The article is organized as follows. We start with short description of bi-infinite symbol space and then give a definition of chaotic map. In Section 4 we consider shift map and prove that every iteration map of shift map is chaotic map as well. In Section 5 we investigate other chaotic maps in symbol space. Finally we give sufficient condition for non-chaotic map in two symbol bi-infinite symbol space.

## 2. Bi-Infinite Symbol Space

**Definition 2.1.** The set of all bi-infinite sequences of 0s and 1s is called a sequence space of 0 and 1 or the *symbolic space* of 0 and 1, i.e.,

$$\Sigma = \{ \ ...s_{-2}s_{-1}\ .s_0 s_1 s_2 ... \ | s_i = 0 \text{ or } s_i = 1 \text{ for every } i \in \mathbf{Z}\}.$$

We will refer to $\Sigma$ as the space of bi-infinite sequence of two symbols. We introduce a metric structure on $\Sigma$ by

$$\forall s = ...s_{-2}s_{-1}\ .s_0 s_1 s_2 ..., \ t = ...t_{-2}t_{-1}\ .t_0 t_1 t_2 ... \in \Sigma : \ d(s,t) = \sum_{i=-\infty}^{+\infty} \frac{|s_i - t_i|}{2^{|i|}}.$$

This indeed is a metric (see, for example, [2]) therefore $(\Sigma, d)$ is a metric space. But this metric is not single. $\Sigma$ forms a metric space if we replace number 2 with $\lambda > 1$ as well (for example, in [11] a case with $\lambda = 3$ and $\lambda = 4$ is examined, in [5], [6] or [14] $\lambda = 2$). In [8] should capture the idea that points are close when large central blocks of their coordinates agree. Specifically, if $s, t \in \Sigma$, then metric is

$$d_1(s,t) = \begin{cases} 0, & \text{if } s = t, \\ \frac{1}{2^k}, & \text{if } s \neq t \text{ and } k \text{ is maximal so that } s_{[-k,k]} = t_{[-k,k]}. \end{cases}$$

$s_{[-k,k]} = t_{[-k,k]}$ means that $s_i = t_i$, $i \in \{-k, -(k-1), ..., -1, 0, 1, 2, ..., k-1, k\}$.

We note again that the two sequences are close if they agree on a long central block in metric space $(\Sigma, d)$ too. The following lemma makes this precise.

**Lemma 2.2.** *Let* $s = ...s_{-2}s_{-1}.s_0s_1s_2...$ *and* $t = ...t_{-2}t_{-1}.t_0t_1t_2...$ *be sequences of* $\Sigma$ *and* $n \in \{0, 1, 2, ...\}$.

*1) If* $s_i = t_i$ *for* $|i| \leq n$*, then* $d(s,t) \leq \frac{1}{2^{n-1}}$.

*2) On the other hand, if* $d(s,t) \leq \frac{1}{2^n}$*, then* $\forall |i| < n: \; s_i = t_i$ *(or* $\forall |i| \leq n-1: \; s_i = t_i$*).*

*Proof.* Literature gives a proof of this lemma for one-sided infinite sequences, for example, [5]. For two-sided or bi-infinite sequence symbol space, for example, in [14] a proof is given in another metric. Therefore we prove our case.

1) Let $s = ...s_{-2}s_{-1}.s_0s_1s_2...$ and $t = ...t_{-2}t_{-1}.t_0t_1t_2...$ be sequences of $\Sigma$. Suppose $s_i = t_i$ for $|i| \leq n$. Then

$$d(s,t) = \sum_{i=-\infty}^{+\infty} \frac{|s_i - t_i|}{2^{|i|}} = \sum_{i=-\infty}^{-(n+1)} \frac{|s_i - t_i|}{2^{|i|}} + \sum_{i=n+1}^{+\infty} \frac{|s_i - t_i|}{2^{|i|}}$$

$$\leq 2 \sum_{i=n+1}^{+\infty} \frac{1}{2^i} = 2 \cdot \frac{1}{2^{n+1}} \cdot (1 + \frac{1}{2} + ...) = 2 \cdot \frac{1}{2^{n+1}} \cdot 2 = \frac{1}{2^{n-1}}.$$

2) On the other hand, if there is $j$ such that $|j| < n$ and $s_j \neq t_j$, then

$$d(s,t) = \sum_{i=-\infty}^{+\infty} \frac{|s_i - t_i|}{2^{|i|}} \geq \frac{1}{2^{|j|}} > \frac{1}{2^n}$$

but this contradicts the hypothesis that $d(s,t) \leq \frac{1}{2^n}$. □

The space $(\Sigma, d)$ has more specific and interesting properties (see, [8] or [14]).

## 3. Definition of Chaotic Map

The term "chaos" in reference to functions was first used in Li and Yorke's paper "Period three implies chaos" (see [9]). We use following definition of R. Devaney (see [2]). Let $(X, \rho)$ be metric space.

**Definition 3.1.** (see [2]) The function $f : X \to X$ is *chaotic* if:

a) the periodic points of $f$ are dense in $X$,

b) $f$ is topologically transitive,

c) $f$ exhibits sensitive dependence on initial conditions.

**Definition 3.2.** The function $f : X \to X$ is *topologically transitive* on $X$ if

$$\forall x, y \in X \; \forall \varepsilon > 0 \; \exists z \in X \; \exists n \in \mathbf{N} : \; \rho(x, z) < \varepsilon \text{ and } \rho(f^n(z), y) < \varepsilon.$$

**Definition 3.3.** The function $f : X \to X$ *exhibits sensitive dependence on initial conditions* if

$$\exists \delta > 0 \; \forall x \in X \; \forall \varepsilon > 0 \; \exists y \in X \; \exists n \in \mathbf{N} : \rho(x, y) < \varepsilon \text{ and } \rho(f^n(x), f^n(y)) > \delta.$$

Devaneys definition is not the only definition of a chaotic map. For example, another definition can be found in [11]. Also mappings with only one property — sensitive dependence on initial conditions — frequently are considered as chaotic (see [3]). Banks, Brooks, Cairns, Davis and Stacey [1] has demonstrated that for continuous functions, the defining charactrristics of chaos are topological transitivity and the density of periodic points.

**Theorem 3.4.** *Let $A$ be an infinite subset of metric space and $f : A \to A$ be continuous. If $f$ is topologically transitive on $A$ and the periodic points of $f$ are dense in $A$, then $f$ is chaotic on $A$.*

This means that we cannot check up exhibits of sensitive dependence on initial conditions of mapping. This property follows from others. Also see [2], Chapter 11.

**Theorem 3.5.** *Let $A$ be a subset of a metric space and $f : A \to A$. If the periodic points of $f$ are dense in $A$ and there is a point whose orbit under iteration of $f$ is dense in the set $A$, then $f$ is topologically transitive on $A$.*

Therefore we conclude

**Corollary 3.6.** *Function $f$ is chaotic in infinite metric space $X$, if following conditions are satisfied:*

*a) function $f$ is continuous in the set $X$,*

*b) the periodic points of $f$ are dense in the set $X$,*

*c) either there exists a point orbit of which by map $f$ is dense in the set $X$, either $f$ is topologically transitive in the set $X$.*

## 4. Shift Map

Now we consider bi-infinite symbol space $(\Sigma, d)$. At first we refer to some results of shift map.

**Definition 4.1.** The map $\sigma : \Sigma \to \Sigma$ is called a *shift map* if

$$\forall\, s = ...s_{-2}s_{-1}\,.s_0s_1s_2... \in \Sigma : \ \sigma(s) = ...s_{-2}s_{-1}s_0\,.s_1s_2....$$

It means that shift map "shifts" one symbol of sequence one step to the left.

The shift map (see, for example, [14]) is:

a) continuous,

b) has a countable infinity of periodic orbits of arbitrarily high period,

c) has an uncountable infinity of non-periodic orbits,

d) has a dense orbit.

Then by Corollary 3.6 the shift map is chaotic. The shift map is well known example of a chaotic map. But it is not unique chaotic map in space $(\Sigma, d)$. At first we generalize idea of shift map.

**Definition 4.2.** The map $\sigma_2 : \Sigma \to \Sigma$ is called a *second shift map* if $\forall\, s = ...s_{-2}s_{-1}\,.s_0s_1s_2... \in \Sigma : \sigma_2(s) = ...s_{-1}s_0s_1\,.s_2s_3....$

In other words, the second shift map "shifts" the two first digits of the sequence. It means that $\sigma_2 = \sigma^2$, but we have not found in literature that iteration of chaotic map is chaotic map therefore we will look at iteration maps are chaotic.

**Definition 4.3.** The map $\sigma_m : \Sigma \to \Sigma$ $(m \geq 1)$ is called a *m-th shift map* if

$$\forall\, s = ...s_{-2}s_{-1}\,.s_0s_1s_2...s_{m-1}s_ms_{m+1}... \in \Sigma :$$
$$\sigma_m(s) = ...s_{-1}s_0s_1s_2s_3...s_{m-1}\,.s_ms_{m+1}....$$

It means that $m$-th shift map transfers any symbol of sequence $m$ steps to the left.

Our aim is to prove that $m$-th shift map is chaotic for every $m \geq 2$.

**Proposition 4.4.** *The m-th shift map is continuous.*

*Proof.* Let $s$ be a point of $\Sigma$ and $\varepsilon > 0$. We must prove that there is $\delta > 0$ such that whenever $d(s,t) < \delta$, then $d(\sigma_m(s), \sigma_m(t)) < \varepsilon$.

We choose $n$ so that $\frac{1}{2^n} < \varepsilon$ and assume that $\delta = \frac{1}{2^{n+m+2}}$. If $d(s,t) < \delta$, then we know from Lemma 2.2 that for all $i$: $|i| < n+m+2$ satisfies $s_i = t_i$. In this case from $m$-th shift map definition follows that in images of $\sigma_m(s)$ and $\sigma_m(t)$ for all $i$: $|i| \leq n+1$ satisfies $s_i = t_i$, and therefore $d(\sigma_m(s), \sigma_m(t)) \leq \frac{1}{2^n} < \varepsilon$. □

**Proposition 4.5.** *The $m$-th shift map $\sigma_m : \Sigma \to \Sigma$ has the following properties:*

*1) the set of periodic points of $m$-th shift map is dense in $\Sigma$;*

*2) there is a point of $\Sigma$ whose orbit is dense in $\Sigma$;*

*3) the set of points that are neither periodic nor eventually periodic is dense in $\Sigma$.*

*Proof.* 1) We assume that $s = ...s_2s_1 .s_0s_1s_2... \in \Sigma$ ia a periodic point of $m$-th shift map $\sigma_m$ with period $k$. Then

$$\sigma_m^n(\sigma_m^k(s)) = \sigma_m^n(s).$$

Since $\sigma_m^n(s)$ shifts first $mn$ symbols to the left in sequence $s$, we see that

$$\begin{aligned} &\sigma_m^n(\sigma_m^k(...s_2s_1 .s_0s_1s_2...)) \\ &\quad = ...s_{mn+mk-3}s_{mn+mk-2}s_{mn+mk-1} .s_{mn+mk}s_{mn+mk+1}s_{mn+mk+2}... \\ &\qquad = \sigma_m^n(...s_2s_1 .s_0s_1s_2...)\,, \end{aligned}$$

and $s_{mn+mk} = s_{mn}$ for all $n$. Conclusion: $s$ is a periodic point of $m$-th shift map with period $k$ if and only if $s$ is a sequence formed by repeating the $mk$ symbols $s_0s_1s_2...s_{mk-1}$ infinitely often:

$$...s_0s_1s_2...s_{mk-1}s_0s_1s_2...s_{mk-1} .s_0s_1s_2...s_{mk-1}s_0s_1s_2...s_{mk-1}... .$$

To prove that the periodic points of $\sigma_m$ are dense in symbol space $\Sigma$, we need to prove that for every point $t$ of symbol space $\Sigma$ and for every $\varepsilon > 0$, there is a periodic point of $\sigma_m$ such that belongs to neigbourhood $N_\varepsilon(t)$.

Let $\varepsilon > 0$ and $t = ...t_{-3}t_{-2}t_{-1} .t_0t_1t_2... \in \Sigma$ be arbitrary. We choose $n \geq 2$ so that $\frac{1}{2^n}$. Then we choose sequence $s$ like this:

$$\begin{aligned} s = ...t_{-mn}...t_{-3}t_{-2}t_{-1}t_0t_1t_2...t_{mn-1}t_{-mn}...t_{-3}t_{-2}t_{-1} \quad &\text{left side to point} \\ .t_0t_1t_2...t_{mn-1}t_{-mn}...t_{-3}t_{-2}t_{-1}t_0t_1t_2...t_{mn-1}... \quad &\text{right side from point} \end{aligned}$$

i.e., a finite sequence of $2mn$ symbols $t_0t_1t_2...t_{mn-1}t_{-mn}...t_{-3}t_{-2}t_{-1}$ is repeated infinitely often to the both sides from the point.

Then $\forall |i| \leq n+1 : \quad s_i = t_i$, Lemma 2.2 implies that $d(s,t) \leq \frac{1}{2^n} < \varepsilon$. Therefore $s \in N_\varepsilon(t)$ and $s$ is a periodic point by construction.

2) The sequence which begins with 0 1 00 01 10 11 and then includes all possible blocks of 0 and 1 with three digits, followed by all possible blocks of

0 and 1 with four digits, and so forth is called the *Morse sequence* (see [2]). The orbit of this sequence is dense in one-sided infinite sequence space by shift map. We construct a one-sided sequence which consists of all possible blocks of 0 and 1 from Morse sequence repeated $m$ times

$$\underbrace{0\,0\,...\,0}_{m \text{ times } 0}\ \ \underbrace{1\,1\,...\,1}_{m \text{ times } 1}\ \underbrace{00\,00\,...\,00}_{m \text{ times } 00}\underbrace{01\,01\,...\,01}_{m \text{ times } 01}\underbrace{10\,10\,...10}_{m \text{ times } 10}\underbrace{11\,11\,...\,11}_{m \text{ times } 11}000\,000\,...$$

We assume that on the left side on bi-infinite sequence $s_m^*$ are 0 (or 1) but on the right side is above construct one-sided sequence. The orbit of this sequence $s_m^*$ by $m$-th shift map is dense in symbol space $\Sigma$.

Let $t \in \Sigma$ and $\varepsilon > 0$ be chosen arbitrary. We need to find a point of $s_m^*$ orbit such that distance from this point to $t$ is $,\varepsilon$.

We choose $n$ such that $n+1$ would be divided with m without remainder and $\frac{1}{2^n} < \varepsilon$. From the construction of $s_m^*$ follows that finite sequence

$$t_{-(n+1)}...t_{-3}t_{-2}t_{-1}t_0t_1t_2...t_{n+1}$$

is located somewhere in sequence $s_m^*$. Therefore there exists an iteration $k$ of the $m$-th shift map $\sigma_m$ such that

$$\sigma_m^k(s_m^*) = ...t_{-(n+1)}...t_{-3}t_{-2}t_{-1}\ .t_0t_1t_2...t_{n+1}...,$$

and from Lemma 2.2 follows that

$$d(t, \sigma_m^k(s_m^*)) \leq \frac{1}{2^n} < \varepsilon.$$

3) Since the set of nonperiodic points includes as a subset the orbit of the $s_m^*$, the truth of part 3) of this proposition follows from part 2). □

**Corollary 4.6.** *The $m$-th shift map is chaotic on $\Sigma$.*

*Proof.* The symbol space $\Sigma$ is infinite metric space. The $m$-th shift map is continuous (Proposition 4.4), its periodic points are dense in symbol space $\Sigma$ (Proposition 4.5.1) ) and there exists a point such that orbit of which by $m$-th shift map is dense in set $\Sigma$ (Proposition 4.5.2) ). From Corollary 3.6 follows that $m$-th shift map is chaotic in symbol space $\Sigma$. □

## 5. Other Chaotic Maps in Symbol Space

In this chapter we will look closer at a map that behaves a bit different from the shift map and which is chaotic as well. At first we will examine the $\alpha$-map.

**Definition 5.1.** The map $\alpha : \Sigma \to \Sigma$ is called an *alpha map* or *$\alpha$-map* if

$$\forall\, s = ...s_{-2}s_{-1}\,.s_0s_1s_2... \in \Sigma :\ \alpha(s) = ...s_{-n}...s_{-3}s_{-2}\,.s_1s_2...s_n....$$

It means that $\alpha$-map throws out from sequence first symbols to the both sides of the point. Second iteration of $\alpha$-map is $\alpha^2(s) = ...s_{-n}...s_{-4}s_{-3}\,.s_2 s_3...s_n....$

We will prove that $\alpha$-map is chaotic.

**Proposition 5.2.** *The $\alpha$-map is continuous.*

*Proof.* Let $s$ be a point from $\Sigma$ and $\varepsilon > 0$. We need to prove that

$$\exists\, \delta > 0 \quad \forall t \in \Sigma : \quad d(s,t) < \delta \ \Rightarrow d(\alpha(s), \alpha(t)) < \varepsilon.$$

We choose $n$ such that $\frac{1}{2^n} < \varepsilon$ and assume that $\delta = \frac{1}{2^{n+3}}$. If $d(s,t) < \delta$, then by second part of Lemma 2.2 it follows that $\forall |i| \le n+2 :\ \ s_i = t_i$. In this case from definition of $\alpha$-map follows that in images $\alpha(s)$ and $\alpha(t)$ for all $i :\ |i| \le n+1$ satisfies $s_i = s_i$, therefore using first part of Lemma 2.2 we conclude

$$d(\alpha(s), \alpha(t)) \le \frac{1}{2^n} < \varepsilon. \qquad \square$$

We introduce a notation $\overline{s} = ...sss...$, where $s$ denotes sequence of 0s and 1s with finite number of elements.

**Proposition 5.3.** *The map $\alpha : \Sigma \to \Sigma$ in the set $\Sigma$ has a countable number of periodic points and the set of periodic points is dense in symbol space $\Sigma$.*

*Proof.* $\alpha$-map has 4 fixed points, i.e., ...00 .00...; ...11 .11...; ...00 .11...; ...11 .00... or $\overline{0}\,.\overline{0}$, $\overline{1}\,.\overline{1}$, $\overline{0}\,.\overline{1}$,$\overline{1}\,.\overline{0}$.

It is easy to see that the orbits of sequences that are periodically repeated are periodic under iterations by $\alpha$-map. For example, let us take a look at sequence $\overline{10}\,.\overline{10}$. Then $\alpha(\overline{10}\,.\overline{10}) = \overline{01}\,.\overline{01}$ and $\alpha(\overline{01}\,.\overline{01} = \overline{10}\,.\overline{10}$. The second iteration of $\alpha$-map is identical to the starting sequence: $\alpha^2(\overline{10}\,.\overline{10}) = \overline{10}\,.\overline{10}$. Therefore the orbit of sequence $\overline{10}\,.\overline{10}$ is an orbit of period two for $\alpha$-map.

From this particular example we can see that if sequence consists of blocks of 0 and 1 in the length $k$, that are periodically repeated, then $\alpha$-map (exactly the same as shift map $\sigma$) $k$-th iteration is identical to the starting sequence. That is, the orbit of this sequence by $\alpha$-map is periodic with period $k$.

We have noticed that $\overline{10}\,.\overline{10}$ is periodic point with period 2 as well. $\alpha$-map has these periodic points with period 2:
$\overline{0}\,.\overline{0}$, $\overline{0}\,.\overline{01}$, $\overline{0}\,.\overline{10}$, $\overline{0}\,.\overline{1}$, $\overline{01}\,.\overline{0}$, $\overline{01}\,.\overline{01}$, $\overline{01}\,.\overline{10}$, $\overline{01}\,.\overline{1}$, $\overline{10}\,.\overline{0}$, $\overline{10}\,.\overline{01}$, $\overline{10}\,.\overline{10}$, $\overline{10}\,.\overline{1}$, $\overline{1}\,.\overline{0}$,

$\overline{1}.\overline{01}$, $\overline{1}.\overline{10}$, $\overline{1}.\overline{1}$.
It means that a number of periodic points is equal to a number of all possible sequences in length 4 that consist of 0s and 1s, that is, $2^4 = 16$. The same situation will be with the periodic points which have a period of 3: we need to find all possible sequences in length of 6 that consist of 0s and 1s, that is, $2^6$. Periodic points with period $m$ will be $2^{2m}$. The number of these sequences is finite, therefore $\alpha$-map has a countable number of periodic points wich have all possible periods.

In order to prove that periodic points of $\alpha$-map are dense in symbol space $\Sigma$, we need to prove that for every point $t$ from symbol space $\Sigma$ and for every $\varepsilon > 0$ there exists a periodic point of $\alpha$-map that belongs to neighbourhood $N_\varepsilon(t)$. It means that we need to find a sequence from $N_\varepsilon(t)$ such that some first symbols are repeated periodically infinite number of times to the both sides of the point.

Let $\varepsilon > 0$ and $t = ...t_{-1}.t_0t_1... \in \Sigma$ be arbitrary. We find $n$ such that $\frac{1}{2^n} < \varepsilon$. We choose $s$

$$s = ...t_{-(n+2)}...t_{-2}t_{-1}t_{-(n+2)}...t_{-2}t_{-1}\,.t_0t_1...t_{n+1}t_0t_1...t_{n+1}...,$$

$$\text{i.e., } s = \overline{t_{-(n+2)}...t_{-2}t_{-1}}\,.\overline{t_0t_1...t_{n+1}}.$$

Since $\forall|i| \leq n+1:\ s_i = t_i$ then from Lemma 2.2 it follows that

$$d(s,t) \leq \frac{1}{2^n} < \varepsilon.$$

Therefore $s \in N_\varepsilon(t)$ and $s$ is a periodic point with period $n+2$ by construction.
□

In order to prove that $\alpha$-map is chaotic, we need to prove that there exists a point such that orbit of this point is dense in symbol space $\Sigma$ or $\alpha$-map is topologically transitive in symbol space $\Sigma$. We prove that $\alpha$-map is topologically transitive.

**Proposition 5.4.** *The $\alpha$-map is topologically transitive in symbol space $\Sigma$.*

*Proof.* By Definition 3.2 we need to prove that

$$\forall s,t \in \Sigma \quad \forall \varepsilon > 0 \quad \exists z \in \Sigma \quad \exists k \in \mathbf{N}:\ d(z,s) < \varepsilon \text{ and } d(\alpha^k(z),t) < \varepsilon.$$

Let $\varepsilon > 0$, $s = ...s_{-n}...s_{-2}s_{-1}\,.s_0s_1s_2...s_n... \in \Sigma$ and $t = ...t_{-n}...t_{-2}t_{-1}\,.t_0t_1 t_2...t_n... \in \Sigma$ be arbitrary. We find $n$ such that $\frac{1}{2^n} < \varepsilon$. We choose sequence $z$ such that $\forall|i| \leq n+1:\ z_i = s_i$, then from Lemma 2.2 it follows that $d(z,s) \leq \frac{1}{2^n} < \varepsilon$. Since $\alpha^k(z) = ...z_{-(k+2)}z_{-(k+1)}\,.z_kz_{k+1}...$ and we need to find

$k \in \mathbf{N}$ such that $d(\alpha^k(z), t) < \varepsilon$ then

$$\alpha^k(z) = ...t_{-(n+2)}...t_{-2}t_{-1}\,.t_0t_1t_2...t_{n+1}...\,.$$

Therefore we choose $z$:

$$z = ......t_{-(n+2)}...t_{-2}t_{-1}s_{-(n+2)}...s_{-2}s_{-1}\,.s_0s_1s_2...s_{n+1}t_0t_1t_2...t_{n+1}...$$

and $k = n + 2$ (where $n$ was chosen at the beginning such that $\frac{1}{2^n} < \varepsilon$). □

From Propositions 5.2, 5.3, 5.3 and Corollary 3.6 follows that

**Theorem 5.5.** *The $\alpha$ -map is chaotic in symbol space $\Sigma$.*

Now we generalize concept of $\alpha$-map. We consider $\alpha_{mk}$-map.

**Definition 5.6.** The map $\alpha_{mk} : \Sigma \to \Sigma$ $(m, k \in \mathbf{N})$ is called a $\alpha_{mk}$*-map* if
$\forall\ s = ...s_{-2}s_{-1}\,.s_0s_1s_2... \in \Sigma : \quad \alpha_{mk}(s) = ...s_{-(m+2)}s_{-(m+1)}\,.s_ks_{k+1}....$

It means that $\alpha_{mk}$-map throws out from the left side first $m$ symbols and from the right side first $k$ symbols. The second iteration of $\alpha_{mk}$-map is

$$\alpha_{mk}(s) = ...s_{-(2m+2)}s_{-2m+1)}\,.s_{2k}s_{2k+1}....$$

This mapping is chaotic too.

**Proposition 5.7.** *The $\alpha_{mk}$-map is continuous.*

*Proof.* Let $s$ be a point from $\Sigma$ and $\varepsilon > 0$.

We choose $n$ such that $\frac{1}{2^n} < \varepsilon$ and assume that

$$\delta = \frac{1}{2^{n+2+\max\{m,k\}}}.$$

If $d(s, t) < \delta$, then by Lemma 2.2 it follows that

$$\forall |i| \le n + 1 + \max\{m, k\} : \ s_i = t_i.$$

From definition of $\alpha_{mk}$-map follows that in images $\alpha_{mk}(s)$ and $\alpha_{mk}(t)$ for all $i : \ |i| \le n + 1$ satisfies $s_i = s_i$, therefore using Lemma 2.2 we conclude

$$d(\alpha_{mk}(s), \alpha_{mk}(t)) \le \frac{1}{2^n} < \varepsilon.$$ □

**Proposition 5.8.** *The set of periodic points of $\alpha_{mk}$-map is dense in symbol space $\Sigma$.*

*Proof.* $\alpha_{mk}$-map has 4 fixed points, i.e., ...00.00...; ...11.11...; ...00.11...; ...11.00... or $\overline{0}.\overline{0}$, $\overline{1}.\overline{1}$, $\overline{0}.\overline{1}$,$\overline{1}.\overline{0}$. But set of fixed points is bigger. The bi-infinite sequence

$$s = ...s_{-m}...s_{-3}s_{-2}s_{-1}s_{-m}...s_{-3}s_{-2}s_{-1}\,.s_0s_1s_2...s_{k-1}s_0s_1s_2...s_{k-1}... \in \Sigma$$

$$\text{or } s = \overline{s_{-m}...s_{-3}s_{-2}s_{-1}}\,.\overline{s_0s_1s_2...s_{k-1}}$$

is fixed point of $\alpha_{mk}$-map, $s_{-m},...,s_{-3},s_{-2},s_{-1},s_0,s_1,s_2,...,s_{k-1} \in \{0,1\}$. From 2 symbols 0 and 1 it is possible to construct $2^m$ (respectively $2^k$) finite sequences with length $m$ (respectively $k$). Therefore $\alpha_{mk}$-map has $2^{mk}$ fixed points.

The periodic point of $\alpha_{mk}$-map with period two is

$$s = \overline{s_{-2m}...s_{-2}s_{-1}}\,.\overline{s_0s_1s_2...s_{2k-1}} \in \Sigma,$$

i.e., a finite sequence $s_{-2m}...s_{-2}s_{-1}$ is repeated infinitely often on the left side from the point and a finite sequence $s_0s_1s_2...s_{2k-1}$ is repeated infinitely often on the right side from the point.

We conclude that the bi-infinite sequence

$$s = \overline{s_{-pm}...s_{-2}s_{-1}}\,.\overline{s_0s_1s_2...s_{pk-1}} \in \Sigma$$

is periodic point of $\alpha_{mk}$-map with period $p$.

We show that the set of periodic points of $\alpha_{mk}$-map is dense in symbol space $\Sigma$.

Let $\varepsilon > 0$ and $t = ...t_{-2}t_{-1}\,.t_0t_1... \in \Sigma$. We find $n$ such that $n+2$ divides with $m$ and $k$ without reminder and $\frac{1}{2^n} < \varepsilon$. We choose $s \in \Sigma$:

$$s = \overline{t_{-(n+2)}...t_{-2}t_{-1}}\,.\overline{t_0t_1t_2...t_{n+1}}.$$

The sequence $s$ is periodic point of $\alpha_{mk}$-map with period $\frac{(n+2)^2}{mk}$. Since $\forall |i| \leq n+1: \; s_i + t_i$ then by Lemma 2.2 it follows that $d(s,t) \leq \frac{1}{2^n} < \varepsilon$. □

**Proposition 5.9.** *The $\alpha_{mk}$-map is topologically transitive in symbol space $\Sigma$.*

*Proof.* Let $\varepsilon > 0$, $s = ...s_{-n}...s_{-2}s_{-1}\,.s_0s_1s_2...s_n... \in \Sigma$ and $t = ...t_{-n}...t_{-2}t_{-1}\,.t_0t_1t_2...t_n... \in \Sigma$ be arbitrary. We prove that $\exists z \in \Sigma$ and $\exists l \in \mathbf{N}$ such that $d(z,s) < \varepsilon$ and $d(\alpha_{mk}^j(z),t) < \varepsilon$.

We choose $n$ such that $\frac{1}{2^n} < \varepsilon$. We choose sequence $z$ such that $\forall |i| \leq n+1: \; z_i = s_i$, then from Lemma 2.2 it follows that $d(z,s) \leq \frac{1}{2^n} < \varepsilon$. Since $\alpha_{mk}^l(z) = ...z_{-(lk+2)}z_{-(lk+1)}\,.z_{kl}z_{lk+1}...$ and we need to find $l \in \mathbf{N}$ such that $d(\alpha_{mk}^l(z),t) < \varepsilon$ then

$$\alpha_{mk}^l(z) = ...t_{-(n+2)}...t_{-2}t_{-1}.t_0t_1t_2...t_{n+1}... \,.$$

Therefore we choose $z$:

$$z = ...t_{-(n+2)}...t_{-2}t_{-1}s_{-m(n+2)}...s_{-2}s_{-1}\,.s_0s_1s_2...s_{k(n+2)-1}t_0t_1t_2...t_{n+1}...$$

and $l = n+2$ (where $n$ was chosen at the beginning such that $\frac{1}{2^n} < \varepsilon$). □

From Propositions 5.7, 5.8, 5.9 and Corollary 3.6 it follows

**Theorem 5.10.** *The $\alpha_{mk}$-map is chaotic in symbol space $\Sigma$.*

It is clear that $\alpha$-map is special case of $\alpha_{mk}$-map with $m = k = 1$. But in this generalized case we have only one aim — to show that $\alpha_{mk}$-map is chaotic. Since it is not necessary to prove a large amount of periodic points in order to prove Proposition 5.8 we have to clean up the appearance of periodic points and we have used this idea in proof of density of periodic points.

Now we consider one different map.

**Definition 5.11.** The map $\beta : \Sigma \to \Sigma$ is called a $\beta$-*map* if

$\forall\, s = ...s_{-2}s_{-1}\,.s_0s_1s_2... \in \Sigma : \quad \beta(s) = ...s_{-n}...s_{-5}s_{-4}s_{-2}.s_1s_3s_4...s_n....$

It means that $\beta$-map throws out of the sequence first and third symbols to the both sides of the point. Second iteration of $\beta$-map is $\beta^2(s) = ...s_{-n}...s_{-6}s_{-4}.s_3s_5...s_n....$

We will have to prove that $\beta$-map is chaotic and we will start just as we did before with continuity.

**Proposition 5.12.** *The $\beta$-map is continuous.*

*Proof.* Let $s$ be a point from $\Sigma$ and $\varepsilon > 0$.

We choose $n$ such that $\frac{1}{2^n} < \varepsilon$ and assume that $\delta = \frac{1}{2^{n+4}}$. If $d(s,t) < \delta$, then by Lemma 2.2 it follows that $\forall |i| \le n+3 : \ s_i = t_i$. In this case from definition of $\beta$-map it follows that in images $\beta(s)$ and $\beta(t)$ for all $i : \ |i| \le n+1$ satisfies $s_i = t_i$, therefore using Lemma 2.2

$$d(\beta(s), \beta(t)) \le \frac{1}{2^n} < \varepsilon. \qquad \square$$

**Proposition 5.13.** *The map $\beta : \Sigma \to \Sigma$ in the set $\Sigma$ has a countable number of periodic points and the set of periodic points is dense in symbol space $\Sigma$.*

*Proof.* It is obvious that $\beta$-map has 4 fixed points $\overline{0}\,.\overline{0}$, $\overline{1}\,.\overline{1}$,$\overline{0}\,.\overline{1}$,$\overline{1}\,.\overline{0}$. But these points are not the only ones.

We find $\beta$-map periodic point with period $k$. Let $s = ...s_{-2}s_{-1}\,.s_0s_1s_2... \in \Sigma$. Then

$$\beta(s) = ...s_{-5}s_{-4}s_{-2}\,.s_1s_3s_4...\,; \ \beta^2 = ...s_{-7}s_{-6}s_{-4}\,.s_3s_5s_6...\,;$$

$$\beta^3 = ...s_{-9}s_{-8}s_{-6}\,.s_5s_7s_8...\,; ...\,; \ \beta^k = ...s_{-2k-3}s_{-2k-2}s_{-2k}\,.s_{2k-1}s_{2k+1}s_{2k+2}...$$

If $s$ is a periodic point of $\beta$-map with period $k$, then

$$s_0 = s_{2k-1}, \qquad s_n = s_{n+2k},\ n > 0,$$
$$s_{-1} = s_{-2k}, \qquad s_n = s_{n-2k},\ n < -1,$$

i.e.

$$s = \begin{cases} \ldots s_{-(2k+1)}s_{-1}s_{-(2k-1)}\ldots s_{-2}s_{-(2k+1)}s_{-1}s_{-(2k-1)}\ldots s_{-2}s_{-1}\,, & \\ \hfill \text{left side to point}\,, & \\ .s_0s_1s_2\ldots s_{2k-2}s_0s_{2k}s_1s_2\ldots s_{2k-2}s_0s_{2k}\cdots\,, & \\ \hfill \text{right side from point}\,. & \end{cases}$$

Consequently periodic point of $\beta$-map with period 1 will look as follows:

$$s_0 = s_1 = s_{1+2} = s_{3+2} = s_{5+2} = \ldots$$
$$s_2 = s_{2+2} = s_{4+2} = s_{6+2} = s_{8+2} = \ldots$$
$$s_{-1} = s_{-1-1} = s_{-2-2} = s_{-4-2} = s_{-6-2} = \ldots$$
$$s_{-3} = s_{-3-2} = s_{-5-2} = s_{-7-2} = s_{-9-2} = \ldots\,.$$

Since $s_0$, $s_2$, $s_{-1}$, $s_{-3}$ (change makes $4 \times 1$ simbols) may be 0 and 1, then there are $2^4 = 16$ periodic points with period 1:
$\overline{0}\,.\overline{0}$, $\overline{0}\,.\overline{01}$, $\overline{01}1\,.\overline{0}$, $\overline{10}0\,.\overline{0}$, $\overline{01}1\,.0\overline{01}$, $\overline{10}0\,.0\overline{01}$, $\overline{1}\,.\overline{0}$, $\overline{1}\,.0\overline{01}$, $\overline{0}\,.1\overline{10}$, $\overline{0}\,.\overline{1}$, $\overline{01}1\,.1\overline{10}$, $\overline{10}0\,.1\overline{10}$, $\overline{01}1\,.\overline{1}$, $\overline{10}0\,.\overline{1}$, $\overline{1}\,.1\overline{10}$, $\overline{1}\,.\overline{1}$.

The condition complex with period 2 is much longer. In this case change makes 8 symbols (or $4 \times 2$), it means that there are $2^8 = 256$ periodic points with period 2.

Analogically we can look at a periodic point with period $k$. In this case change makes $4k$ simbols (or $4 \times k$). Therefore there are $2^{4k}$ periodic points with period $k$. It is finite number for every $k$, then $\beta$-map has a countable number of periodic points.

Now we prove that periodic points of $\beta$-map are dense in symbol space $\Sigma$.

Let $\varepsilon > 0$ and $t = \ldots t_{-1}\,.t_0t_1\ldots \in \Sigma$ be arbitrary. We find $n > 2$ such that $\frac{1}{2^n} < \varepsilon$. We choose $s$

$$s = \ldots t_{-(2n+1)}t_{-1}$$
$$t_{-(2n-1)}\ldots t_{-2}t_{-(2n+1)}t_{-1}t_{-(2n-1)}\ldots t_{-2}t_{-1} \quad \text{left side to point}$$
$$.t_0t_1t_2\ldots t_{2n-2}t_0t_{2n}t_1t_2\ldots t_{2n-2}t_0t_{2n}\ldots \quad \text{right side from point}$$

Since $\forall |i| \leq n+1:\ s_i = t_i$ then from Lemma 2.2 it follows that

$$d(s,t) \leq \frac{1}{2^n} < \varepsilon.$$

Therefore $s \in N_\varepsilon(t)$ and $s$ is a periodic point with period $n$ by construction.

**Proposition 5.14.** *The $\beta$-map is topologically transitive in symbol space $\Sigma$.*

*Proof.* Let $\varepsilon > 0$ and $s, t \in \Sigma$ be arbitrary. We find $n > 2$ such that $\frac{1}{2^n} < \varepsilon$. We chose sequence $z$ such that $\forall |i| \leq n+1:\ z_i = s_i$, then from Lemma 2.2 it follows that $d(z,s) \leq \frac{1}{2^n} < \varepsilon$. Since

$$\beta^k(z) = ...z_{-(2k+3)}z_{-(2k+2)}z_{-2k} .z_{2k-1}z_{2k+1}z_{2k+2}...$$

and we need to find $k \in \mathbf{N}$ such that $d(\beta^k(z), t) < \varepsilon$, then

$$\beta^k(z) = ...t_{-(n+2)}...t_{-2}t_{-1} .t_0t_1t_2...t_{n+1}... .$$

Therefore we choose $z$:

$$z = ...000t_{-(n+1)}...t_{-2}0t_{-1} \underbrace{000...000}_{n-2 \text{ symbols}} s_{-(n+1)}...s_{-2}s_{-1} \quad \text{left side to point}$$

$$.s_0s_1s_2...s_{n+1} \underbrace{000...000}_{n-3 \text{ symbols}} t_00t_1t_2...t_{n+1}000... \quad \text{right side from point}$$

and $k = n$ (where $n$ was chosen as the beginning such that $\frac{1}{2^n} < \varepsilon$). Then $d(\beta^n(z), t) < \varepsilon$.

For example, if $\varepsilon > 0$ is chosen such that $\frac{1}{2^3} < \varepsilon$, i.e., $n = 3$, then

$$z = ...000t_{-4}t_{-3}t_{-2}0t_{-1}0s_{-4}s_{-3}s_{-2}s_{-1}.s_0s_1s_2s_3s_4t_00t_1t_2t_3t_4000... .$$

It is clear that $d(z,s) \leq \frac{1}{2^3} < \varepsilon$ by Lemma 2.2. Next we find 3-rd iteration of $\beta$-map:

$$\beta(z) = ...000t_{-4}t_{-3}t_{-2}0t_{-1}0s_{-4}s_{-2}.s_1s_3s_4t_00t_1t_2t_3t_4000...,$$
$$\beta^2(z) = ...000t_{-4}t_{-3}t_{-2}0t_{-1}s_{-4}.s_3t_00t_1t_2t_3t_4000...,$$
$$\beta^3(z) = ...000t_{-4}t_{-3}t_{-2}t_{-1}.t_0t_1t_2t_3t_4000... .$$

We see that $d(\beta^3(z), t) \leq \frac{1}{2^3} < \varepsilon$.

We note that in sequence $z$ can be 1s or 0s instead of our 0. It means that $z$ is not unique. □

From Propositions 5.12, 5.13, 5.14 and Corollary 3.6 it follows

**Theorem 5.15.** *The $\beta$-map is chaotic in symbol space $\Sigma$.*

It is clear that shift map and its iterations and there other examined chaotic maps are not unique chaotic maps in bi-infinite symbol space $\Sigma$. The work started by this article could be a great experience in further map dynamics exploration and classification.

## 6. Non-Chaotic Maps in Symbol Space

In this chapter we will look closer at map of set $\Sigma$, denoted by $\gamma$-map and defined as follows:

**Definition 6.1.** The map $\gamma : \Sigma \to \Sigma$ is called a *$\gamma$-map* if

$$\forall\ s = ...s_{-2}s_{-1}\,.s_0s_1s_2... \in \Sigma : \quad \gamma(s) = ...s_{-n}...s_{-3}s_{-2}s_{-1}\,.s_1s_2s_3...s_n....$$

It means that $\gamma$-map throws out of the sequence the first symbol to the right side of the point, but the left side is not changed.

The second iteration of $\gamma$-map is

$$\gamma^2(s) = ......s_{-n}...s_{-3}s_{-2}s_{-1}\,.s_2s_3s_4...s_n....$$

We want to show that this map in symbol spaces is not chaotic. From Definition 3.1 it follows that three conditions must be satisfied in order to prove that the map is chaotic. It means that it is enough to prove that one of these conditions is not satisfied and we can state that the map is not chaotic.

**Proposition 6.2.** *$\gamma$-map is not topologically transitive in symbol space $\Sigma$.*

*Proof.* By Definition 3.2 the mapping $\gamma : \Sigma \to \Sigma$ is topologically transitive in the set $\Sigma$, if

$$\forall s, t \in \Sigma \quad \forall \varepsilon > 0 \quad \exists z \in \Sigma \quad \exists k \in \mathbf{N} : \quad d(z, s) < \varepsilon \text{ and } d(\gamma^k(z), t) < \varepsilon.$$

We need to prove the opposite:

$$\exists s, t \in \Sigma \quad \exists \varepsilon > 0 \quad \forall z \in \Sigma \quad \forall k \in \mathbf{N} : \quad d(z, s) \geq \varepsilon \text{ or } d(\gamma^k(z), t) \geq \varepsilon.$$

We assume that $s = ...000\,.000...$, $t = ...111\,.111...$ and $\varepsilon = \frac{1}{2} > 0$. Let $z \in \Sigma$ be arbitrary. Two cases are possible:

1) if $z_0 \neq 0$, then by definition of metrics $d(z, s) \geq \frac{1}{2^0} = 1 > \frac{1}{2} = \varepsilon$; if $z_0 = 0$, but $z_{-1} \neq 0$ or $z_1 \neq 0$, then $d(z, s) \geq \frac{1}{2^1} = \frac{1}{2} = \varepsilon$ as well.

2) if $z_{-1} = z_0 = z_1 = 0$, then we cannot state that $d(z, s) \geq \varepsilon$, but in this case $\forall k \in \mathbf{N} :\ \gamma^k(z) = ...z_{-3}z_{-2}z_{-1}\,.z_{k+1}z_{k+2}z_{k+3}....$ In this iteration $z_{-1} = 0$, but $t_{-1} = 1$, then $z_{-1} \neq t_{-1}$, therefore $d(\gamma^k(z), t) \geq \frac{1}{2^1} = \frac{1}{2}\varepsilon$. □

**Corollary 6.3.** *$\gamma$-map is not chaotic in symbol space $\Sigma$.*

Exactly as we proved that $\gamma$-map is not chaotic in symbol space $\Sigma$, we can prove that any map that has at least one fixed symbol (not moving to the left or to the right from the point) will not be chaotic in symbol space $\Sigma$.

**Theorem 6.4.** *If map $\overline{\gamma} : \Sigma \to \Sigma$ has the property: $\exists m \in \mathbf{Z}$ such that $\forall s = ...s_{-2}s_{-1}.s_0s_1s_3... \in \Sigma$ the symbol $s_m$ does not change its place by mapping $\overline{\gamma}$ then $\overline{\gamma}$-map is not topologically transitive.*

*Proof.* We choose $s = ...000\,.000...$, $t = ...111\,.111...$ and $\varepsilon = \frac{1}{2^{|m|}} > 0$. Let

$z \in \Sigma$. If the symbol $z_m \neq 0$, then

$$d(z, s) \geq \frac{1}{2^{|m|}} = \varepsilon.$$

If the symbol $z_m = 0$, then by definition of $\overline{\gamma}$ for every iteration $k$ in the sequence $\overline{\gamma}^k(z)$ the symbol $z_m$ is 0 and therefore

$$d(\overline{\gamma}^k(z), t) \geq \frac{1}{2^{|m|}} = \varepsilon. \qquad \square$$

**Corollary 6.5.** *$\overline{\gamma}$-map is not chaotic in symbol space $\Sigma$.*

## References

[1] J. Banks, J. Brooks, G. Cairns, G. Davis, P. Stacey, On Devaney's definition of chaos, *Amer. Math. Monthly*, **99** (1992), 29-39.

[2] R. Devaney, *An Introduction to Chaotic Dynamical Systems*, Benjamin-Cummings, Menlo Park, CA (1986).

[3] D. Gulick, *Encounters with Chaos*, McGraw-Hill (1992).

[4] J. Hadamard, Les surfacews a courbures opposées et leurs lignes geodesiques, *Journal de Mathematiques Pures et Appliqué*, **4** (1898), 27-73.

[5] R.A. Holmgren, A first course in discrete dynamical systems, *Universitext*, Second Edition, Springer-Verlag (1996).

[6] B.P. Kitchens, *Symbolic Dynamics. One-Sided, Two-Sided and Countable State Markov Shifts*, Springer-Verlag (1998).

[7] N. Levinson, A second order differential equation with singular solutions, *Ann. Math.*, **50** (1949), 127-153.

[8] D. Lind, B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press (1995).

[9] T.Y. Li, J.A. Yorke, Period three implies chaos, *Amer. Math. Monthly*, **82**, No. 12 (1975), 985-992.

[10] M. Morse, G. Hedlund, Symbolic dynamics, *Amer. J. Math.*, **60** (1938), 815-866.

[11] C. Robinson, *Dynamical Systems. Stability, Symbolic Dynamics, and Chaos*, CRS Press (1995).

[12] S. Smale, Diffeomorphisms with many periodic points, In: *Differential and Combinatorial Topology* (Ed. S.S. Cairns), Princeton University Press (1963), 63-80.

[13] S. Smale, *The Mathematics of Time: Essays on Dynamical Systems; Economic Processes, and Related Topics*, Springer-Verlag (1980).

[14] S. Wiggins, *Global Bifurcations and Chaos. Analytical Methods*, Applied Mathematical Sciences, Volume 73, Springer-Verlag (1988).